

Stochastic evolution equations of jump type: existence, uniqueness and large deviation principles

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Abstract

This paper has two parts. In part I, existence and uniqueness results are established for solutions of stochastic evolution equations driven both by Brownian motion and by Poisson point processes. Exponential integrability of the solution are also proved. In part II, a large deviation principle is obtained for stochastic evolution equations driven by additive Lévy noise.

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1 Introduction

Stochastic evolution equations and stochastic partial differential equations driven by Wiener processes have been studied by many people. There exists a great amount of literature on the subject, see, for example the monograph [DZ]. In contrast, there has not been very much study of stochastic partial differential equations driven by jump processes. However, it begun to gain attention recently. In [AWZ] we obtained existence and uniqueness for solutions of stochastic reaction diffusion equations driven by Poisson random measures. In [F], Malliavin calculus was applied to study the absolute continuity of the law of the solutions of stochastic reaction diffusion equations driven by Poisson random measures. In [MC], a minimal solution was obtained for the stochastic heat equation driven by non-negative Lévy noise with coefficients of polynomial growth. In [ML], a weak solution is established for the stochastic heat equation driven by stable noise with coefficients of polynomial growth.

In this paper, we consider the following evolution equation:

$$dY_t = -AY_t dt + b(Y_t)dt + \sigma(Y_t)dB_t + \int_X f(Y_{t-}, x)\tilde{N}(dt, dx), \quad (1.1)$$

$$Y_0 = h \in H, \quad (1.2)$$

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in the framework of a Gelfand triple :

$$V \subset H \cong H^* \subset V^*, \tag{1.3}$$

where H, V are Hilbert spaces, A is the infinitesimal generator of a strongly continuous semigroup, b, σ, f are measurable mappings from H into H . The solutions are considered to be weak solutions (in the PDE sense) in the space V and not as mild solutions in H as is more common in the literature. The stochastic evolution equations of this type driven by Wiener processes were first studied by E.Pardoux in [P] and subsequently in [KR]. For stochastic equations with general Hilbert space valued semimartingales replacing the Brownian motion we refer to [GK1], [GK2], and [G]. A large deviation principle for this type of stochastic evolution equations driven by Wiener process was obtained by P.Chow in [C].

The purpose of this paper is twofold. The first one is to establish the existence and uniqueness for solutions of equation (1.1). Our approach is similar to the one in [P], [GK1], [GK2], [G]. We, however, don't use Galerkin approximations. Instead, we get the solution via successive approximations. Secondly we will study the large deviation principle and exponential integrability of the solutions. We will prove that the solution is exponentially integrable both as a random variable in $D([0, 1] \rightarrow H)$ and in $L^2([0, 1] \rightarrow V)$. These estimates are of their own interest and also necessary for the study of large deviations. For the large deviation principle, we confine ourselves to the case of additive Lévy noise. The situation is quite different from the Gaussian case. If $X_t, t \geq 0$ is a Wiener process, the solution of the equation:

$$dY_t^n = -AY_t^n dt + \frac{1}{n}dX_t$$

is still Gaussian. The large deviations of Y^n follows from the well known large deviations of Gaussian processes. However, if $X_t, t \geq 0$, is a Lévy process, the solution Y^n is no longer a Lévy process. The additive noise case is already quite involved. Large deviations for stochastic evolution equations and stochastic partial differential equations have been investigated in many papers, see e.g. [CM2],[CR], [CW], [Z]. To the best of our knowledge, this is the first to study large deviations for stochastic evolution equations driven by Lévy processes. Large deviations for Lévy processes on Banach spaces and large deviations for solutions of stochastic differential equations driven by Poisson measures were studied by de Acosta in [A1], [A2].

The rest of the paper is organized as follows. In Section 2 we present our framework. Existence and uniqueness are proved in Section 3. In Section 4, exponential integrability is established for the solutions. Section 5 is devoted to the large deviation principle.

2 Framework

Let V, H be two separable Hilbert spaces such that V is continuously, densely imbedded in H . Identifying H with its dual we have

$$V \subset H \cong H^* \subset V^*, \quad (2.1)$$

where V^* stands for the topological dual of V . Let A be a bounded linear operator from V to V^* satisfying the following coercivity hypothesis: There exist constants $\alpha > 0$ and $\lambda_0 \geq 0$ such that

$$2\langle Au, u \rangle + \lambda_0 \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad \text{for all } u \in V. \quad (2.2)$$

$\langle Au, u \rangle = Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.

We remark that A is generally not bounded as an operator from H into H . Let (Ω, \mathcal{F}, P) be a probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let $\{B_t, t \geq 0\}$ be a real-valued \mathcal{F}_t -Brownian motion. $\{W_t, t \geq 0\}$ will denote an H -valued \mathcal{F}_t -Brownian motion with covariance operator Q which could be degenerate. Let $(X, \mathcal{B}(X))$ be a measurable space and $\nu(dx)$ a σ -finite measure on it. Let $p = (p(t)), t \in D_p$ be a stationary \mathcal{F}_t -Poisson point process on X with characteristic measure ν . See [IW] for the details on Poisson point processes. Denote by $N(dt, dx)$ the Poisson counting measure associated with p , i.e., $N(t, A) = \sum_{s \in D_p, s \leq t} I_A(p(s))$. Let $\tilde{N}(dt, dx) := N(dt, dx) - dt\nu(dx)$ be the compensated Poisson measure. Let b, σ be measurable mappings from H into H , and $f(y, x)$ a measurable mapping from $H \times X$ into H . For a separable Hilbert space L , we denote by $M^2([0, T], L)$ the Hilbert space of progressively measurable, square integrable, L -valued processes equipped with the inner product $\langle a, b \rangle_M = E[\int_0^T \langle a_t, b_t \rangle_L dt]$. Denote by $M^{\nu, 2}([0, T] \times X, L)$ the collection of predictable mappings:

$$g(s, x, \omega) : [0, T] \times X \times \Omega \rightarrow L$$

such that $E[\int_0^T \int_X |g(s, x, \omega)|_L^2 ds \nu(dx)] < \infty$. Denote by $D([0, T], L)$ the space of all cadlag paths from $[0, T]$ into L . Consider the stochastic evolution equation:

$$dY_t = -AY_t dt + b(Y_t) dt + \sigma(Y_t) dB_t + \int_X f(Y_{t-}, x) \tilde{N}(dt, dx), \quad (2.3)$$

$$Y_0 = h \in H. \quad (2.4)$$

We introduce

(H.1) There exists a constant $C < \infty$ such that

$$|b(y)|_H^2 + |\sigma(y)|_H^2 + \int_X |f(y, x)|_H^2 \nu(dx) \leq C(1 + |y|_H^2), \quad (2.5)$$

for all $y \in H$.

(H.2) There exists a constant $C < \infty$ such that

$$\begin{aligned} & |b(y_1) - b(y_2)|_H^2 + |\sigma(y_1) - \sigma(y_2)|_H^2 \\ & + \int_X |f(y_1, x) - f(y_2, x)|_H^2 \nu(dx) \\ & \leq C |y_1 - y_2|_H^2, \end{aligned} \quad (2.6)$$

for all $y_1, y_2 \in H$.

We finish this section by two examples.

Example 2.1 Let $H = L^2(\mathbf{R}^d)$, and set

$$V = H_2^1(\mathbf{R}^d) = \{u \in L^2(R^d); \nabla u \in L^2(\mathbf{R}^d \rightarrow \mathbf{R}^d)\}.$$

Denote by $a(x) = (a_{ij}(x))$ a matrix-valued function on \mathbf{R}^d satisfying the uniform ellipticity condition:

$$\frac{1}{c} I_d \leq a(x) \leq c I_d \quad \text{for some constant } c \in (0, \infty).$$

Let $b(x)$ be a vector field on \mathbf{R}^d with $b \in L^p(\mathbf{R}^d)$ for some $p > d$. Define

$$Au = -\operatorname{div}(a(x)\nabla u(x)) + b(x) \cdot \nabla u(x).$$

Then (2.2) is fulfilled for (H, V, A) .

Example 2.2 Stochastic evolution equations associated with fractional Laplacian:

$$dY_t = \Delta_\alpha Y_t dt + dL_t, \quad (2.7)$$

$$Y_0 = h \in H, \quad (2.8)$$

where Δ_α denotes the generator of the symmetric α -stable process in R^d , $0 < \alpha \leq 2$. Δ_α is called the fractional Laplace operator. L_t stands for a Lévy process. It is well known that the Dirichlet form associated with Δ_α is given by

$$\mathcal{E}(u, v) = K(d, \alpha) \int \int_{R^d \times R^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy,$$

$$D(\mathcal{E}) = \{u \in L^2(R^d) : \int \int_{R^d \times R^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} dx dy < \infty\},$$

where $K(d, \alpha) = \alpha 2^{\alpha-3} \pi^{-\frac{d+\alpha}{2}} \sin(\frac{\alpha\pi}{2}) \Gamma(\frac{d+\alpha}{2}) \Gamma(\frac{\alpha}{2})$. To study equation (2.7), we choose $H = L^2(\mathbf{R}^d)$, and $V = D(\mathcal{E})$ with the inner product $\langle u, v \rangle = \mathcal{E}(u, v) + (u, v)_{L^2(R^d)}$.

Define

$$Au = -\Delta_\alpha u.$$

Then (2.2) is fulfilled for (H, V, A) . See [FOT] for details about the fractional Laplace operator.

3 Existence and uniqueness

In [GK1], [GK2] and [G] it is explained how under certain conditions results on existence and uniqueness of solutions for equations with random measures as driving term can be derived from there results. In this section we prove existence and uniqueness for solutions of our equation (1.1), however, directly by a different method.

Proposition 3.1 *Let $b \in M^2([0, T], H)$, $\sigma \in M^2([0, T], H)$ and $f \in M^{\nu,2}([0, T] \times X, H)$. There exists a unique solution $Y_t, t \geq 0$ to the following equation:*

$$\begin{aligned} Y &\in M^2([0, T], V) \cap D([0, T], H), \\ dY_t &= -AY_t dt + b(t, \omega)dt + \sigma(t, \omega)dB_t \\ &\quad + \int_X f(t, x, \omega) \tilde{N}(dt, dx), \end{aligned} \tag{3.1}$$

$$Y_0 = h \in H. \tag{3.2}$$

Proof. We prove the existence in two steps.

Step 1. Assume $b \in M^2([0, T], V)$, $\sigma \in M^2([0, T], V)$ and $f \in M^{\nu,2}([0, T] \times X, V)$. Set

$$U_t = \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t \int_X f(s, x) \tilde{N}(ds, dx).$$

It is easy to see that $U \in M^2([0, T], V)$. Consider the random equation:

$$\begin{aligned} dv_t &= (-Av_t - AU_t)dt, \\ v_0 &= h. \end{aligned} \tag{3.3}$$

It is known from [L] that there exists a unique solution v to equation (3.3) such that $v \in M^2([0, T], V) \cap C([0, T], H)$. Set $Y_t := v_t + U_t$. Then $Y \in M^2([0, T], V) \cap D([0, T], H)$. Moreover, it solves equation (3.1).

Step 2. General case.

Choose $b_n \in M^2([0, T], V)$, $\sigma_n \in M^2([0, T], V)$ and $f_n \in M^{\nu,2}([0, T] \times X, V)$ such that $b_n \rightarrow b$, $\sigma_n \rightarrow \sigma$ in $M^2([0, T], H)$ and $f_n \rightarrow f$ in $M^{\nu,2}([0, T] \times X, H)$ as $n \rightarrow \infty$. Denote by Y_t^n the unique solution to equation (3.1) with b, σ, f replaced by b_n, σ_n, f_n . Such a Y^n exists by step 1. By Ito's formula,

we have

$$\begin{aligned}
& |Y_t^n - Y_t^m|_H^2 \\
= & -2 \int_0^t \langle Y_s^n - Y_s^m, A(Y_s^n - Y_s^m) \rangle ds \\
& + 2 \int_0^t \langle Y_s^n - Y_s^m, b_n(s) - b_m(s) \rangle ds + 2 \int_0^t \langle Y_s^n - Y_s^m, \sigma_n(s) - \sigma_m(s) \rangle dB_s \\
& + \int_0^t |\sigma_n(s) - \sigma_m(s)|_H^2 ds \\
& + \int_0^t \int_X \left(|f_n(s, x) - f_m(s, x)|_H^2 + 2 \langle Y_{s-}^n - Y_{s-}^m, f_n(s, x) - f_m(s, x) \rangle_H \right) \tilde{N}(ds, dx) \\
& + \int_0^t \int_X |f_n(s, x) - f_m(s, x)|_H^2 ds \nu(dx). \tag{3.4}
\end{aligned}$$

In the following, C will denote a generic constant whose values might change from line to line. Set

$$M_t = \int_0^t \int_X \left(|f_n(s, x) - f_m(s, x)|_H^2 + 2 \langle Y_{s-}^n - Y_{s-}^m, f_n(s, x) - f_m(s, x) \rangle_H \right) \tilde{N}(ds, dx).$$

Then,

$$\begin{aligned}
& [M, M]_t^{\frac{1}{2}} = \\
& \left\{ \sum_{s \in D_p, s \leq t} \left(|f_n(s, p(s)) - f_m(s, p(s))|_H^2 + 2 \langle Y_{s-}^n - Y_{s-}^m, f_n(s, p(s)) - f_m(s, p(s)) \rangle_H \right)^2 \right\}^{\frac{1}{2}} \\
& \leq C \left(\sum_{s \in D_p, s \leq t} |f_n(s, p(s)) - f_m(s, p(s))|_H^4 \right)^{\frac{1}{2}} \\
& + C \left(\sum_{s \in D_p, s \leq t} |Y_{s-}^n - Y_{s-}^m|_H^2 |f_n(s, p(s)) - f_m(s, p(s))|_H^2 \right)^{\frac{1}{2}} \\
& \leq C \sum_{s \in D_p, s \leq t} |f_n(s, p(s)) - f_m(s, p(s))|_H^2 \\
& + C \sup_{0 \leq s \leq t} (|Y_{s-}^n - Y_{s-}^m|_H) \left(\sum_{s \in D_p, s \leq t} |f_n(s, p(s)) - f_m(s, p(s))|_H^2 \right)^{\frac{1}{2}} \\
& \leq C \sum_{s \in D_p, s \leq t} |f_n(s, p(s)) - f_m(s, p(s))|_H^2 + \frac{1}{4} \sup_{0 \leq s \leq t} (|Y_{s-}^n - Y_{s-}^m|_H^2).
\end{aligned}$$

By Burkholder's inequality,

$$E \left[\sup_{0 \leq s \leq t} |M_s| \right] \leq CE([M, M]_t^{\frac{1}{2}})$$

$$\begin{aligned}
&\leq CE\left[\sum_{s \in D_p, s \leq t} |f_n(s, p(s)) - f_m(s, p(s))|_H^2\right] + \frac{1}{4}E\left[\sup_{0 \leq s \leq t} |Y_{s-}^n - Y_{s-}^m|_H^2\right] \\
&= CE\left[\int_0^t \int_X |f_n(s, x) - f_m(s, x)|_H^2 ds \nu(dx)\right] + \frac{1}{4}E\left[\sup_{0 \leq s \leq t} |Y_s^n - Y_s^m|_H^2\right]. \quad (3.5)
\end{aligned}$$

It follows from (3.4) and (2.2) that

$$\begin{aligned}
&E\left[\sup_{0 \leq s \leq t} |Y_s^n - Y_s^m|_H^2\right] \leq -\alpha E\left[\int_0^t \|Y_s^n - Y_s^m\|_V^2 ds\right] \\
&\quad + (\lambda_0 + C)E\left[\int_0^t |Y_s^n - Y_s^m|_H^2 ds\right] + CE\left[\int_0^t |b_n(s) - b_m(s)|_H^2 ds\right] \\
&+ CE\left[\left(\int_0^t \langle Y_s^n - Y_s^m, \sigma_n(s) - \sigma_m(s) \rangle_H ds\right)^{\frac{1}{2}}\right] + CE\left[\int_0^t |\sigma_n(s) - \sigma_m(s)|_H^2 ds\right] \\
&\quad + CE([M, M]_t^{\frac{1}{2}}) + CE\left[\int_0^t \int_X |f_n(s, x) - f_m(s, x)|_H^2 ds \nu(dx)\right].
\end{aligned}$$

Applying (3.5) we have

$$\begin{aligned}
&E\left[\sup_{0 \leq s \leq t} |Y_s^n - Y_s^m|_H^2\right] \\
&\leq -\alpha E\left[\int_0^t \|Y_s^n - Y_s^m\|_V^2 ds\right] + CE\left[\int_0^t |Y_s^n - Y_s^m|_H^2 ds\right] \\
&\quad + \frac{1}{2}E\left[\sup_{0 \leq s \leq t} (|Y_{s-}^n - Y_{s-}^m|_H^2) + CE\left[\int_0^t |b_n(s) - b_m(s)|_H^2 ds\right]\right] \\
&+ CE\left[\int_0^t |\sigma_n(s) - \sigma_m(s)|_H^2 ds\right] + CE\left[\int_0^t \int_X |f_n(s, x) - f_m(s, x)|_H^2 ds \nu(dx)\right]. \quad (3.6)
\end{aligned}$$

By Gronwall's inequality, this implies that

$$\begin{aligned}
&E\left[\sup_{0 \leq s \leq t} |Y_s^n - Y_s^m|_H^2\right] \leq Ce^{Ct} \left\{ E\left[\int_0^t |b_n(s) - b_m(s)|_H^2 ds\right] \right. \\
&\quad \left. + CE\left[\int_0^t |\sigma_n(s) - \sigma_m(s)|_H^2 ds\right] + CE\left[\int_0^t \int_X |f_n(s, x) - f_m(s, x)|_H^2 ds \nu(dx)\right] \right\}. \quad (3.7)
\end{aligned}$$

Therefore,

$$\lim_{n, m \rightarrow \infty} E\left[\sup_{0 \leq s \leq t} |Y_s^n - Y_s^m|_H^2\right] = 0. \quad (3.8)$$

This further implies by (3.6) that $Y^n, n \geq 1$ is also a Cauchy sequence in $M^2([0, T], V)$. Let $Y_t, t \geq 0$ denote an element in $M^2([0, T], V)$ such that

$$\lim_{n \rightarrow \infty} E\left[\sup_{0 \leq s \leq t} |Y_s^n - Y_s|_H^2\right] = 0,$$

and

$$\lim_{n \rightarrow \infty} E \left[\int_0^t \|Y_s^n - Y_s\|_V^2 ds \right] = 0.$$

Letting $n \rightarrow \infty$, we see that $Y_t, t \geq 0$ is a solution to equation (3.1).

Uniqueness: If X_t, Y_t are two solutions to equation (3.1), then

$$\begin{cases} \frac{d(X_t - Y_t)}{dt} = -A(X_t - Y_t), \\ X_0 - Y_0 = 0. \end{cases}$$

By the chain rule, we have

$$\begin{aligned} |X_t - Y_t|_H^2 &= -2 \int_0^t \langle X_s - Y_s, A(X_s - Y_s) \rangle ds \\ &\leq -\alpha \int_0^t \|X_s - Y_s\|_V^2 ds + \lambda_0 \int_0^t |X_s - Y_s|_H^2 ds \end{aligned}$$

By Gronwall's inequality, we obtain that $Y_t = X_t$, which completes the proof.

Theorem 3.2 *Assume (H.1) and (H.2). Then there exists a unique H -valued progressively measurable process (Y_t) such that*

- (i) $Y \in M^2(0, T; V) \cap D(0, T; H)$ for any $T > 0$,
- (ii) $Y_t = h - \int_0^t AY_s ds + \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) dB_s + \int_0^{t+} \int_X f(Y_{s-}, x) \tilde{N}(ds, dx)$

a.s.,

- (iii) $Y_0 = h \in H$.

Proof.

Existence of solution.

Let $Y_t^0 := h, t \geq 0$. For $n \geq 0$, define $Y^{n+1} \in M^2(0, T; V) \cap D(0, T; H)$ to be the unique solution to the following equation:

$$\begin{aligned} dY_t^{n+1} &= -AY_t^{n+1} dt + b(Y_t^n) dt + \sigma(Y_t^n) dB_t \\ &\quad + f(Y_{t-}^n, x) \tilde{N}(dt, dx), \end{aligned} \tag{3.9}$$

$$Y_0^n = h. \tag{3.10}$$

The solution Y^{n+1} of the above equation exists according to Proposition 3.1. We are going to show that $\{Y^n, n \geq 1\}$ forms a Cauchy sequence. Using Itô's

formula, we find that

$$\begin{aligned}
& |Y_t^{n+1} - Y_t^n|_H^2 \\
&= -2 \int_0^t \langle Y_s^{n+1} - Y_s^n, A(Y_s^{n+1} - Y_s^n) \rangle ds \\
&+ 2 \int_0^t \langle Y_s^{n+1} - Y_s^n, b(Y_s^n) - b(Y_s^{n-1}) \rangle ds \\
&+ 2 \int_0^t \langle Y_s^{n+1} - Y_s^n, \sigma(Y_s^n) - \sigma(Y_s^{n-1}) \rangle dB_s \\
&+ \int_0^t |\sigma(Y_s^n) - \sigma(Y_s^{n-1})|_H^2 ds \\
&+ \int_0^{t+} \int_X [|f(Y_{s-}^n, x) - f(Y_{s-}^{n-1}, x)|_H^2 + 2 \langle Y_s^{n+1} - Y_s^n, f(Y_{s-}^n, x) - f(Y_{s-}^{n-1}, x) \rangle] \tilde{N}(ds, dx) \\
&+ \int_0^t \int_X |f(Y_s^n, x) - f(Y_s^{n-1}, x)|_H^2 ds \nu(dx). \tag{3.11}
\end{aligned}$$

By a similar calculation as in Proposition 3.1, it follows from (3.11) that

$$\begin{aligned}
& E[\sup_{0 \leq s \leq t} |Y_s^{n+1} - Y_s^n|_H^2] \\
&\leq -\alpha E[\int_0^t \|Y_s^{n+1} - Y_s^n\|_V^2 ds] + CE[\int_0^t |Y_s^{n+1} - Y_s^n|_H^2 ds] \\
&+ \frac{1}{2} E[\sup_{0 \leq s \leq t} |Y_s^{n+1} - Y_s^n|_H^2] + CE[\int_0^t |b(Y_s^n) - b(Y_s^{n-1})|_H^2 ds] \\
&+ CE[\int_0^t |\sigma(Y_s^n) - \sigma(Y_s^{n-1})|_H^2 ds] + CE[\int_0^t \int_X |f(Y_s^n, x) - f(Y_s^{n-1}, x)|_H^2 ds \nu(dx)]. \tag{3.12}
\end{aligned}$$

By virtue of (H.1), this implies that

$$\begin{aligned}
E[\sup_{0 \leq s \leq t} |Y_s^{n+1} - Y_s^n|_H^2] &\leq CE[\int_0^t |Y_s^{n+1} - Y_s^n|_H^2 ds] \\
&+ CE[\int_0^t |Y_s^n - Y_s^{n-1}|_H^2 ds]. \tag{3.13}
\end{aligned}$$

Define

$$g_t^n = E[\sup_{0 \leq s \leq t} |Y_s^n - Y_s^{n-1}|_H^2], \quad G_t^n = \int_0^t g_s^n ds.$$

We have

$$g_t^{n+1} \leq CG_t^{n+1} + CG_t^n. \tag{3.14}$$

Multiplying above inequality by e^{-Ct} , we get that

$$\frac{d(G_t^{n+1} e^{-Ct})}{dt} \leq C e^{-Ct} G_t^n. \tag{3.15}$$

Therefore,

$$G_t^{n+1} \leq C e^{Ct} \int_0^t e^{-Cs} G_s^n ds \leq C e^{Ct} G_t^n. \quad (3.16)$$

Combining (3.14) and (3.16) we see that for a fixed $T > 0$, and $t \leq T$,

$$g_t^{n+1} \leq C^2 e^{Ct} G_t^n + C G_t^n \leq C_T \int_0^t g_s^n ds, \quad (3.17)$$

for some constant C_T . Iterating (3.17), we obtain that

$$E\left[\sup_{0 \leq s \leq T} |Y_s^{n+1} - Y_s^n|_H^2\right] \leq C \frac{(C_T T)^n}{n!}.$$

This implies that there exists $Y \in D([0, T], H)$ such that

$$\lim_{n \rightarrow \infty} E\left[\sup_{0 \leq s \leq T} |Y_s^n - Y_s|_H^2\right] = 0.$$

In view of (3.12) we see that Y^n also converges to Y in $M^2(0, T; V)$. Letting $n \rightarrow \infty$ in (3.9) it is seen that Y is a solution to equation (ii) in the statement of the theorem.

Uniqueness.

Let X, Y be two solutions to (ii) in $M^2(0, T; V) \cap D(0, T; H)$. By Ito's formula, we have

$$\begin{aligned} & |Y_t - X_t|_H^2 \\ &= -2 \int_0^t \langle Y_s - X_s, A(Y_s - X_s) \rangle ds \\ &+ 2 \int_0^t \langle Y_s - X_s, b(Y_s) - b(X_s) \rangle ds \\ &+ 2 \int_0^t \langle Y_s - X_s, \sigma(Y_s) - \sigma(X_s) \rangle dB_s \\ &+ \int_0^t |\sigma(Y_s) - \sigma(X_s)|_H^2 ds \\ &+ \int_0^t \int_X [|f(Y_{s-}, x) - f(X_{s-}, x)|_H^2 + 2 \langle Y_s - X_s, f(Y_{s-}, x) - f(X_{s-}, x) \rangle] \tilde{N}(ds, dx) \\ &+ \int_0^t \int_X |f(Y_s, x) - f(X_s, x)|_H^2 ds \nu(dx). \end{aligned} \quad (3.18)$$

By virtue of (H.2), it follows that

$$\begin{aligned} & E[|Y_t - X_t|_H^2] \\ & \leq -\alpha E\left[\int_0^t \|Y_s - X_s\|_V^2 ds\right] + C E\left[\int_0^t |Y_s - X_s|_H^2 ds\right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} E \left[\sup_{0 \leq s \leq t} |Y_s - X_s|_H^2 \right] + CE \left[\int_0^t |b(Y_s) - b(X_s)|_H^2 ds \right] \\
& + CE \left[\int_0^t |\sigma(Y_s) - \sigma(X_s)|_H^2 ds \right] + CE \left[\int_0^t \int_X |f(Y_s, x) - f(X_s, x)|_H^2 ds \nu(dx) \right] \\
& \leq CE \left[\int_0^t |Y_s - X_s|_H^2 ds \right]. \tag{3.19}
\end{aligned}$$

Hence, $X_t = Y_t$.

Next we move to a more general equation which includes terms involving also Poisson measures. Let U be a set in $B(X)$ such that $\nu(X \setminus U) < \infty$. Let $g(y, x)$ be a measurable mapping from $H \times X$ into H . Introduce the following conditions:

(H.3) There exists a constant $C < \infty$ such that

$$|b(y)|_H^2 + |\sigma(y)|_H^2 + \int_U |g(y, x)|_H^2 \nu(dx) \leq C(1 + |y|_H^2) \tag{3.20}$$

for all $y \in H$.

(H.4) There exists a constant $C < \infty$ such that

$$\begin{aligned}
& |b(y_1) - b(y_2)|_H^2 + |\sigma(y_1) - \sigma(y_2)|_H^2 \\
& + \int_U |g(y_1, x) - g(y_2, x)|_H^2 \nu(dx) \\
& \leq C|y_1 - y_2|_H^2
\end{aligned} \tag{3.21}$$

for all $y_1, y_2 \in H$.

Consider the stochastic evolution equation:

$$\begin{aligned}
Y_t &= h - \int_0^t AY_s ds + \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) dB_s \\
& + \int_0^t \int_U g(Y_{t-}, x) \tilde{N}(dt, dx) + \int_0^t \int_{X \setminus U} g(Y_{t-}, x) N(dt, dx). \tag{3.23}
\end{aligned}$$

Theorem 3.3 *Assume (H.3) and (H.4). Then there exists a unique H -valued progressively measurable process (Y_t) such that*

(i) $Y \in M^2(0, T; V) \cap D(0, T; H)$ for any $T > 0$,

(ii)

$$\begin{aligned}
Y_t &= h - \int_0^t AY_s ds + \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) dB_s \\
& + \int_0^t \int_U g(Y_{t-}, x) \tilde{N}(dt, dx) + \int_0^t \int_{X \setminus U} g(Y_{t-}, x) N(dt, dx), \tag{3.24}
\end{aligned}$$

(iii) $Y_0 = h \in H$.

Proof. Having Theorem 3.2 in hand, this theorem can be proved in the same way as in the finite dimensional case (see [IW]). For completeness we sketch the proof. Let $\tau_1 < \tau_2 < \dots$ be the enumeration of all elements in $D = \{s \in D_p; p(s) \in X \setminus U\}$. It is clear that τ_n is an (\mathcal{F}_t) -stopping time and $\lim_{n \rightarrow \infty} \tau_n = \infty$. First we solve the equation on the time interval $[0, \tau_1]$. Consider the equation

$$\begin{aligned} X_t = & h - \int_0^t AX_s ds + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s \\ & + \int_0^t \int_U g(X_{t-}, x) \tilde{N}(dt, dx). \end{aligned} \quad (3.25)$$

Following the same proof as that of Theorem 3.2, it is seen that there exists a unique solution $X_t, t \geq 0$ to equation (3.25). Set

$$Y_t^1 = X_t, 0 \leq t < \tau_1, \quad = Y_{\tau_1-} + g(Y_{\tau_1-}, p(\tau_1)), t = \tau_1.$$

Clearly the process $\{Y_t^1\}_{t \in [0, \tau_1]}$ is the unique solution to equation (3.24). Now, set $\hat{B}_t = B_{t+\tau_1} - B_{\tau_1}$, $\hat{p}(s) = p(s + \tau_1)$. We can construct the process Y_t^2 on $[0, \hat{\tau}_1]$ with respect to the initial value $Y_0^2 = Y_{\tau_1}^1$, Brownian motion \hat{B} and Poisson point process \hat{p} in the same way as Y_t^1 . Note that $\hat{\tau}_1$ defined with respect to \hat{p} coincides with $\tau_2 - \tau_1$. Define

$$Y_t = Y_t^1, t \in [0, \tau_1], \quad = Y_{t-\tau_1}, t \in [\tau_1, \tau_2].$$

It is easy to see that $\{Y_t\}_{t \in [0, \tau_2]}$ is the unique solution to equation (3.24) in the interval $[0, \tau_2]$. Continuing this procedure successively, we get the unique solution Y to equation (3.24).

4 Exponential integrability

(H.5) There exists a measurable function \bar{f} on X satisfying

$$\sup_{y \in H} |f(y, x)|_H \leq \bar{f}(x), \quad (4.1)$$

and

$$\int_X (\bar{f}(x))^2 \exp(a\bar{f}(x)) \nu(dx) < \infty, \quad \text{for all } a > 0. \quad (4.2)$$

In this section, for simplicity we assume that $b = 0, \sigma = 0$ in equation (1.1). Again we denote the solution of (1.1) by Y_t .

Lemma 4.1 For $g \in C_b^2(H)$, $M_t^g = \exp(g(Y_t) - g(h) - \int_0^t h(Y_s) ds)$ is an \mathcal{F}_t -local martingale, where

$$h(y) = \langle -Ay, g'(y) \rangle + \int_X (\exp[g(y+f(y, x)) - g(y)] - 1 - \langle g'(y), f(y, x) \rangle) \nu(dx).$$

Proof. Applying Itô's formula first to $\exp(g(Y_t))$ and then to $\exp(g(Y_t) - g(h))\exp(-\int_0^t h(Y_s)ds)$ proves the lemma.

Proposition 4.2 *Assume (2.2) with $\lambda_0 = 0$ and also (H.5). Then for $r > 0$ and any $\lambda > 0$, there exists a constant C_λ such that*

$$P(\sup_{0 \leq t \leq 1} |Y_t|_H > r) \leq C_\lambda e^{-(1+\lambda r^2)^{\frac{1}{2}}}.$$

Proof. For $\lambda > 0$, set $g(y) = (1 + \lambda|y|_H^2)^{\frac{1}{2}}$. Then

$$g'(y) = \lambda(1 + \lambda|y|_H^2)^{-\frac{1}{2}}y,$$

$$g''(y) = -\lambda^2(1 + \lambda|y|_H^2)^{-\frac{3}{2}}y \times y + \lambda(1 + \lambda|y|_H^2)^{-\frac{1}{2}}I_H.$$

where I_H stands for the identity operator. It is easy to see that

$$\sup_y |g''(y)| \leq \lambda, \quad \sup_y |g'(y)| \leq \lambda^{\frac{1}{2}}.$$

Moreover,

$$\langle -Ay, g'(y) \rangle = \lambda(1 + \lambda|y|_H^2)^{-\frac{1}{2}} \langle -Ay, y \rangle \leq 0 \quad (4.3)$$

for $y \in V$. Write $G(y) = e^{g(y)}$. By Taylor's expansion, there exists θ between 0 and 1 such that

$$\begin{aligned} & \exp[g(y + f(y, x)) - g(y)] - 1 - \langle g'(y), f(y, x) \rangle \\ &= e^{-g(y)} [G(y + f(y, x)) - G(y) - G(y) \langle g'(y), f(y, x) \rangle] \\ &= \frac{1}{2} e^{-g(y)} \langle G''(y + \theta f(y, x)), f(y, x) \times f(y, x) \rangle. \end{aligned} \quad (4.4)$$

Note that

$$G''(y) = G(y)g'(y) \times g'(y) + G(y)g''(y).$$

It follows that

$$|G''(y)|_{L(H)} \leq \lambda G(y), \quad \text{for all } y \in H. \quad (4.5)$$

By (4.4),

$$\begin{aligned} & |\exp[g(y + f(y, x)) - g(y)] - 1 - \langle g'(y), f(y, x) \rangle| \\ & \leq \lambda \exp(g(y + \theta f(y, x)) - g(y)) |f(y, x)|_H^2 \\ & = \lambda \exp(\langle g'(y + \theta f(y, x)), \theta f(y, x) \rangle) |f(y, x)|_H^2 \\ & \leq \lambda \exp(\lambda^{\frac{1}{2}} |f(y, x)|_H) |f(y, x)|_H^2. \end{aligned} \quad (4.6)$$

Applying Lemma 4.1, with the above choice of g , $M_t^g = \exp(g(Y_t) - g(y) - \int_0^t h(Y_s) ds)$ is an \mathcal{F}_t -local martingale, where

$$\begin{aligned} h(y) &= \langle -Ay, g'(y) \rangle + \int_X (\exp[g(y+f(y,x)) - g(y)] - 1 - \langle g'(y), f(y,x) \rangle) \nu(dx) \\ &\leq \int_X \lambda \exp(\lambda^{\frac{1}{2}} |f(y,x)|_H) |f(y,x)|_H^2 \nu(dx) \\ &\leq \int_X \lambda \exp(\lambda^{\frac{1}{2}} |\bar{f}(x)|_H) |\bar{f}(x)|_H^2 \nu(dx) = M_\lambda < \infty. \end{aligned} \quad (4.7)$$

We have

$$\begin{aligned} P(\sup_{0 \leq t \leq 1} |Y_t|_H > r) &= P(\sup_{0 \leq t \leq 1} g(Y_t) \geq (1 + \lambda r^2)^{\frac{1}{2}}) \\ &= P(\sup_{0 \leq t \leq 1} (g(Y_t) - g(h) - \int_0^t h(Y_s) ds + g(h) + \int_0^t h(Y_s) ds) \geq (1 + \lambda r^2)^{\frac{1}{2}}) \\ &\leq P(\sup_{0 \leq t \leq 1} (g(Y_t) - g(h) - \int_0^t h(Y_s) ds) + g(h) + M_\lambda \geq (1 + \lambda r^2)^{\frac{1}{2}}) \\ &= P(\sup_{0 \leq t \leq 1} (g(Y_t) - g(h) - \int_0^t h(Y_s) ds) \geq (1 + \lambda r^2)^{\frac{1}{2}} - g(h) - M_\lambda) \\ &\leq E[\sup_{0 \leq t \leq 1} M_t^g] \exp(-(1 + \lambda r^2)^{\frac{1}{2}} + g(h) + M_\lambda). \end{aligned} \quad (4.8)$$

Since M_t^g is a non-negative local martingale (hence, a supermartingale), $E[\sup_{0 \leq t \leq 1} M_t^g] \leq 1$. Therefore the assertion follows with $C_\lambda = \exp(g(h) + M_\lambda)$.

Corollary 4.3 *Assume (2.2) with $\lambda_0 = 0$ and also (H.5). Then for any $l > 0$,*

$$E[\exp(l \sup_{0 \leq t \leq 1} |Y_t|_H)] < \infty.$$

Proposition 4.4 *Assume (2.2) with $\lambda_0 = 0$ and also (H.5). Then for any $l > 0$,*

$$E[\exp(l \|Y\|_{L^2([0,1] \rightarrow V)})] < \infty.$$

Proof. Let $Z_\lambda = \int_0^1 (1 + \lambda |Y_s|_H^2)^{-\frac{1}{2}} \|Y_s\|_V^2 ds$. We first prove that

$$P(Z_\lambda > r) \leq \exp(-\alpha \lambda r + M_\lambda + (1 + \lambda |h|_H^2)^{\frac{1}{2}}), \quad (4.9)$$

where M_λ is the same constant as in (4.7). For $\lambda > 0$, define $g(y) = (1 + \lambda |y|_H^2)^{\frac{1}{2}}$. In view of (2.2) we have

$$\langle -Ay, g'(y) \rangle = \lambda (1 + \lambda |y|_H^2)^{-\frac{1}{2}} \langle -Ay, y \rangle$$

$$\leq -\alpha\lambda(1 + \lambda|y|_H^2)^{-\frac{1}{2}}\|y\|_V^2. \quad (4.10)$$

So the estimate in (4.7) can be strengthened as follows:

$$h(y) \leq -\alpha\lambda(1 + \lambda|y|_H^2)^{-\frac{1}{2}}\|y\|_V^2 + M_\lambda. \quad (4.11)$$

Let $M_t^g, t \geq 0$ be defined as in the proof of Proposition 4.2. By (4.11), we have

$$\begin{aligned} P(Z_\lambda > r) &= P(\alpha\lambda \int_0^1 (1 + \lambda|Y_s|_H^2)^{-\frac{1}{2}}\|Y_s\|_V^2 ds > \alpha\lambda r) \\ &\leq P(g(Y_1) + \alpha\lambda \int_0^1 (1 + \lambda|Y_s|_H^2)^{-\frac{1}{2}}\|Y_s\|_V^2 ds > \alpha\lambda r) \\ &= P(g(Y_1) - g(h) - \int_0^1 h(Y_s) ds + g(h) + \int_0^1 h(Y_s) ds \\ &\quad + \alpha\lambda \int_0^1 (1 + \lambda|Y_s|_H^2)^{-\frac{1}{2}}\|Y_s\|_V^2 ds > \alpha\lambda r) \\ &\leq P(g(Y_1) - g(h) - \int_0^1 h(Y_s) ds + g(h) + M_\lambda > \alpha\lambda r) \\ &= P(g(Y_1) - g(h) - \int_0^1 h(Y_s) ds > \alpha\lambda r - g(h) - M_\lambda) \\ &\leq E[M_1^g] \exp(-\alpha\lambda r + g(h) + M_\lambda) \\ &\leq \exp(-\alpha\lambda r + g(h) + M_\lambda) \end{aligned} \quad (4.12)$$

which proves (4.9). It is easy to see from (4.9) that for any $l > 0$, one can choose $\lambda_l > 0$ large enough so that $E[\exp(lZ_{\lambda_l})] < \infty$. Now for every $\lambda > 0$,

$$\begin{aligned} \|Y\|_{L^2([0,1] \rightarrow V)} &= \left(\int_0^1 \|Y_s\|_V^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 (1 + \lambda|Y_s|_H^2)^{-\frac{1}{2}}\|Y_s\|_V^2 ds \right)^{\frac{1}{2}} \left(1 + \lambda \sup_{0 \leq s \leq 1} |Y_s|_H^2 \right)^{\frac{1}{4}} \\ &\leq \frac{1}{2} Z_\lambda + \frac{1}{2} \left(1 + \lambda \sup_{0 \leq s \leq 1} |Y_s|_H^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.13)$$

By Hölder's inequality, for $l > 0$,

$$\begin{aligned} &E[\exp(l\|Y\|_{L^2([0,1] \rightarrow V)})] \\ &\leq E[\exp(\frac{1}{2}lZ_\lambda) \exp(\frac{1}{2}l \left(1 + \lambda \sup_{0 \leq s \leq 1} |Y_s|_H^2 \right)^{\frac{1}{2}})] \\ &\leq \left(E[\exp(lZ_\lambda)] \right)^{\frac{1}{2}} \times \left(E[\exp(l \left(1 + \lambda \sup_{0 \leq s \leq 1} |Y_s|_H^2 \right)^{\frac{1}{2}})] \right)^{\frac{1}{2}}. \end{aligned} \quad (4.14)$$

According to (4.9), we can choose λ such that $E[\exp(lZ_\lambda)] < \infty$. On the other hand $E[\exp(l\left(1 + \lambda \sup_{0 \leq s \leq 1} |Y_s|_H^2\right)^{\frac{1}{2}})] < \infty$ for all $\lambda > 0$ according to Corollary 4.3. So we conclude that $E[\exp(l\|Y\|_{L^2([0,1] \rightarrow V)})] < \infty$ proving the assertion.

5 Large deviations

In this section we consider the following Lévy process:

$$L_t = bt + W_t + \int_0^t \int_X f(x) \tilde{N}(ds, dx),$$

where W is the H -valued Brownian motion introduced in Section 2, b is a constant vector in H and f is a measurable mapping from X into H . The following Ornstein-Uhlenbeck type stochastic evolution equation was first studied in [CM1] and subsequently by many other authors (cf. e.g. [FR]).

$$dY_t = -AY_t dt + dL_t, \quad (5.1)$$

$$Y_0 = h \in H \quad (5.2)$$

The following Theorem can be proved similarly as in Section 3. See also [CM1].

Theorem 5.1 *There exists a unique H -valued progressively measurable process (Y_t) such that*

- (i) $Y \in M^2(0, T; V) \cap D(0, T; H)$ for any $T > 0$,
- (ii) $Y_t = h - \int_0^t AY_s ds + L_t$ a.s.,
- (iii) $Y_0 = h \in H$.

To get large deviations estimates, it is natural to impose the following exponential integrability. Assume throughout this section that

$$\int_X |f(x)|_H^2 \exp(a|f(x)|_H) \nu(dx) < \infty, \quad \text{for all } a > 0. \quad (5.3)$$

Consider the stochastic evolution equation:

$$Y_t^n = x - \int_0^t AY_s^n ds + bt + \frac{1}{n^{\frac{1}{2}}} W_t + \frac{1}{n} \int_0^t \int_X f(x) \tilde{N}_n(ds, dx), \quad (5.4)$$

where $\tilde{N}_n(ds, dx)$ denotes the compensated Poisson measure with intensity measure $n\nu$. The purpose is to establish a large deviation principle for the law μ_n of $Y_t^n, t \geq 0$ on $D([0, 1] \rightarrow H)$. To this end, we first do some preparations.

For $g \in D([0, 1] \rightarrow V)$, define $\phi(g) \in D([0, 1] \rightarrow H) \cap L^2([0, 1] \rightarrow V)$ as the solution to the following equation:

$$\phi_t(g) = x - \int_0^t A\phi_s(g) ds + g(t). \quad (5.5)$$

Lemma 5.2 *The mapping ϕ from $D([0, 1] \rightarrow V)$ into $\phi(g) \in D([0, 1] \rightarrow H) \cap L^2([0, 1] \rightarrow V)$ is continuous in the topology of uniform convergence.*

Proof. Let $v_t(g) = \phi_t(g) - g(t)$. it is easy to see that $v(g)$ satisfies the equation:

$$v_t(g) = x - \int_0^t Av_s(g)ds - \int_0^t Ag(s)ds.$$

It suffices to show that the mapping

$$v(\cdot) : D([0, 1] \rightarrow V) \rightarrow D([0, 1] \rightarrow H) \cap L^2([0, 1] \rightarrow V)$$

is continuous. Taking $\beta < \alpha$, where α is the constant in (2.2), by the chain rule and (2.2),

$$\begin{aligned} |v_t(g_n) - v_t(g)|_H^2 &= -2 \int_0^t \langle A(v_s(g_n) - v_s(g)), v_s(g_n) - v_s(g) \rangle ds \\ &\quad - 2 \int_0^t \langle A(g_n - g)(s), v_s(g_n) - v_s(g) \rangle ds \\ &\leq -\alpha \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds + \lambda_0 \int_0^t |v_s(g_n) - v_s(g)|_H^2 ds \\ &\quad + 2 \int_0^t \|v_s(g_n) - v_s(g)\|_V \|A(g_n - g)(s)\|_{V^*} ds \\ &\leq -\alpha \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds + \lambda_0 \int_0^t |v_s(g_n) - v_s(g)|_H^2 ds \\ &\quad + \beta \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds + C_\beta \int_0^t \|A(g_n - g)(s)\|_{V^*}^2 ds. \end{aligned}$$

This gives that

$$\begin{aligned} &|v_t(g_n) - v_t(g)|_H^2 + (\alpha - \beta) \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds \\ &\leq \lambda_0 \int_0^t |v_s(g_n) - v_s(g)|_H^2 ds + C_\beta \|A\| \int_0^t \|g_n - g(s)\|_V^2 ds \end{aligned}$$

Applying Gronwall's inequality it is easily seen that the mapping $v(\cdot)$ is continuous, which completes the proof.

For $l \in H$, define

$$F(l) = \int_X [\exp(\langle f(x), l \rangle) - 1 - \langle f(x), l \rangle] \nu(dx) + \langle Ql, l \rangle + \langle b, l \rangle.$$

Set, for $z \in H$,

$$F^*(z) = \sup_{l \in H} [\langle z, l \rangle - F(l)]. \quad (5.6)$$

Define a functional $I_0(\cdot)$ on $D([0, 1] \rightarrow H)$ as follows: if $g \in D([0, 1] \rightarrow H)$ and $g' \in L^1([0, 1] \rightarrow H)$, $I_0(g) = \int_0^1 F^*(g'(s))ds$; otherwise $I(g) = \infty$.

Lemma 5.3 *Let $a > 0$. Then $\mathcal{G} = \{|g'|; I_0(g) \leq a\}$ is uniformly integrable on the probability space $([0, 1], \mathcal{B}, m)$, where m denotes the Lebesgue measure.*

Proof. Recall that \mathcal{G} is uniformly integrable if and only if

- (i) \mathcal{G} is equi-absolutely continuous, i.e., for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $m(A) < \delta$ implies $\int_A |g'|_H m(ds) < \varepsilon$ for all $g \in \mathcal{G}$.
- (ii) $\sup_{g \in \mathcal{G}} \int_0^1 |g'|_H m(ds) < \infty$.

We will modify the proof of Theorem 3.1 in [A2] to get (i) and (ii). Let $a_i, b_i, i = 1, \dots, n$ be any given numbers such that $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n \leq 1$. For any partition $\tau^i = \{t_0^i = a_i < t_1^i < \dots < t_{i_m}^i = b_i\}$ of $[a_i, b_i]$ and any $\eta_k^i \in H$ with $|\eta_k^i|_H \leq 1$, define $\beta \in M([0, 1], H)$ by

$$\beta = \sum_{i=1}^n \sum_{k=0}^{i_m} \eta_k^i (\delta_{t_k^i} - \delta_{t_{k-1}^i}),$$

where $M([0, 1], H)$ denotes the space of H -valued vector measures on $([0, 1], \mathcal{B})$. Let μ be the law of $\int_0^1 \int_X f(x) \tilde{N}(ds, dx)$ on H . Denote the characteristic functional of μ by $\hat{\mu}$. Then,

$$\int_0^1 \log \hat{\mu}(\beta(s, 1)) ds = \sum_{i=1}^n \sum_{k=0}^{i_m} \log \hat{\mu}(\eta_k^i)(t_k^i - t_{k-1}^i).$$

Let $\rho > 0$. By the characterization of I_0 in [A2], for $g \in \mathcal{G}$, we have

$$\begin{aligned} \rho \int_0^1 \langle g, d\beta \rangle &= \rho \sum_{i=1}^n \sum_{k=0}^{i_m} \langle g(t_k^i) - g(t_{k-1}^i), \eta_k^i \rangle \\ &\leq \int_0^1 \log \hat{\mu}(\rho \beta(s, 1)) ds + I_0(g) \\ &\leq \sup_{i,k} |\log \hat{\mu}(\rho \eta_k^i)| \sum_{i=1}^n \sum_{k=0}^{i_m} (t_k^i - t_{k-1}^i) + I_0(g) \\ &\leq \log \left(\int_H \exp(\rho |x|_H) \mu(dx) \right) \sum_{i=1}^n (b_i - a_i) + a. \end{aligned} \quad (5.7)$$

Taking sup in (5.7) over all possible $\eta_k^i \in H$ with $|\eta_k^i|_H \leq 1$ we get

$$\sum_{i=1}^n \sum_{k=0}^{i_m} |g(t_k^i) - g(t_{k-1}^i)|_H \leq \rho^{-1} \log \left(\int_H \exp(\rho |x|_H) \mu(dx) \right) \sum_{i=1}^n (b_i - a_i) + \rho^{-1} a. \quad (5.8)$$

Let $V(g)[a, b]$ denote the total variation of g over the interval $[a, b]$. Taking sup in (5.8) over all possible partitions we obtain

$$\sum_{i=1}^n V(g)[a_i, b_i] = \sum_{i=1}^n \int_{a_i}^{b_i} |g'(s)|_H ds = \int_{\cup_{i=1}^n (a_i, b_i)} |g'(s)|_H ds$$

$$\leq \rho^{-1} \log \left(\int_H \exp(\rho|x|_H) \mu(dx) \right) \sum_{i=1}^n (b_i - a_i) + \rho^{-1} a \quad (5.9)$$

For every $\varepsilon > 0$, choose first ρ_0 large enough such that $\rho_0^{-1} a \leq \frac{\varepsilon}{2}$. Set $\delta = \frac{1}{3} [\rho_0^{-1} \log(\int_H \exp(\rho|x|_H) \mu(dx))]^{-1} \varepsilon$. If $\cup_{i=1}^n (a_i, b_i) \subset [0, 1]$ with $m(\cup_{i=1}^n (a_i, b_i)) < \delta$, by (5.9) we have $\int_{\cup_{i=1}^n (a_i, b_i)} |g'(s)|_H ds < \varepsilon$, for all $g \in \mathcal{G}$. This implies (i). Take particularly $a_1 = 0, b_1 = 1$ in the above proof to see that (ii) also holds.

Let $T_t, t \geq 0$ denote the semigroup generated by $-A$. For $g \in L^1([0, 1] \rightarrow H)$, define the operator

$$Rg(t) = \int_0^t T_{t-s} g(s) ds, \quad t \geq 0,$$

which is the mild solution of the equation:

$$\phi(t) = - \int_0^t A\phi(s) ds + \int_0^t g(s) ds.$$

Proposition 5.4 *Assume that $T_t, t > 0$ are compact operators. If $\mathcal{G} \subset L^1([0, 1] \rightarrow H)$ is uniformly integrable, then $\mathcal{S} = R(\mathcal{G})$ is relatively compact in $C([0, 1] \rightarrow H)$.*

Proof. The proof is a modification of the proof of Proposition 8.4 in [DZ]. According to the Ascoli-Arzelà theorem we need to show

- (i) for every $t \in [0, 1]$ the set $\{Rg(t); g \in \mathcal{G}\}$ is relatively compact in H ;
- (ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \leq s \leq t \leq 1, t - s \leq \delta$,

$$|Rg(t) - Rg(s)|_H \leq \varepsilon \quad \text{for all } g \in \mathcal{G}. \quad (5.10)$$

To prove (i), fix $t \in (0, 1]$ and define for $\varepsilon > 0$ $R^\varepsilon g(t) = \int_0^{t-\varepsilon} T_{t-s} g(s) ds$. Since

$$R^\varepsilon g(t) = T_\varepsilon \int_0^{t-\varepsilon} T_{t-\varepsilon-s} g(s) ds$$

and $T_\varepsilon, \varepsilon > 0$ is compact, $\{R^\varepsilon g(t), g \in \mathcal{G}\}$ is relatively compact in H for every $\varepsilon > 0$. On the other hand,

$$|R^\varepsilon g(t) - Rg(t)|_H \leq M \int_{t-\varepsilon}^t |g(s)|_H ds, \quad (5.11)$$

where $M = \sup_{t \in [0, 1]} \|T_t\|$. Since \mathcal{G} is uniformly integrable, (4.52) implies that

$$\lim_{\varepsilon \rightarrow 0} \sup_{g \in \mathcal{G}} |R^\varepsilon g(t) - Rg(t)|_H = 0$$

which furthermore implies that $\{Rg(t); g \in \mathcal{G}\}$ is also relatively compact. Let us now prove (ii). For $0 \leq t \leq t + u \leq 1$, we have

$$|Rg(t + u) - Rg(t)|_H$$

$$\begin{aligned} &\leq \int_0^t \|T_{t+u-s} - T_{t-s}\| |g(s)|_H ds + \int_t^{t+u} \|T_{t+u-s}\| |g(s)|_H ds \\ &:= I_g^u + II_g^u. \end{aligned}$$

By the uniform integrability of \mathcal{G} , it is clear that

$$\limsup_{u \rightarrow 0} \sup_{g \in \mathcal{G}} II_g^u \leq M \limsup_{u \rightarrow 0} \sup_{g \in \mathcal{G}} \int_t^{t+u} |g(s)|_H ds = 0.$$

Since the semigroup T is compact, $\|T_{t+u-s} - T_{t-s}\| \rightarrow 0$ for any $t - s > 0$ as $u \rightarrow 0$. By the dominated convergence theorem, we have that

$$\lim_{u \rightarrow 0} \int_0^t \|T_{t+u-s} - T_{t-s}\| ds = 0. \quad (5.12)$$

Now we prove

$$\limsup_{u \rightarrow 0} \sup_{g \in \mathcal{G}} I_g^u = 0. \quad (5.13)$$

For given $\varepsilon > 0$, since \mathcal{G} is uniformly integrable, one can choose $\rho > 0$ such that $2M \int_{|g|>\rho} |g(s)|_H ds < \frac{\varepsilon}{2}$ for all $g \in \mathcal{G}$. For the fixed $\rho > 0$ above, there exists $\delta > 0$ such that $u \leq \delta$ implies that

$$\rho \int_0^t \|T_{t+u-s} - T_{t-s}\| ds \leq \frac{\varepsilon}{2}$$

for all $t \in [0, 1]$. Therefore if $u \leq \delta$, for all $g \in \mathcal{G}$, $t \in [0, 1]$,

$$\begin{aligned} I_g^u &= \int_{|g|>\rho} \|T_{t+u-s} - T_{t-s}\| |g(s)|_H ds + \int_{|g|\leq\rho} \|T_{t+u-s} - T_{t-s}\| |g(s)|_H ds \\ &\leq 2M \int_{|g|>\rho} |g(s)|_H ds + \rho \int_0^t \|T_{t+u-s} - T_{t-s}\| ds \\ &\leq \varepsilon. \end{aligned} \quad (5.14)$$

This proves (ii), hence the assertion.

Theorem 5.5 *Assume that A has discrete spectrum with eigenvalues*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \rightarrow +\infty$$

and H -normalized eigenfunctions $\{e_i\}$. Suppose that the Brownian motion W_t admits the following representation:

$$W_t = \sum_{i=1}^{\infty} q_i \beta_i(t) e_i, \quad (5.15)$$

where $\beta_i(t), i \geq 1$ are independent standard Brownian motions, and $q_i, i \geq 1$ are non-negative real numbers satisfying $\sum_{i=1}^{\infty} q_i^2 < \infty$. Then $\{\mu_n, n \geq 1\}$

satisfies a large deviation principle on $D([0, 1] \rightarrow H)$ (equipped with the topology of uniform convergence) with a rate functional I given for $k \in D([0, 1] \rightarrow H)$ by

$$\begin{aligned} I(k) &= \inf\{I_0(g); g \in D([0, 1] \rightarrow H) \text{ satisfying} \\ &k(t) = T_t x + \int_0^t T_{t-s} g'(s) ds\}. \end{aligned} \quad (5.16)$$

Proof.

Denote by $P_m : H \rightarrow H$ the projection operator defined by

$$P_m x = \sum_{i=1}^m \langle x, e_i \rangle e_i.$$

As $\{e_i\} \subset V$, we have $\text{Rang}(P_m) \subset V$. For any integer $m \geq 1$, introduce a mapping $\phi^m(\cdot)$ from $D([0, 1] \rightarrow H)$ into $D([0, 1] \rightarrow H)$ as follows: for $g \in D([0, 1] \rightarrow H)$ define $\phi_t^m(g)$ as the solution of the following equation:

$$\phi_t^m(g) = x - \int_0^t A \phi_s^m(g) ds + P_m g(t). \quad (5.17)$$

By Lemma 5.2 the mapping $\phi^m(\cdot)$ is continuous. Let

$$L_t^n = bt + \frac{1}{n^{\frac{1}{2}}} W_t + \frac{1}{n} \int_0^t \int_X f(x) \tilde{N}_n(ds, dx).$$

Then it is easy to see that $Y_t^{n,m} := \phi_t^m(L_t^n)$ is the solution to the following equation:

$$Y_t^{n,m} = x - \int_0^t A Y_s^{n,m} ds + b^m t + \frac{1}{n^{\frac{1}{2}}} W_t^m + \frac{1}{n} \int_0^t \int_X f^m(x) \tilde{N}_n(ds, dx), \quad (5.18)$$

where $f^m(x) = P_m f(x) = \sum_{i=1}^m \langle f(x), e_i \rangle e_i$, $W_t^m = P_m W_t$ and $b^m = P_m b$. Let ν_n be the law of L_t^n . It was proved in [A2] that $\{\nu_n, n \geq 1\}$ satisfies a large deviation principle with rate function I_0 . Applying the contraction principle, we see that $\{Y_t^{n,m}\}$ satisfies a large deviation principle on $D([0, 1] \rightarrow H)$ with a rate functional I_m given by, for $k \in D([0, 1] \rightarrow H)$,

$$\begin{aligned} I_m(k) &= \inf\{I_0(g); g \in D([0, 1] \rightarrow H) \text{ satisfying} \\ &k(t) = T_t x + \int_0^t T_{t-s} P_m g'(s) ds\}. \end{aligned} \quad (5.19)$$

According to the generalized contraction principle Theorem 4.2 in [DZ], the theorem now follows from the following two lemmas.

Lemma 5.6 For any $\delta > 0$,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \log P(\sup_{0 \leq t \leq 1} |Y_t^{n,m} - Y_t^n|_H > \delta) = -\infty. \quad (5.20)$$

Proof. Set $X_t^{n,m} = n(Y_t^{n,m} - Y_t)$. Then it can be seen that

$$\begin{aligned} X_t^{n,m} &= - \int_0^t AX_s^{n,m} ds + \int_0^t \int_X (f^m(x) - f(x)) \tilde{N}_n(ds, dx) \\ &\quad + n(b^m - b)t + n^{\frac{1}{2}}(W_t^m - W_t) \end{aligned} \quad (5.21)$$

For $\lambda > 0$, set $g(y) = (1 + \lambda|y|_H^2)^{\frac{1}{2}}$. As in Section 4, we know that $M_t^g = \exp(g(X_t^{n,m}) - g(0) - \int_0^t h(X_s^{n,m}) ds)$ is an \mathcal{F}_t -local martingale, where

$$\begin{aligned} h(y) &= n \int_X (\exp[g(y + f^m(x) - f(x)) - g(y)] - 1 - \langle g'(y), f^m(x) - f(x) \rangle) \nu(dx) \\ &\quad - \langle Ay, g'(y) \rangle + n \langle b^m - b, g'(y) \rangle \\ &\quad + n \sum_{i=m+1}^{\infty} q_i^2 \langle (g'(y) \otimes g'(y) + g''(y)) e_i, e_i \rangle. \end{aligned} \quad (5.22)$$

Furthermore (See Section 4), we have

$$h(y) \leq c_{\lambda,m} n, \quad (5.23)$$

where

$$\begin{aligned} c_{\lambda,m} &= \lambda \int_X \exp(\lambda^{\frac{1}{2}} |f^m(x) - f(x)|_H) (|f^m(x) - f(x)|_H)^2 \nu(dx) \\ &\quad + \lambda^{\frac{1}{2}} |b^m - b|_H + 2\lambda \sum_{i=m+1}^{\infty} q_i^2. \end{aligned} \quad (5.24)$$

We have

$$\begin{aligned} &P(\sup_{0 \leq t \leq 1} |X_t^{n,m}|_H > r) = P(\sup_{0 \leq t \leq 1} g(X_t^{n,m}) \geq (1 + \lambda r^2)^{\frac{1}{2}}) \\ &= P(\sup_{0 \leq t \leq 1} (g(X_t^{n,m}) - g(0) - \int_0^t h(X_s^{n,m}) ds + 1 + \int_0^t h(X_s^{n,m}) ds) \geq (1 + \lambda r^2)^{\frac{1}{2}}) \\ &\leq P(\sup_{0 \leq t \leq 1} (g(X_t^{n,m}) - g(0) - \int_0^t h(X_s^{n,m}) ds) + 1 + c_{\lambda,m} n \geq (1 + \lambda r^2)^{\frac{1}{2}}) \\ &= P(\sup_{0 \leq t \leq 1} (g(X_t^{n,m}) - g(0) - \int_0^t h(X_t^{n,m}) ds) \geq (1 + \lambda r^2)^{\frac{1}{2}} - 1 - c_{\lambda,m} n) \\ &\leq E[\sup_{0 \leq t \leq 1} M_t^g] \exp(-(1 + \lambda r^2)^{\frac{1}{2}} + 1 + c_{\lambda,m} n). \end{aligned} \quad (5.25)$$

This gives that

$$\begin{aligned} &P(\sup_{0 \leq t \leq 1} |Y_t^{n,m} - Y_t^n|_H > \delta) = P(\sup_{0 \leq t \leq 1} |X_t^{n,m}|_H > n\delta) \\ &\leq \exp(-(1 + \lambda(n\delta)^2)^{\frac{1}{2}} + 1 + c_{\lambda,m} n). \end{aligned}$$

Therefore,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \frac{1}{n} \log P \left(\sup_{0 \leq t \leq 1} |Y_t^{n,m} - Y_t^n|_H > \delta \right) \\
& \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \left[-(1 + \lambda(n\delta)^2)^{\frac{1}{2}} + 1 + c_{\lambda,m}n \right] \\
& \leq -\lambda\delta + c_{\lambda,m}.
\end{aligned} \tag{5.26}$$

Note that by the dominated convergence theorem, for fixed λ , $\lim_{m \rightarrow \infty} c_{\lambda,m} = 0$. Taking $m \rightarrow \infty$ in (5.26) we get that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \log P \left(\sup_{0 \leq t \leq 1} |Y_t^{n,m} - Y_t^n|_H > \delta \right) \leq -\lambda\delta.$$

Let $\lambda \rightarrow \infty$ to get (5.20).

For $g \in D([0, 1] \rightarrow H)$, let $\phi_t(g), t \geq 0$, be defined as in (5.5).

Lemma 5.7 For any $r > 0$,

$$\lim_{m \rightarrow \infty} \sup_{\{f; I_0(f) \leq r\}} \sup_{0 \leq t \leq 1} |\phi_t^m(f) - \phi_t(f)| = 0. \tag{5.27}$$

Proof. For $f \in D([0, 1] \rightarrow H)$ with $I_0(f) < \infty$, we note that

$$\phi_t^m(f) = \int_0^t T_{t-s} P_m f'(s) ds = P_m \int_0^t T_{t-s} f'(s) ds = P_m \phi_t(f), \tag{5.28}$$

where we have used the fact that A has a discrete spectrum to exchange P_m and T_s . Since by Lemma 5.3 $L_r := \{f'; I_0(f) \leq r\}$ is uniformly integrable on the probability space $([0, 1], \mathcal{B}, m)$, it follows from Proposition 5.4 that $\mathcal{S} = \{\phi(f); I_0(f) \leq r\}$ is relatively compact in $C([0, 1] \rightarrow H)$. Therefore, for any $\varepsilon > 0$, there exist $f_1, f_2, \dots, f_N \in \{f; I_0(f) \leq r\}$ such that $\mathcal{S} \subset \cup_{k=1}^N B(\phi(f_k), \frac{\varepsilon}{3})$, where $B(\phi(f_k), \frac{\varepsilon}{3})$ stands for the ball centered at $\phi(f_k)$ with radius $\frac{\varepsilon}{3}$ in $C([0, 1] \rightarrow H)$. Since $\lim_{m \rightarrow \infty} \sup_{0 \leq t \leq 1} |\phi_t^m(f_k) - \phi_t(f_k)| = 0$ for every k , there exists $m_0 \geq 1$ such that

$$\sup_{0 \leq t \leq 1} |\phi_t^m(f_k) - \phi_t(f_k)| \leq \frac{\varepsilon}{3} \quad \text{for all } k \leq N, m \geq m_0. \tag{5.29}$$

Fix any f with $I_0(f) \leq r$. Then there is $k \leq N$ such that $\phi(f) \in B(\phi(f_k), \frac{\varepsilon}{3})$. Hence, if $m \geq m_0$,

$$\begin{aligned}
& \sup_{0 \leq t \leq 1} |\phi_t^m(f) - \phi_t(f)| \\
& \leq \sup_{0 \leq t \leq 1} |\phi_t^m(f) - P_m \phi_t(f_k)| + \sup_{0 \leq t \leq 1} |\phi_t^m(f_k) - \phi_t(f_k)| \\
& \quad + \sup_{0 \leq t \leq 1} |\phi_t(f_k) - \phi_t(f)| \\
& \leq 2 \sup_{0 \leq t \leq 1} |\phi_t(f_k) - \phi_t(f)| + \sup_{0 \leq t \leq 1} |\phi_t^m(f_k) - \phi_t(f_k)| \\
& \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned} \tag{5.30}$$

which proves (5.27).

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References

- [A1] A.de Acosta: A general non-convex large deviation result with applications to stochastic equations, *Probability Theory and Related Fields* 2000 483-521.
- [A2] A.de Acosta: Large deviations for vector valued Lévy processes, *Stochastic Processes and Their Applications* 51 (1994) 75-115.
- [AWZ] S.Albeverio, J.L.Wu and T.S.Zhang: Parabolic SPDEs driven by Poisson White Noise, *Stochastic Processes and Their Applications* 74 (1998) 21-36.
- [C] P. Chow: Large deviation problem for some parabolic Itô equations, *Communications on Pure and Applied Mathematics Vol. XLV* (1992) 97-120.
- [CM1] A. Chojnowska-Michalik: On processes of Ornstein-Uhlenbeck type in Hilbert space, *Stochastics* vol.21 (1987) 251-286.
- [CM2] F. Chenal and A. Millet: Uniform large deviations for parabolic SPDEs and applications. *Stochastic Process. Appl.* 72 (1997), 161-186.
- [CR] S. Cerrai and M. Röckner: Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. *Ann. Probab.* 32:1 (2004), 1100-1139.
- [CW] C. Cardon-Weber: Large deviations for a Burgers'-type SPDE. *Stochastic processes and their applications* 84 (1999), 53-70.
- [DZ] A. Dembo and O. Zeitouni: *Large deviations techniques and applications*, Jones and Bartlett Publishers, Boston, London 1992.
- [F] N. Fournier: Malliavin calculus for parabolic SPDEs with jumps, *Stochastic Processes and Their Applications* 87:1 (2000) 115-147.
- [FR] M. Fuhrmann and M. Röckner: Generalized Mehler semigroups: the non-Gaussian case. *Potential Analysis* 12 (2000), 1-47.
- [FOT] M. Fukushima, Y.Oshima and M.Takeda: *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, Berlin. New York 1994.

- [G] I. Gyöngy: On stochastic equations with respect to semimartingales. III. *Stochastics* 7 (1982), 231–254.
- [GK1] I. Gyöngy and N.V. Krylov: On stochastic equations with respect to semimartingales. I. *Stochastics* 4 (1980/81), 1–21.
- [GK2] I. Gyöngy and N.V. Krylov: On stochastic equations with respect to semimartingales. II. Itô formula in Banach spaces. *Stochastics* 6 (1981/82), 153–173.
- [IW] N.Ikeda and S.Watanabe: *Stochastic Differential Equations and Diffusion Processes*, North-Holland/Kodansha Amsterdam. Oxford.New York 1989.
- [KR] N.V. Krylov and B.L. Rosowskii: *Stochastic Evolution Equations*. Current problems in Mathematics, vol. 14 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979, pp. 71–147.
- [L] J.L.Lions: *Equations différentielles opérationnelles et problèmes aux limites*, Springer-Verlag Berlin, 1961.
- [MC] C.Mueller: The heat equation with Levy noise, *Stochastic Processes and Their Applications* 74:1 (1998) 67-82.
- [ML] L.Mytnik: Stochastic partial differential equations driven by stable noise, *Probability Theory and Related Fields* 123:2 (2002) 157-201.
- [P] E. Pardoux: Stochastic partial differential equations and filtering of diffusion processes. *Stochastics* **3** (1979) 127–167.
- [PZ] G.D. Prato and J. Zabczyk: *Stochastic equations in infinite dimensions*. Cambridge University Press, 1992.
- [PP] P.Protter: *Stochastic Integration and Differential Equations*, Springer-Verlag 1990.
- [Z] T.S.Zhang: On small time asymptotics of diffusions on Hilbert spaces. *Ann. Probab.* **28:2** (2000), 537-557.