Equilibrium Glauber and Kawasaki dynamics of continuous particle systems

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Abstract

We construct two types of equilibrium dynamics of infinite particle systems in a Riemannian manifold X. These dynamics are analogs of the Glauber, respectively Kawasaki dynamics of lattice spin systems. The Glauber dynamics now is a process where interacting particles randomly appear and disappear, i.e., it is a birth-and-death process in X, while in the Kawasaki dynamics interacting particles randomly jump over X. We establish conditions on *a priori* explicitly given symmetrizing measures and generators of both dynamics under which corresponding conservative Markov processes exist.

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1 Introduction

In the classical d-dimensional Ising model with spin space $S=\{-1,1\}$, the Glauber dynamics means that particles randomly change their spin value, which is called a spin-flip. The generator of this dynamics is given by

$$(H_{\mathrm{G}}f)(\sigma)=\sum_{x\in\mathbb{Z}^d}a(x,\sigma)(\nabla_x f)(\sigma),$$

where

$$(\nabla_x f)(\sigma)=f(\sigma^x)-f(\sigma),$$

σ^x denoting the configuration σ in which the particle at site x has changed its spin value. On the other hand, in the Kawasaki dynamics, pairs of neighboring particles with different spins randomly exchange their spin values. A generator of this dynamics is given by

$$(H_{\mathrm{K}}f)(\sigma) = \sum_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d,\,|x-y|=1} c(x,y,\sigma)(\nabla_{xy}f)(\sigma),$$

where

$$(\nabla_{xy}f)(\sigma) = f(\sigma^{xy}) - f(\sigma),$$

σ^{xy} denoting the configuration σ in which the particles at sites x and y have exchanged their spin values. Under appropriate conditions on the coefficients $a(x,\sigma)$ and $c(x,y,\sigma)$, the corresponding dynamics has a Gibbs measure as symmetrizing (hence invariant) measure. We refer, e.g., to [4, 21, 25] for a discussion of the Glauber and Kawasaki dynamics of lattice spin systems.

Let us now interpret a lattice system with spin space $S = \{-1, 1\}$ as a model of a lattice gas. Then $\sigma(x) = 1$ means that there is a particle at site x, while $\sigma(x) = -1$ means that the site x is empty. The Glauber dynamics of such a system means that, at each site x, a particle randomly appears and disappears. Hence, this dynamics may be interpreted as a birth-and-death process on $\mathbb{Z}^d$. A similar interpretation of the Kawasaki dynamics yields that particles randomly jump from one site to another.

If we consider a continuous particle system, i.e., a system of particles which can take any position in the Euclidean space $\mathbb{R}^d$, then an analog of the Glauber dynamics should be a process in which particles randomly appear and disappear in the space, i.e., a spatial birth-and-death process. The generator of such a process is informally given by the formula

$$(H_{\mathrm{G}}F)(\gamma) = \sum_{x\in\gamma} d(x,\gamma)(D_x^- F)(\gamma) + \int_{\mathbb{R}^d} b(x,\gamma)(D_x^+ F)(\gamma)\,dx,$$

where

$$(D_x^- F)(\gamma) = F(\gamma\setminus x) - F(\gamma), \quad (D_x^+ F)(\gamma) = F(\gamma\cup x) - F(\gamma). \tag{1.1}$$

Here and below, for simplicity of notations, we just write x instead of $\{x\}$. The coefficient $d(x,\gamma)$ describes the rate at which the particle x of the configuration γ dies, while $b(x,\gamma)$ describes the rate at which, given the configuration γ, a new particle is born at x.

Furthermore, an analog of the Kawasaki dynamics of continuous particles should be a process in which particles randomly jump over the space $\mathbb{R}^d$. The generator of such a process is then informally given by

$$(H_{\mathrm{K}}F)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x,y,\gamma)(D_{xy}^{-+} F)(\gamma)\,dy,$$

where

$$(D_{xy}^{-+}F)(\gamma) = F(\gamma \setminus x \cup y) - F(\gamma) \tag{1.2}$$

and the coefficient $c(x, y, \gamma)$ describes the rate at which the particle x of the configuration γ jumps to y.

Spatial birth-and-death processes were first discussed by Preston in [28]. Under some conditions on the birth and death rates, Preston proved the existence of such processes in a bounded domain in $\mathbb{R}^d$. Though the number of particles can be arbitrarily large in this case, the total number of particles remains finite at any moment of time. The problem of convergence of these processes to equilibrium was later studied in [22, 26].

The problem of construction of a spatial birth-and-death process in the infinite volume was initiated by Holley and Stroock in [13]. In fact, in that paper, birth-and-death processes in bounded domains were analyzed in detail. Only in a very special case of nearest neighbor birth-and-death processes on the real line, the existence of a corresponding process on the whole space was proved and its properties were studied. See also [5] for an extension of the uniqueness result of [13].

Glötzl [9, 10] derived conditions on the coefficients $d(x, \gamma)$, $b(x, \gamma)$, respectively $c(x, y, \gamma)$, under which the Glauber and Kawasaki generators are symmetric in the space $L^2(\mu)$, where μ is a given Gibbs measure. However, the problem of existence of such dynamics was left open.

In the recent paper [3], Bertini, Cancrini, and Cesi studied the problem of existence of a spectral gap for the Glauber dynamics in a bounded domain in $\mathbb{R}^d$. They considered a positive, finite range pair potential ϕ and an activity $z > 0$ which satisfy the condition of low activity-high temperature regime. Then, with any bounded domain $\Lambda \subset \mathbb{R}^d$ and a boundary condition η outside Λ, one may associate a finite volume Gibbs measure $\mu_{\Lambda,\eta}$. Bertini *et al.* considered the Glauber dynamics with death coefficient $d(x, \gamma) = 1$ and which has $\mu_{\Lambda,\eta}$ as symmetrizing measure (which uniquely determines the birth coefficient $b(x, \gamma)$). It was shown that the generator of this dynamics has a spectral gap on $L^2(\mu_{\Lambda,\eta})$ which is uniformly positive with respect to all bounded domains Λ and boundary conditions η. A ramification of this result and its extension to hard core potentials have been proposed by L. Wu [33].

By using the theory of Dirichlet forms [23, 24], an analog of the Glauber dynamics from [3], but on the whole space (thus, involving infinite configurations) and for a quite general pair potential ϕ, has been constructed in [15]. The coefficients of the generator of this dynamics are given by

$$d(x, \gamma) = 1, \quad b(x, \gamma) = \exp\left[-\sum_{y\in\gamma} \phi(x, y)\right],$$

and this dynamics has a Gibbs measure corresponding to the pair potential ϕ as symmetrizing (hence invariant) measure. The result about the spectral gap for a positive

ϕ has also been extended in [15] to the infinite volume. We also refer to [11] for a discussion of a scaling limit of equilibrium fluctuations of this dynamics.

It should be stressed that, till now, there has not been any proof of existence of a Kawasaki-type dynamics of interacting continuous particle systems.

Thus, the aim of this paper is to present a general theorem on the existence of the Glauber and Kawasaki dynamics of continuous particle systems which have a Gibbs measure as symmetrizing (hence invariant) measure, and consider some examples of these dynamics.

In Section 2, we fix a Riemannian manifold X as underlying space (the position space of the particles) and the space Γ of all locally finite configurations in X. The restriction to the Riemannian manifold case is mainly motivated by the necessity to have constructive conditions for the existence of equilibrium states for interacting particle systems in X. Let us stress that all general statements of the paper (with minor changes) remain valid for much more general underlying spaces.

We next recall the definition of a Gibbs measure μ on Γ which corresponds to a relative energy $E(x, \gamma)$ of the interaction between a particle x and a configuration γ. About the measure μ we assume that it has correlation functions which satisfy the classical Ruelle bound. We also present some examples of a Gibbs measure corresponding to a pair potential ϕ. It should be mentioned that, though in all examples we deal with a pair potential ϕ, our general theory for existence of dynamics holds for a general relative energy $E(x, \gamma)$.

Next, in Section 3, under mild conditions on $E(x, \gamma)$, we prove that there exist Hunt processes $\mathbf{M}_{\mathrm{G}}$ and $\mathbf{M}_{\mathrm{K}}$ on Γ which are properly associated with the Dirichlet form of the Glauber, respectively Kawasaki dynamics. In particular, $\mathbf{M}_{\mathrm{G}}$ and $\mathbf{M}_{\mathrm{K}}$ are conservative Markov process on Γ with *cadlag* paths, and have μ as symmetrizing, hence invariant measure. We also characterize these processes in terms of corresponding martingale problems. Furthermore, we discuss the explicit form of the $L^2(\mu)$-generator of this process on the set of continuous bounded cylinder functions. However, this generator can only be written down under stronger conditions on $E(x, \gamma)$, which however still admit a singularilty of $E(x, \gamma)$. In this section, we use the theory of Dirichlet forms [23], and in particular, some ideas and techniques developed in [15, 16, 24, 29].

Throughout Section 3, we formulate our results for both dynamics, while proving them only in the case of the Kawasaki dynamics. This is connected with the fact that the proofs in the Glauber case are quite similar to, and simpler than the corresponding proofs for the Kawasaki dynamics.

Finally, in Section 4 we consider some examples of Glauber and Kawasaki dynamics.

Let us conclude this section with the following remarks. First, we note that, in a bounded domain, both the Glauber and Kawasaki dynamics can be described as jump Markov processes. However, in the infinite volume, both dynamics do not belong to this class, since in any time interval $[0, t]$, each dynamics has an infinite number of jumps.

We also note that, though the construction of the Glauber dynamics and that of the Kawasaki dynamics look quite similar, there is a drastic difference between them in that (at least heuristically) the law of conservation of the number of particles holds for the Kawasaki dynamics, and does not for the Glauber dynamics. We, therefore, cannot expect a spectral gap for the generator of the Kawasaki dynamics in the infinite volume.

Furthermore, the Glauber and Kawasaki dynamics have different sets of symmetrizing measures. Indeed, the set of symmetrizing measures of a given Glauber dynamics consists of all grand-canonical Gibbs measures corresponding to a given relative energy of interaction and a *fixed* activity parameter $z > 0$. On the other hand, the set of symmetrizing measures for a given Kawasaki dynamics consists of all Gibbs measures as above, but corresponding to *any* activity parameter $z > 0$. This fact makes it especially interesting to study the hydrodynamic behavior of the Kawasaki dynamics, cf. [7, 32].

Finally, let us note a similarity between the Kawasaki dynamics and the diffusion dynamics of continuous particle systems [1, 24, 12, 16]. Namely, both types of dynanics have conserved particle numbers and the same set of symmetrizing measures. Therefore, just as in the diffusion case, it is natural to study the scaling limit of equilibrium fluctuations for the Kawasaki dynamics, which is the subject of [17].

2 Gibbs measures on configuration spaces

Let X be a connected oriented C^∞ manifold. We denote the Riemannian distance on X by dist. Let $\mathcal{B}(X)$ denote the Borel σ-algebra on X and m the volume measure on X.

The configuration space $\Gamma := \Gamma_X$ over X is defined as the set of all subsets of X which are locally finite:

$$\Gamma := \big\{ \gamma \subset X : |\gamma_\Lambda| < \infty \text{ for each compact } \Lambda \subset X \big\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(X)$, where ε_x is the Dirac measure with mass at x, $\sum_{x\in\varnothing} \varepsilon_x$:=zero measure, and $\mathcal{M}(X)$ stands for the set of all positive Radon measures on $\mathcal{B}(X)$. The space Γ can be endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ with respect to which all maps

$$\Gamma \ni \gamma \mapsto \langle f, \gamma\rangle := \int_X f(x)\,\gamma(dx) = \sum_{x\in\gamma} f(x), \qquad f \in C_0(X),$$

are continuous. Here, $C_0(X)$ is the space of all continuous real-valued functions on X with compact support. We shall denote the Borel σ-algebra on Γ by $\mathcal{B}(\Gamma)$.

Now we proceed to consider Gibbs measures on Γ. For $\gamma \in \Gamma$ and $x \in X$, we consider a relative energy $E(x,\gamma) \in (-\infty, +\infty]$ of interaction between a particle located at x and the configuration γ. We suppose that the mapping E is measurable.

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a (grand-canonical) Gibbs measure corresponding to activity $z > 0$ and the relative energy E if it satisfies the Georgii–Nguyen–Zessin identity ([27, Theorem 2], see also [19, Theorem 2.2.4]):

$$\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) F(x,\gamma) = \int_\Gamma \mu(d\gamma) \int_X zm(dx) \exp\left[-E(x,\gamma)\right] F(x, \gamma \cup x) \tag{2.1}$$

for any measurable function $F : X \times \Gamma \to [0, +\infty]$. Let $\mathcal{G}(z, E)$ denote the set of all Gibbs measures corresponding to z and E.

In particular, if $E(x,\gamma) \equiv 0$, then (2.1) is the Mecke identity, which holds if and only if μ is the Poisson measure π_z with intensity measure $zm(dx)$.

We assume that

$$E(x,\gamma) \in \mathbb{R} \quad \text{for } m \otimes \mu\text{-a.e. } (x,\gamma) \in X \times \Gamma. \tag{2.2}$$

Furthermore, we assume that, for any $n \in \mathbb{N}$, there exists a non-negative measurable symmetric function $k_\mu^{(n)}$ on X^n such that, for any measurable symmetric function $f^{(n)}$: $X^n \to [0, \infty]$,

$$\begin{aligned} &\int_\Gamma \sum_{\{x_1,\dots,x_n\} \subset \gamma} f^{(n)}(x_1,\dots,x_n)\, \mu(d\gamma) \\ &\quad = \frac{1}{n!} \int_{X^n} f^{(n)}(x_1,\dots,x_n) k_\mu^{(n)}(x_1,\dots,x_n)\, m(dx_1) \cdots m(dx_n), \end{aligned} \tag{2.3}$$

and

$$\forall (x_1,\dots,x_n) \in X^n : \quad k_\mu^{(n)}(x_1,\dots,x_n) \le \xi^n, \tag{2.4}$$

where $\xi > 0$ is independent of n. The functions $k_\mu^{(n)}$, $n \in \mathbb{N}$, are called the correlation functions of the measure μ, while (2.4) is called the Ruelle bound.

Notice that any probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ satisfyng the Ruelle bound has all local moments finite, i.e.,

$$\int_\Gamma \langle f, \gamma \rangle^n \, \mu(d\gamma) < \infty, \qquad f \in C_0(X),\ f \ge 0,\ n \in \mathbb{N}. \tag{2.5}$$

Let us give examples of a Gibbs measure corresponding to a pair potential ϕ and satisfying the above assumptions.

Let $\phi : X^2 \to (-\infty, +\infty]$ be a symmetric measurable function such that $\phi(x,y) \in \mathbb{R}$ for any $x, y \in X$, $x \ne y$. For each $x \in X$ and $\gamma \in \Gamma$, we define

$$E(x,\gamma) := \begin{cases} \sum_{y\in\gamma} \phi(x,y), & \text{if } \sum_{y\in\gamma} |\phi(x,y)| < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us formulate some conditions on the pair potential ϕ.

(S) (*Stability*) There exists $B \geq 0$ such that, for any $\gamma \in \Gamma$, $|\gamma| < \infty$,

$$\sum_{\{x,y\}\subset\gamma} \phi(x,y) \geq -B|\gamma|.$$

(I) (*Integrability*) We have

$$C := \sup_{x\in X} \int_X |\exp[-\phi(x,y)] - 1|\, m(dy) < \infty.$$

(F) (*Finite range*) There exists $R > 0$ such that

$$\phi(x,y) = 0 \quad \text{if } \operatorname{dist}(x,y) \geq R.$$

Note that if ϕ satisfies (F), then $E(x,\gamma) \in \mathbb{R}$ for any $\gamma \in \Gamma$ and $x \in X \setminus \gamma$.

Theorem 2.1 ([14, 19, 20]) 1) *Let* (S), (I), *and* (F) *hold, and let* $z > 0$ *be such that*

$$z < \frac{1}{2e}\,(e^{2B}C)^{-1},$$

where B *and* C *are as in* (S) *and* (I), *respectively. Then there exists a Gibbs measure* $\mu \in \mathcal{G}(z,E)$ *whose correlation functions* $k_\mu^{(n)}$ *exist and satisfy the Ruelle bound.*

2) *Let* ϕ *be a non-negative potential which fulfills* (I) *and* (F). *Then for each* $z > 0$, *there exists a Gibbs measure* $\mu \in \mathcal{G}(z,E)$ *whose correlation functions* $k_\mu^{(n)}$ *exist and satisfy the Ruelle bound.*

Assume now that $X = \mathbb{R}^d$, $d \in \mathbb{N}$, and assume that ϕ is translation invariant, i.e., $\phi(x,y) = \tilde{\phi}(x-y)$, where $\tilde{\phi} : \mathbb{R} \to (-\infty,\infty]$ is such that $\tilde{\phi}(x) \in \mathbb{R}$ for $x \neq 0$ and $\tilde{\phi}(-x) = \tilde{\phi}(x)$ for all $x \in \mathbb{R}^d$. In this case, the conditions on z and ϕ can be significantly weakened. First, we note that the condition (I) now looks as follows:

$$C := \int_{\mathbb{R}^d} |\exp[-\tilde{\phi}(x)] - 1|\, m(dx) < \infty.$$

For the notion of a superstable, lower regular potential and the notion of a tempered Gibbs measure, appearing in the following theorem, see [31].

Theorem 2.2 ([30, 31]) *Assume that* $X = \mathbb{R}^d$ *and* ϕ *is translation invariant.*

1) *Let* (S) *and* (I) *hold and let* $z > 0$ *be such that*

$$z < \frac{1}{e}\,(e^{2B}C)^{-1},$$

where B and C are as in (S) *and* (I), *respectively. Then there exists a Gibbs measure $\mu \in \mathcal{G}(z,E)$ whose correlation functions exist and satisfy the Ruelle bound.*

2) *Let ϕ be a non-negative potential which fulfills* (I). *Then, for each $z>0$, there exists a Gibbs measure $\mu \in \mathcal{G}(z,E)$ whose correlation functions exist and satisfy the Ruelle bound.*

3) *Let ϕ satisfy* (I) *and additionally let ϕ be a superstable, lower regular potential. Then the set $\mathcal{G}_{\mathrm{temp}}(z,E)$ of all tempered Gibbs measures is non-empty and each measure from $\mathcal{G}_{\mathrm{temp}}(z,E)$ has correlation functions which satisfy the Ruelle bound.*

We also have the following lemma, which follows from (the proof of) [16, Lemma 3.1].

Lemma 2.1 *Let $X=\mathbb{R}^d$ and let ϕ, z, and $\mu \in \mathcal{G}(z,E)$ be as in one of the statements of Theorem* 2.2. *Assume additionally that there exists $r>0$ such that*

$$\sup_{x\in B(r)^c} \tilde{\phi}(x) < \infty,$$

where $B(r)$ denotes the ball in $\mathbb{R}^d$ of radius r centered at the origin. Then (2.2) *holds.*

3 Existence results

In what follows, we shall consider a Gibbs measure $\mu \in \mathcal{G}(z,E)$ as in Section 2, i.e., a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ which satisfies (2.1)–(2.4). We introduce the set $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ of all functions of the form

$$\Gamma \ni \gamma \mapsto F(\gamma) = g_F(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma \rangle),$$

where $N\in\mathbb{N}$, $\varphi_1,\dots,\varphi_N \in C_0(X)$ and $g_F \in C_{\mathrm{b}}(\mathbb{R}^N)$, where $C_{\mathrm{b}}(\mathbb{R}^N)$ denotes the set of all continuous bounded functions on $\mathbb{R}^N$.

We consider measurable mappings

$$X\times\Gamma \ni (x,\gamma) \mapsto d(x,\gamma) \in [0,\infty),$$
$$X\times X\times\Gamma \ni (x,y,\gamma) \mapsto c(x,y,\gamma) \in [0,\infty).$$

We assume that, for each compact $\Lambda \subset X$,

$$\int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx) d(x,\gamma) < \infty, \tag{3.1}$$

$$\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X m(dy) c(x,y,\gamma)(\mathbf{1}_\Lambda(x) + \mathbf{1}_\Lambda(y)) < \infty, \tag{3.2}$$

where $\mathbf{1}_\Lambda$ denotes the indicator of Λ.

For each function $F:\Gamma\to\mathbb{R}$, $\gamma\in\Gamma$, and $x,y\in X$, we recall the notations (1.1), (1.2).

We define bilinear forms

$$\mathcal{E}_{\mathrm{G}}(F,G):=\int_\Gamma \mu(d\gamma)\int_X \gamma(dx)d(x,\gamma)(D_x^- F)(\gamma)(D_x^- G)(\gamma), \tag{3.3}$$

$$\mathcal{E}_{\mathrm{K}}(F,G):=\int_\Gamma \mu(d\gamma)\int_X \gamma(dx)\int_X zm(dy)c(x,y,\gamma)(D_{xy}^{-+} F)(\gamma)(D_{xy}^{-+} G)(\gamma), \tag{3.4}$$

where $F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$. Below we shall show that $\mathcal{E}_{\mathrm{K}}$ corresponds to a Glauber dynamics and $\mathcal{E}_{\mathrm{K}}$ to a Kawasaki dynamics.

We note that, for any $F\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$, there exist a compact $\Lambda\subset X$ and $C_1>0$ such that

$$|(D_x^- F)(\gamma)|\le C_1\mathbf{1}_\Lambda(x),\quad |(D_{xy}^{-+} F)(\gamma)|\le C_1(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y)),\qquad \gamma\in\Gamma,\ x,y\in X.$$

Therefore, by (3.1) and (3.2), the right hand sides of formulas (3.3) and (3.4) are well-defined and finite.

Lemma 3.1 *We have $\mathcal{E}_\sharp(F,G)=0$ for all $F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ such that $F=0$ μ-a.e., $\sharp=\mathrm{G},\mathrm{K}$.*

Proof. It suffices to show that, for $F\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$, $F=0$ μ-a.e., we have $(D_{x,y}^{-+}F)(\gamma)=0$ $\tilde\mu$-a.e., where $\tilde\mu$ is the measure on $X\times X\times\Gamma$ defined by

$$\tilde\mu(dx,dy,d\gamma):=\gamma(dx)\,zm(dy)\,\mu(d\gamma). \tag{3.5}$$

Let Λ be a compact subset of X. We have:

$$\int_\Gamma \mu(d\gamma)\int_\Lambda \gamma(dx)\int_\Lambda zm(dy)|F(\gamma)|=\int_\Gamma \mu(d\gamma)|F(\gamma)|\int_\Lambda\gamma(dx)\int_\Lambda zm(dy)=0,$$

which implies that $F(\gamma)=0$ $\tilde\mu$-a.e. Next, by (2.1) and (2.2),

$$\int_\Gamma \mu(d\gamma)\int_\Lambda \gamma(dx)\int_\Lambda zm(dy)|F(\gamma\setminus x\cup y)|$$
$$=\int_\Gamma \mu(d\gamma)|F(\gamma)|\int_\Lambda\gamma(dx)\int_\Lambda zm(dy)\exp[-E(y,\gamma)+E(x,\gamma\setminus x\cup y)]. \tag{3.6}$$

Since F is bounded, by (2.5), the integrals in (3.6) are finite. Therefore,

$$|F(\gamma)|\exp[-E(y,\gamma)+E(x,\gamma\setminus x\cup y)]<\infty\quad\text{for } \tilde\mu\text{-a.e. } (x,y,\gamma)\in X\times X\times\Gamma. \tag{3.7}$$

Since $F=0$ μ-a.e., by (3.6) and (3.7), $F(\gamma\setminus x\cup y)=0$ $\tilde\mu$-a.e. $\square$

Thus, $(\mathcal{E}_{\mathrm{G}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ and $(\mathcal{E}_{\mathrm{K}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ are well-defined bilinear forms on $L^2(\Gamma,\mu)$.

Lemma 3.2 *The bilinear forms* $(\mathcal{E}_{\mathrm{G}}, \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ *and* $(\mathcal{E}_{\mathrm{K}}, \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ *are closable on* $L^2(\Gamma,\mu)$ *and their closures will be denoted by* $(\mathcal{E}_{\mathrm{G}}, D(\mathcal{E}_{\mathrm{G}}))$, $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$, *respectively.*

Proof. Let $(F_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ such that $\|F_n\|_{L^2(\mu)} \to 0$ as $n \to \infty$ and

$$\mathcal{E}_{\mathrm{K}}(F_n - F_k) \to 0 \quad \text{as } n,k \to \infty. \tag{3.8}$$

To prove the closability of $\mathcal{E}_{\mathrm{K}}$ it suffices to show that there exists a subsequence $\{F_{n_k}\}_{k=1}^\infty$ such that $\mathcal{E}_{\mathrm{K}}(F_{n_k}) \to 0$ as $k \to \infty$.

Let Λ be a compact subset of X. By (2.5), we have

$$\int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx)\, |F_n(\gamma)| \le \|F_n\|_{L^2(\mu)} \left(\int_\Gamma \langle \mathbf{1}_\Lambda, \gamma\rangle^2 \, \mu(d\gamma)\right)^{1/2} \to 0 \quad \text{as } n \to \infty.$$

Therefore, there exists a subsequence of $(F_n)_{n=1}^\infty$, denoted by $(F_n^{(1)})_{n=1}^\infty$, such that $F_n^{(1)}(\gamma) \to 0$ for $\gamma(dx)\mu(d\gamma)$-a.e. $(x,\gamma) \in \Lambda \times \Gamma$. Hence, there exists a subsequence $(F_n^{(2)})_{n=1}^\infty$ of $(F_n^{(1)})_{n=1}^\infty$ such that $F_n^{(2)}(\gamma) \to 0$ for $\gamma(dx)\mu(d\gamma)$-a.e. $(x,\gamma) \in X \times \Gamma$.

Next, analogously to (3.6),

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx) \int_\Lambda zm(dy)\, \exp[-E(y,\gamma) + E(x, \gamma\setminus x \cup y)]\, |F_n^{(2)}(\gamma\setminus x\cup y)| \\
&\quad = \int_\Gamma \mu(d\gamma) \int_\Lambda zm(dx) \int_\Lambda \gamma(dy) |F_n(\gamma)| \\
&\quad \le \|F_n^{(2)}\|_{L^2(\mu)} zm(\Lambda) \left(\int_\Gamma \langle \mathbf{1}_\Lambda, \gamma\rangle^2 \, \mu(d\gamma)\right)^{1/2} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

By virtue of (2.2),

$$\exp[-E(y,\gamma) + E(x,\gamma\setminus x\cup y)] \in (0,+\infty] \quad \text{for } \tilde{\mu}\text{-a.e. } (x,y,\gamma) \in X \times X \times \Gamma.$$

Therefore, there exists a subsequence $(F_n^{(3)})_{n=1}^\infty$ of $(F_n^{(2)})_{n=1}^\infty$ such that $F_n^{(3)}(\gamma\setminus x\cup y) \to 0$ for $\tilde{\mu}$-a.e. $(x,y,\gamma) \in X \times X \times \Gamma$, where the measure $\tilde{\mu}$ is defined by (3.5).

Thus,

$$(D_{xy}^{-+} F_n^{(3)})(\gamma) \to 0 \quad \text{as } n \to \infty \text{ for } \tilde{\mu}\text{-a.e. } (x,y,\gamma) \in X \times X \times \Gamma. \tag{3.9}$$

Now, by (3.9) and Fatou's lemma

$$\begin{aligned}
\mathcal{E}_{\mathrm{K}}(F_n^{(3)}) &= \int c(x,y,\gamma) (D_{xy}^{-+} F_n^{(3)})(\gamma)^2 \, \tilde{\mu}(dx,dy,d\gamma) \\
&= \int c(x,y,\gamma) \left((D_{xy}^{-+} F_n^{(3)})(\gamma) - \lim_{m\to\infty} (D_{xy}^{-+} F_m^{(3)})(\gamma)\right)^2 \tilde{\mu}(dx,dy,d\gamma)
\end{aligned}$$

$$\leq \liminf_{m\to\infty} \int c(x,y,\gamma)((D^{-+}_{xy}F^{(3)}_n)(\gamma) - (D^{-+}_{xy}F^{(3)}_m)(\gamma))^2\,\tilde\mu(dx,dy,d\gamma)$$
$$= \liminf_{m\to\infty} \mathcal{E}_{\mathrm{K}}(F^{(3)}_n - F^{(3)}_m),$$

which by (3.8) can be made arbitrarily small for n large enough. □

For the notion of a Dirichlet form, appearing in the following lemma, we refer to e.g. [23, Chap. I, Sect. 4].

Lemma 3.3 *$(\mathcal{E}_{\mathrm{G}}, D(\mathcal{E}_{\mathrm{G}})$ and $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$ are Dirichlet form on $L^2(\Gamma,\mu)$.*

Proof. On $D(\mathcal{E}_{\mathrm{K}})$ we consider the norm $\|F\|_{D(\mathcal{E}_{\mathrm{K}})} := (\|F\|^2_{L^2(\mu)} + \mathcal{E}_{\mathrm{K}}(F))^{1/2}$, $F\in D(\mathcal{E}_{\mathrm{K}})$. For any $F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$, we define

$$S(F,G)(x,y,\gamma) := c(x,y,\gamma)(D^{-+}_{xy}F)(\gamma)(D^{-+}_{xy}G)(\gamma), \qquad x,y\in X, \gamma\in\Gamma.$$

Using the Cauchy inequality, we conclude that S extends to a bilinear continuous map from $(D(\mathcal{E}_{\mathrm{K}}),\|\cdot\|_{D(\mathcal{E}_{\mathrm{K}})})\times(D(\mathcal{E}_{\mathrm{K}}),\|\cdot\|_{D(\mathcal{E}_{\mathrm{K}})})$ into $L^1(X\times X\times\Gamma,\tilde\mu)$. Let $F\in D(\mathcal{E}_{\mathrm{K}})$ and consider any sequence $(F_n)_{n=1}^\infty$ in $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ such that $F_n\to F$ in $(D(\mathcal{E}_{\mathrm{K}}),\|\cdot\|_{D(\mathcal{E}_{\mathrm{K}})})$. In particular, $F_n\to F$ in $L^2(\mu)$. Then, analogously to the proof of Lemma 3.2, for some subsequence $(F_{n_k})_{k=1}^\infty$, we get

$$(D^{-+}_{xy}F_{n_k})(\gamma)\to(D^{-+}_{xy}F)(\gamma) \quad \text{for } \tilde\mu\text{-a.e. } (x,y,\gamma)\in X\times X\times\Gamma.$$

Therefore, for any $F,G\in D(\mathcal{E}_{\mathrm{K}})$,

$$S(F,G)(x,y,\gamma) := c(x,y,\gamma)(D^{-+}_{xy}F)(\gamma)(D^{-+}_{xy}G)(\gamma) \quad \text{for } \tilde\mu\text{-a.e. } (x,y,\gamma)\in X\times X\times\Gamma \tag{3.10}$$

and

$$\mathcal{E}_{\mathrm{K}}(F,G) = \int S(F,G)(x,y,\gamma)\,\tilde\mu(dx,dy,d\gamma). \tag{3.11}$$

Define $\mathbb{R}\ni x\mapsto g(x):=(0\vee x)\wedge 1$. We again fix any $F\in D(\mathcal{E}_{\mathrm{K}})$ and let $(F_n)_{n=1}^\infty$ be a sequence of functions from $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ such that $F_n\to F$ in $(D(\mathcal{E}_{\mathrm{K}}),\|\cdot\|_{D(\mathcal{E}_{\mathrm{K}})})$. Consider the sequence $(g(F_n))_{n\in\mathbb{N}}$. We evidently have: $g(F_n)\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ for each $n\in\mathbb{N}$ and, by the dominated convergence theorem, $g(F_n)\to g(F)$ as $n\to\infty$ in $L^2(\mu)$. Next, by the above argument, we have, for some subsequence $(F_{n_k})_{k=1}^\infty$, $(D^{-+}_{xy}g(F_{n_k}))(\gamma)\to(D^{-+}_{xy}g(F))(\gamma)$ as $n\to\infty$ for $\tilde\mu$-a.e. (x,y,γ).

For any $x,y\in\mathbb{R}$, we evidently have

$$|g(x)-g(y)|\leq|x-y|. \tag{3.12}$$

Therefore, the sequence $c(x,y,\gamma)^{1/2}(D^{-+}_{xy}g(F_n))(\gamma)$, $n\in\mathbb{N}$, is $\tilde\mu$-uniformly square-integrable, since so is the sequence $c(x,y,\gamma)^{1/2}(D^{-+}_{xy}F_n)(\gamma)$, $n\in\mathbb{N}$. Hence

$$c(x,y,\gamma)^{1/2}(D^{-+}_{xy}g(F_{n_k}))(\gamma)\to c(x,y,\gamma)^{1/2}(D^{-+}_{xy}g(F))(\gamma) \quad \text{as } k\to\infty \text{ in } L^2(\tilde\mu).$$

By (3.10) and (3.11), this yields: $g(F) \in D(\mathcal{E}_{\mathrm{K}})$.

Finally, by (3.10)–(3.12), $\mathcal{E}_{\mathrm{K}}(g(F)) \le \mathcal{E}_{\mathrm{K}}(F)$, which means that $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$ is a Dirichlet form. $\square$

We shall now need the bigger space $\ddot{\Gamma}$ consisting of all $\mathbb{Z}_+ \cup \{\infty\}$-valued Radon measures on X (which is Polish, see e.g. [18]). Since $\Gamma \subset \ddot{\Gamma}$ and $\mathcal{B}(\ddot{\Gamma}) \cap \Gamma = \mathcal{B}(\Gamma)$, we can consider μ as a measure on $(\ddot{\Gamma}, \mathcal{B}(\ddot{\Gamma}))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a Dirichlet form on $L^2(\ddot{\Gamma}, \mu)$.

For the notion of a quasi-regular Dirichlet form, appearing in the following lemma, we refer to [23, Chap. IV, Sect. 3].

Lemma 3.4 *$(\mathcal{E}_{\mathrm{G}}, D(\mathcal{E}_{\mathrm{G}}))$ and $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$ are quasi-regular Dirichlet forms on $L^2(\ddot{\Gamma}, \mu)$.*

Proof. Analogously to [24, Proposition 4.1], it suffices to show that there exists a bounded, complete metric ρ on $\ddot{\Gamma}$ generating the vague topology such that, for all $\gamma_0 \in \ddot{\Gamma}$, $\rho(\cdot, \gamma_0) \in D(\mathcal{E}_{\mathrm{K}})$ and

$$\int_X \gamma(dx) \int_X zm(dy)\, S(\rho(\cdot, \gamma_0))(x, y, \gamma) \le \eta(\gamma) \quad \mu\text{-a.e.}$$

for some $\eta \in L^1(\ddot{\Gamma}, \mu)$ (independent of γ_0). Here, $S(F){:=}S(F, F)$. The proof below is a modification of the proof of [24, Proposition 4.8] and the proof of [15, Proposition 3.2].

Fix any $x_0 \in X$, let $B(r)$ denote the open ball in X of radius $r > 0$ centered at x_0. For each $k \in \mathbb{N}$, we define

$$g_k(x) := \frac{2}{3}\left(\frac{1}{2} - \operatorname{dist}(x, B(k)) \wedge \frac{1}{2}\right), \qquad x \in X,$$

where $\operatorname{dist}(x, B(k))$ denotes the distance from the point x to the ball $B(k)$. Next, we set

$$\phi_k(x){:=}3g_k(x), \qquad x \in X,\ k \in \mathbb{N}.$$

Let ζ be a function in $C_{\mathrm{b}}^1(\mathbb{R})$ such that $0 \le \zeta \le 1$ on $[0, \infty)$, $\zeta(t) = t$ on $[-1/2, 1/2]$, $\zeta' \in [0, 1]$ on $[0, \infty)$. For any fixed $\gamma_0 \in \ddot{\Gamma}$ and for any $k, n \in \mathbb{N}$, (the restriction to Γ of) the function

$$\zeta\left(\sup_{j \le n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\right)$$

belongs to $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ (note that $\langle \phi_k g_j, \gamma_0\rangle$ is a constant). Furthermore, taking into account that $\zeta' \in [0, 1]$ on $[0, \infty)$, we get from the mean value theorem, for each $\gamma \in \Gamma$, $x \in \gamma$, and $x \in X \setminus \gamma$,

$$S\left(\zeta\left(\sup_{j \le n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\right)\right)(x, y, \gamma)$$

$$
\begin{aligned}
&\le c(x,y,\gamma)\bigg(\sup_{j\le n}|\langle\phi_k g_j,\gamma\rangle-\langle\phi_k g_j,\gamma_0\rangle-(\phi_k g_j)(x)+(\phi_k g_j)(y)| \\
&\quad -\sup_{j\le n}|\langle\phi_k g_j,\gamma\rangle-\langle\phi_k g_j,\gamma_0\rangle|\bigg)^2 \\
&\le c(x,y,\gamma)\sup_{j\le n}|-(\phi_k g_j)(x)+(\phi_k g_j)(y)|^2 \\
&\le 2c(x,y,\gamma)\bigg(\sup_{j\le n}(\phi_k g_j)(x)^2+\sup_{j\le n}(\phi_k g_j)(y)^2\bigg) \\
&\le 2c(x,y,\gamma)(\mathbf{1}_{B(k+1/2)}(x)+\mathbf{1}_{B(k+1/2)}(y)).
\end{aligned}
\tag{3.13}
$$

For each $k\in\mathbb{N}$, we define

$$
F_k(\gamma,\gamma_0):=\zeta\left(\sup_{j\in\mathbb{N}}|\langle\phi_k g_j,\gamma\rangle-\langle\phi_k g_j,\gamma_0\rangle|\right),\qquad \gamma,\gamma_0\in\ddot{\Gamma}.
$$

Then, for a fixed $\gamma_0\in\ddot{\Gamma}$,

$$
\zeta\left(\sup_{j\le n}|\langle\phi_k g_j,\gamma\rangle-\langle\phi_k g_j,\gamma_0\rangle|\right)\to F_k(\gamma,\gamma_0)
$$

as $n\to\infty$ for each $\gamma\in\ddot{\Gamma}$ and in $L^2(\mu)$. Hence, by (3.13) and the Banach–Alaoglu and the Banach–Saks theorems (see e.g. [23, Appendix A.2]), $F_k(\cdot,\gamma_0)\in D(\mathcal{E}_{\mathrm{K}})$ and

$$
S(F_k(\cdot,\gamma_0))(x,y,\gamma)\le 2c(x,y,\gamma)(\mathbf{1}_{B(k+1/2)}(x)+\mathbf{1}_{B(k+1/2)}(y))\quad \tilde{\mu}\text{-a.e.}
$$

Define

$$
c_k:=\left(1+2\int c(x,y,\gamma)(\mathbf{1}_{B(k+1/2)}(x)+\mathbf{1}_{B(k+1/2)}(y))\,\tilde{\mu}(dx,dy,d\gamma)\right)^{-1/2}2^{-k/2},\qquad k\in\mathbb{N},
$$

which are finite positive numbers by (3.2), and furthermore, $c_k\to 0$ as $k\to\infty$.

We define

$$
\rho(\gamma_1,\gamma_2):=\sup_{k\in\mathbb{N}}\big(c_kF_k(\gamma_1,\gamma_2)\big),\qquad \gamma_1,\gamma_2\in\ddot{\Gamma}.
$$

By [24, Theorem 3.6], ρ is a bounded, complete metric on $\ddot{\Gamma}$ generating the vague topology.

Analogously to the above, we now conclude that, for any fixed $\gamma_0\in\ddot{\Gamma}$, $\rho(\cdot,\gamma_0)\in D(\mathcal{E}_{\mathrm{K}})$ and

$$
\int_X\gamma(dx)\int_X zm(dy)S(\rho(\cdot,\gamma_0))(x,y,\gamma)\le\eta(\gamma)\quad \mu\text{-a.e.},
$$

where

$$
\eta(\gamma):=2\sup_{k\in\mathbb{N}}\left(c_k^2\int_X\gamma(dx)\int_X zm(dy)c(x,y,\gamma)(\mathbf{1}_{B(k+1/2)}(x)+\mathbf{1}_{B(k+1/2)}(y))\right).
$$

Finally,

$$\int_\Gamma \eta(\gamma)\,\mu(d\gamma) \le 2\sum_{k=1}^\infty c_k^2 \int c(x,y,\gamma)(\mathbf{1}_{B(k+1/2)}(x)+\mathbf{1}_{B(k+1/2)}(y))\,\tilde\mu(dx,dy,d\gamma)$$
$$\le \sum_{k=1}^\infty 2^{-k} = 1.$$

Thus, the lemma is proved. □

For the notion of an exceptional set, appearing in the next proposition, we refer e.g. to [23, Chap. III, Sect. 2].

Lemma 3.5 1) *Assume that, for any compact $\Lambda\subset X$, we have*

$$\int_\Gamma \mu(d\gamma)\int_\Lambda zm(dx)\exp[-2E(x,\gamma)]d(x,\gamma\cup x)^2<\infty. \tag{3.14}$$

Then the set $\ddot\Gamma\setminus\Gamma$ is $\mathcal{E}_{\mathrm{G}}$–exceptional.

2) *Assume that $c(x,y,\gamma)$ may be represented in the form*

$$c(x,y,\gamma)=a(x,y)\varkappa(x,y,\gamma), \tag{3.15}$$

where $a: X\times X\to[0,\infty]$ is measurable and satisfies

$$\sup_{x\in\Lambda}\int_X a(x,y)\,m(dy)<\infty,\quad \sup_{y\in\Lambda}\int_X a(x,y)\,m(dx)<\infty \tag{3.16}$$

for any compact $\Lambda\subset X$, and $\varkappa: X\times X\times\Gamma\to[0,\infty]$ is measurable and satisfies

$$\int_\Gamma \mu(d\gamma)\int_X zm(dx)\int_X zm(dy)a(x,y)\exp[-2E(x,\gamma)]\varkappa(x,y,\gamma\cup x)^2(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))<\infty \tag{3.17}$$

for any compact $\Lambda\subset X$. Then the set $\ddot\Gamma\setminus\Gamma$ is $\mathcal{E}_{\mathrm{K}}$-exceptional.

Remark 3.1 Notice that, by (2.1) and the Cauchy inequality, condition (3.14), respectively (3.15)–(3.17), is stronger than condition (3.1), respectively (3.2).

Proof of Lemma 3.5. We modify the proof of [29, Proposition 1 and Corollary 1] and the proof of [15, Proposition 3.3] according to our situation.

It suffices to prove the result locally, i.e., to show that, for any fixed $a\in X$, there exists a closed set B_a that is the closure of an open neighborhood of a and such that the set

$$N_a:=\{\gamma\in\ddot\Gamma:\sup_{x\in B_a}\gamma(\{x\})\ge 2\}$$

is $\mathcal{E}_{\mathrm{K}}$-exceptional. By [29, Lemma 1], we need to prove that there exists a sequence $u_n \in D(\mathcal{E}_{\mathrm{K}})$, $n \in \mathbb{N}$, such that each u_n is a continuous function on $\ddot{\Gamma}$, $u_n \to \mathbf{1}_{N_a}$ pointwise as $n \to \infty$, and $\sup_{n\in\mathbb{N}} \mathcal{E}_{\mathrm{K}}(u_n) < \infty$.

So, we fix $a \in X$. There exists an open neighborhood $\tilde{B}_a$ of a which is diffeomorphic to the open cube $(-3, +3)^d$ in $\mathbb{R}^d$. We fix the corresponding coordinate system in $\tilde{B}_a$ and we set $B_a := [-1, 1]^d$.

Let $f \in C_0(\mathbb{R})$ be such that $\mathbf{1}_{[0,1]} \le f \le \mathbf{1}_{[-1/2,3/2)}$. For any $n \in \mathbb{N}$ and $i = (i_1, \dots, i_d) \in \mathcal{A}_n := \mathbb{Z}^d \cap [-n, n]^d$, we define a function $f_i^{(n)} \in C_0(X)$ by

$$f_i^{(n)}(x) := \begin{cases} \prod_{k=1}^d f(nx_k - i_k), & x \in \tilde{B}_a, \\ 0, & \text{otherwise.} \end{cases}$$

Let also

$$I_i^{(n)}(x) := \begin{cases} \prod_{k=1}^d \mathbf{1}_{[-1/2,3/2)}(nx_k - i_k), & x \in \tilde{B}_x, \\ 0, & \text{otherwise,} \end{cases}$$

and note that $f_i^{(n)} \le I_i^{(n)}$.

Let $\psi \in C_{\mathrm{b}}^1(\mathbb{R})$ be such that $\mathbf{1}_{[2,\infty)} \le \psi \le \mathbf{1}_{[1,\infty)}$ and $0 \le \psi' \le 2\,\mathbf{1}_{(1,\infty)}$. We define continuous functions

$$\ddot{\Gamma} \ni \gamma \mapsto u_n(\gamma) := \psi\left(\sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma\rangle\right), \qquad n \in \mathbb{N},$$

whose restriction to Γ belongs to $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$. Evidently, $u_n \to \mathbf{1}_{N_a}$ pointwise as $n \to \infty$.

By the mean value theorem, we have, for each $\gamma \in \Gamma$, $x \in \gamma$, $y \in X \setminus \gamma$, and for some point $T_n(x, y, \gamma)$ between $\sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma \setminus x \cup y\rangle$ and $\sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma\rangle$:

$$\begin{aligned}
S(u_n)(x, y, \gamma) &= c(x, y, \gamma)\psi'(T_n(x, y, \gamma))^2 \left(\sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma \setminus x \cup y\rangle - \sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma\rangle\right)^2 \\
&\le c(x, y, \gamma)\psi'(T_n(x, y, \gamma))^2 \sup_{i\in\mathcal{A}_n} |\langle f_i^{(n)}, \gamma \setminus x \cup y\rangle - \langle f_i^{(n)}, \gamma\rangle|^2 \\
&\le 2c(x, y, \gamma)\psi'(T_n(x, y, \gamma))^2 \left(\sup_{i\in\mathcal{A}_n} f_i^{(n)}(x)^2 + \sup_{i\in\mathcal{A}_n} f_i^{(n)}(y)^2\right) \\
&\le 2c(x, y, \gamma)\psi'(T_n(x, y, \gamma))^2 (\mathbf{1}_{N_a}(x) + \mathbf{1}_{N_a}(y)) \\
&\le 8c(x, y, \gamma)\mathbf{1}_{(1,\infty)}(T_n(x, y, \gamma))(\mathbf{1}_{N_a}(x) + \mathbf{1}_{N_a}(y)) \\
&\le 8c(x, y, \gamma)\mathbf{1}_{(1,\infty)}\left(\sup_{i\in\mathcal{A}_n} (\langle f_i^{(n)}, \gamma\rangle + f_i^{(n)}(y))\right)(\mathbf{1}_{N_a}(x) + \mathbf{1}_{N_a}(y)) \\
&\le 8c(x, y, \gamma)\mathbf{1}_{(1,\infty)}\left(\sup_{i\in\mathcal{A}_n} (\langle I_i^{(n)}, \gamma\rangle + I_i^{(n)}(y))\right)(\mathbf{1}_{N_a}(x) + \mathbf{1}_{N_a}(y))
\end{aligned}$$

$$
\begin{aligned}
&= 8c(x,y,\gamma)\mathbf{1}_{[2,\infty)}\Big(\sup_{i\in\mathcal{A}_n}(\langle I_i^{(n)},\gamma\rangle + I_i^{(n)}(y))\Big)(\mathbf{1}_{N_a}(x)+\mathbf{1}_{N_a}(y))\\
&\le 8c(x,y,\gamma)\sum_{i\in\mathcal{A}_n}\mathbf{1}_{[2,\infty)}(\langle I_i^{(n)},\gamma\rangle + I_i^{(n)}(\gamma))(\mathbf{1}_{N_a}(x)+\mathbf{1}_{N_a}(y))\\
&\le 8c(x,y,\gamma)\sum_{i\in\mathcal{A}_n}(\mathbf{1}_{[2,\infty)}(\langle I_i^{(n)},\gamma\rangle)+\mathbf{1}_{[1,\infty)}(\langle I_i^{(n)},\gamma\rangle)\mathbf{1}_{\operatorname{supp} I_i^{(n)}}(y))(\mathbf{1}_{N_a}(x)+\mathbf{1}_{N_a}(y)),
\end{aligned}
\tag{3.18}
$$

where we used that $I_i^{(n)}$ is integer-valued. Since $|\mathcal{A}_n| = (2n+1)^d$, we get from (3.18):

$$
\begin{aligned}
\mathcal{E}_{\mathrm{K}}(u_n) &\le 8(2n+1)^d \sup_{i\in\mathcal{A}_n}\int c(x,y,\gamma)(\mathbf{1}_{[2,\infty)}(\langle I_i^{(n)},\gamma\rangle)+\mathbf{1}_{[1,\infty)}(\langle I_i^{(n)},\gamma\rangle)\mathbf{1}_{\operatorname{supp} I_i^{(n)}}(y))\\
&\qquad \times(\mathbf{1}_{N_a}(x)+\mathbf{1}_{N_a}(y))\,\tilde{\mu}(dx,dy,d\gamma)\\
&= 8(2n+1)^d\sup_{i\in\mathcal{A}_n}\int_\Gamma \mu(d\gamma)\int_X zm(dx)\exp[-E(x,\gamma)]\int_X zm(dy)c(x,y,\gamma\cup x)\\
&\qquad (\mathbf{1}_{[2,\infty)}(\langle I_i^{(n)},\gamma\cup x\rangle)+\mathbf{1}_{[1,\infty)}(\langle I_i^{(n)},\gamma\cup x\rangle)\mathbf{1}_{\operatorname{supp} I_i^{(n)}}(y))\\
&\le 8(2n+1)^d\sup_{i\in\mathcal{A}_n}\int_\Gamma \mu(d\gamma)\int_X zm(dx)\exp[-E(x,\gamma)]\int_X zm(dy)\,c(x,y,\gamma\cup x)\\
&\qquad \times(\mathbf{1}_{[2,\infty)}(\langle I_i^{(n)},\gamma\rangle)+\mathbf{1}_{[1,\infty)}(\langle I_i^{(n)},\gamma\rangle)\mathbf{1}_{\operatorname{supp} I_i^{(n)}}(x)\\
&\quad +\mathbf{1}_{[1,\infty)}(\langle I_i^{(n)},\gamma\rangle)\mathbf{1}_{\operatorname{supp} I_i^{(n)}}(y)+\mathbf{1}_{\operatorname{supp} I_i^{(n)}}(x)\mathbf{1}_{\operatorname{supp} I_i^{(n)}}(y))(\mathbf{1}_{N_a}(x)+\mathbf{1}_{N_a}(y)).
\end{aligned}
\tag{3.19}
$$

By using the Ruelle bound and the representation of the factorial moment densities through correlation functions, see e.g. [6], Section 5.4, in particular formula (5.4.12), we easily conclude that the following statement holds: There exist constants $C_2, C_3 > 0$ such that, for any set $A \in \mathcal{B}(X)$, $A \subset \tilde{B}_a$, we have:

$$
\begin{aligned}
\mu(|\gamma_A|\ge 1) &\le C_2 m(A),\\
\mu(|\gamma_A|\ge 2) &\le C_3 m(A)^2.
\end{aligned}
$$

Therefore, there exist constants $C_4, C_5 > 0$ such that,

$$
\begin{aligned}
\mu(|\gamma_{\operatorname{supp} I_i^{(n)}}|\ge 1) &\le C_4\left(\frac{2}{n}\right)^d,\\
\mu(|\gamma_{\operatorname{supp} I_i^{(n)}}|\ge 2) &\le C_5\left(\frac{2}{n}\right)^{2d}, \qquad n\in\mathbb{N},\ i\in\mathcal{A}_n.
\end{aligned}
\tag{3.20}
$$

By (3.15)–(3.17), (3.19), (3.20), and the Cauchy inequality, we now easily conclude that

$$
\sup_{n\in\mathbb{N}}\mathcal{E}_{\mathrm{K}}(u_n) < \infty,
$$

which implies the lemma. $\square$

We now have the main result of this paper.

Theorem 3.1 *Let* $\sharp = \mathrm{G}, \mathrm{K}$. *We have:*

1) *Assume that* (3.14), *respectively* (3.15)–(3.17), *holds. Then there exists a conservative Hunt process*

$$\mathbf{M}^\sharp = (\mathbf{\Omega}^\sharp, \mathbf{F}^\sharp, (\mathbf{F}^\sharp_t)_{t\ge 0}, (\mathbf{\Theta}^\sharp_t)_{t\ge 0}, (\mathbf{X}^\sharp(t))_{t\ge 0}, (\mathbf{P}^\sharp_\gamma)_{\gamma\in\Gamma})$$

on Γ *(see e.g.* [23, p. 92]*) which is properly associated with* $(\mathcal{E}_\sharp, D(\mathcal{E}_\sharp))$, *i.e., for all* ($\mu$*-versions of*) $F \in L^2(\Gamma, \mu)$ *and all* $t > 0$ *the function*

$$\Gamma \ni \gamma \mapsto p^\sharp_t F(\gamma) := \int_{\mathbf{\Omega}} F(\mathbf{X}^\sharp(t))\, d\mathbf{P}^\sharp_\gamma \tag{3.21}$$

is an $\mathcal{E}_\sharp$*-quasi-continuous version of* $\exp(-tH_\sharp)F$, *where* $(H_\sharp, D(H_\sharp))$ *is the generator of* $(\mathcal{E}_\sharp, D(\mathcal{E}_\sharp))$. $\mathbf{M}^\sharp$ *is up to* μ*-equivalence unique* (*cf.* [23, Chap. IV, Sect. 6]). *In particular,* $\mathbf{M}^\sharp$ *is* μ*-symmetric (i.e.,* $\int G\, p^\sharp_t F\, d\mu = \int F\, p^\sharp_t G\, d\mu$ *for all* $F, G : \Gamma \to \mathbb{R}_+$, $\mathcal{B}(\Gamma)$*-measurable), so has* μ *as an invariant measure.*

2) $\mathbf{M}^\sharp$ *from* 1) *is up to* μ*-equivalence* (*cf.* [23, Definition 6.3]) *unique between all Hunt processes* $\mathbf{M}' = (\mathbf{\Omega}', \mathbf{F}', (\mathbf{F}'_t)_{t\ge 0}, (\mathbf{\Theta}'_t)_{t\ge 0}, (\mathbf{X}'(t))_{t\ge 0}, (\mathbf{P}'_\gamma)_{\gamma\in\Gamma})$ *on* Γ *having* μ *as invariant measure and solving the martingale problem for* $(-H_\sharp, D(H_\sharp))$, *i.e., for all* $G \in D(H_\sharp)$

$$\widetilde{G}(\mathbf{X}'(t)) - \widetilde{G}(\mathbf{X}'(0)) + \int_0^t (H_\sharp G)(\mathbf{X}'(s))\, ds, \qquad t \ge 0,$$

is an $(\mathbf{F}'_t)$*-martingale under* $\mathbf{P}'_\gamma$ *for* $\mathcal{E}_\sharp$*-q.e.* $\gamma \in \Gamma$. *(Here,* $\widetilde{G}$ *denotes an* $\mathcal{E}_\sharp$*-quasi-continuous version of* G, *cf.* [23, Ch. IV, Proposition 3.3].)

Remark 3.2 In Theorem 3.1, $\mathbf{M}^\sharp$ can be taken canonical, i.e., $\mathbf{\Omega}^\sharp$ is the set of all *cadlag* functions $\omega : [0, \infty) \to \Gamma$ (i.e., ω is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$), $\mathbf{X}^\sharp(t)(\omega) := \omega(t)$, $t \ge 0$, $\omega \in \mathbf{\Omega}^\sharp$, $(\mathbf{F}^\sharp_t)_{t\ge 0}$ together with $\mathbf{F}^\sharp$ is the corresponding minimum completed admissible family (cf. [8, Section 4.1]) and $\mathbf{\Theta}^\sharp_t$, $t \ge 0$, are the corresponding natural time shifts.

Proof of Theorem 3.1. The first part of the theorem follows from Lemmas 3.3–3.5, the fact that $1 \in D(\mathcal{E}_\sharp)$ and $\mathcal{E}_\sharp(1, 1) = 0$, $\sharp = \mathrm{G}, \mathrm{K}$, and [23, Chap. IV, Theorem 3.5 and Chap. V, Proposition 2.15]. The second part follows directly from (the proof of) [2, Theorem 3.5]. $\square$

Remark 3.3 It follows from Lemmas 3.3, 3.4 and the proof of Theorem 3.1 that, if instead of (3.14), respectively (3.15)–(3.17), we only demand the weaker condition (3.1), respectively (3.2), then Theorem 3.1 remains true with Γ replaced by $\ddot{\Gamma}$.

Let us now derive an explicit formula for the generator of $\mathcal{E}_{\rm G}$, respectively $\mathcal{E}_{\rm K}$. However, this can only be done under stronger conditions on the coefficients $d(x,\gamma)$ and $c(x,y,\gamma)$.

Using (2.1) and (2.2), we have, for $F\in\mathcal{F}C_{\rm b}(C_0(X),\Gamma)$,

$$\begin{aligned}
&\mathcal{E}_{\rm K}(F):=\mathcal{E}_{\rm K}(F,F)\\
&=\int_\Gamma\mu(d\gamma)\int_X zm(dy)\exp[-E(y,\gamma)+E(y,\gamma)]\int_X\gamma(dx)\,c(x,y,\gamma)(D^{-+}_{xy}F)(\gamma)^2\\
&=\int_\Gamma\mu(d\gamma)\int_X\gamma(dy)\exp[E(y,\gamma\setminus y)]\int_X(\gamma\setminus y)(dx)c(x,y,\gamma\setminus y)(F(\gamma\setminus x)-F(\gamma\setminus y))^2\\
&=\int_\Gamma\mu(d\gamma)\int_X\gamma(dx)\int_X(\gamma\setminus x)(dy)\exp[E(y,\gamma\setminus y)]c(x,y,\gamma\setminus y)(F(\gamma\setminus x)-F(\gamma\setminus y))^2\\
&=\int_\Gamma\mu(d\gamma)\int_X zm(dx)\exp[-E(x,\gamma)]\int_X\gamma(dy)\exp[E(y,\gamma\setminus y\cup x)]\\
&\qquad\times c(x,y,\gamma\setminus y\cup x)(F(\gamma)-F(\gamma\setminus y\cup x))^2\\
&=\int_\Gamma\mu(d\gamma)\int_X\gamma(dx)\int_X zm(dy)\,c(y,x,\gamma\setminus x\cup y)\\
&\qquad\times\exp[-E(y,\gamma)+E(x,\gamma\setminus x\cup y)](D^{-+}_{xy}F)(\gamma)^2. \qquad (3.22)
\end{aligned}$$

By (3.4) and (3.22), we have, for any $F,G\in\mathcal{F}C_{\rm b}(C_0(X),\Gamma)$,

$$\mathcal{E}_{\rm K}(F,G)=\int_\Gamma\mu(d\gamma)\int_X\gamma(dx)\int_X zm(dy)\,\tilde c(x,y,\gamma)(D^{-+}_{xy}F)(\gamma)(D^{-+}_{xy}G)(\gamma),$$

where

$$\tilde c(x,y,\gamma)=\frac12\big(c(x,y,\gamma)+c(y,x,\gamma\setminus x\cup y)\exp[-E(y,\gamma)+E(x,\gamma\setminus x\cup y)]\big). \qquad (3.23)$$

As easily seen, $\tilde c$ again satisfies the condition (3.2). Furthermore, $\tilde c$ evidently satisfies the following identity:

$$\tilde c(x,y,\gamma)=\tilde c(y,x,\gamma\setminus x\cup y)\exp[-E(y,\gamma)+E(x,\gamma\setminus x\cup y)],$$

so that $\tilde{\tilde c}=\tilde c$.

Theorem 3.2 1) *Assume that, for each compact* $\Lambda\subset X$,

$$\begin{aligned}
&\int_\Lambda\gamma(dx)d(x,\gamma)\in L^2(\Gamma,\mu),\\
&\int_\Lambda zm(dx)b(x,\gamma)\in L^2(\Gamma,\mu), \qquad (3.24)
\end{aligned}$$

where

$$b(x,\gamma) := \exp[-E(x,\gamma)]d(x,\gamma\cup x), \qquad x\in X, \gamma\in\Gamma. \tag{3.25}$$

Then

$$\mathcal{E}_{\mathrm{G}}(F,G) = \int_\Gamma (H_{\mathrm{G}}F)(\gamma)G(\gamma)\,\mu(d\gamma), \qquad F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma), \tag{3.26}$$

where

$$(H_{\mathrm{G}}F)(\gamma) = -\int_X zm(dx)\,b(x,\gamma)(D_x^+F)(\gamma) - \int_X \gamma(dx)\,d(x,\gamma)(D_x^-F)(\gamma) \qquad \mu\text{-a.e.} \tag{3.27}$$

and $H_{\mathrm{G}}F\in L^2(\Gamma,\mu)$. *The Friedrichs' extension of the operator* $(H_{\mathrm{G}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ *in* $L^2(\Gamma,\mu)$ *is* $(H_{\mathrm{G}},D(H_{\mathrm{G}}))$.

2) *Assume that, for each compact* $\Lambda\subset X$,

$$\int_X \gamma(dx)\int_X zm(dy)\,\tilde{c}(x,y,\gamma)(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))\in L^2(\Gamma,\mu). \tag{3.28}$$

Then

$$\mathcal{E}_{\mathrm{K}}(F,K) = \int_\Gamma (H_{\mathrm{K}}F)(\gamma)G(\gamma)\,\mu(d\gamma), \qquad F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma), \tag{3.29}$$

where

$$(H_{\mathrm{K}}F)(\gamma) = -2\int_X \gamma(dx)\int_X zm(dy)\tilde{c}(x,y,\gamma)(D_{xy}^{-+}F)(\gamma) \qquad \mu\text{-a.e.} \tag{3.30}$$

and $H_{\mathrm{K}}F\in L^2(\Gamma,\mu)$. *The Friedrichs' extension of the operator* $(H_{\mathrm{K}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ *in* $L^2(\Gamma,\mu)$ *is* $(H_{\mathrm{K}},D(H_{\mathrm{K}}))$.

Proof. Formulas (3.26), (3.27), (3.29), and (3.30) follow from (2.1), (3.3), (3.4), (3.23), and (3.25), analogously to (3.22). The facts that $H_{\mathrm{G}}F\in L^2(\Gamma,\mu)$, $H_{\mathrm{K}}F\in L^2(\Gamma,\mu)$ trivially follow from (3.24) and (3.28), respectively. □

Remark 3.4 Let us fix an *arbitrary* activity $z'>0$ and suppose that there exists $\nu\in\mathcal{G}(z',E)$ which satisfies our assumptions on a Gibbs measure. Then, it follows from Theorem 3.2 that the operator $(H_{\mathrm{K}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ constructed for the *fixed* $z>0$ is symmetric on $L^2(\Gamma,\nu)$ and hence has a Friedrichs' extension on this space. On the other hand, by (3.27), the corresponding statement in the Glauber case case only holds when $z'=z$.

4 Examples

Throughout this section, we shall always assume that a pair potential ϕ, an activity z, and a corresponding Gibbs measure $\mu \in \mathcal{G}(z, E)$ are either as in Theorem 2.1 or as in Theorem 2.2. Furthermore, in the case $X = \mathbb{R}^d$, we shall also suppose that the condition of Lemma 2.1 is satisfied. Thus, in any case we have that ϕ is bounded from below, satisfies (I), and

$$\sum_{y\in\gamma} |\phi(x,y)| < \infty \quad \text{for } m \otimes \mu\text{-a.e. } (x,\gamma) \in X \times \Gamma.$$

By (2.1), the latter easily implies that, for μ-a.e. $\gamma \in \Gamma$ and for each $x \in \gamma$,

$$\sum_{y\in\gamma\setminus x} |\phi(x,y)| < \infty.$$

We shall now consider some examples of coefficients $d(x,\gamma)$ and $c(x,y,\gamma)$ for which the conditions of Theorems 3.1, 3.2 are satisfied.

4.1 Glauber dynamics

For each $s \in [0,1]$, we define

$$d(x,\gamma) = d_s(x,\gamma) := \exp\Bigg[s \sum_{y\in\gamma\setminus x} \phi(x,y)\Bigg].$$

By (3.25), we then have

$$b(x,\gamma) = b_s(x,\gamma) = \exp\Bigg[(s-1)\sum_{y\in\gamma} \phi(x,y)\Bigg].$$

In particular, for $s = 0$,

$$d_0(x,\gamma) = 1, \quad b_0(x,\gamma) = \exp\Bigg[-\sum_{y\in\gamma} \phi(x,y)\Bigg],$$

and for $s = 1$,

$$d_1(x,\gamma) = \exp\Bigg[\sum_{y\in\gamma\setminus x} \phi(x,y)\Bigg], \quad b_1(x,\gamma) = 1.$$

Proposition 4.1 1) *For each $s \in [0,1]$, the coefficient d_s satisfies* (3.14).

2) *For each* $s \in [0, 1/2]$, *the coefficients* d_s, b_s *satisfy* (3.24). *Furthermore, for each* $s \in (1/2, 1]$, (3.24) *is satisfied if additionally*

$$\sup_{x \in \Lambda} \int_X |e^{(2s-1)\phi(x,y)} - 1| \, m(dy) < \infty \tag{4.1}$$

for each compact $\Lambda \subset X$.

Proof. 1) By (2.3), we have for each $s \in [0, 1]$:

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_\Lambda zm(dx) \exp[-2E(x,\gamma)] d_s(x, \gamma \cup x)^2 \\
&= \int_\Lambda zm(dx) \int_\Gamma \mu(d\gamma) \exp\left[2(s-1) \sum_{y \in \gamma} \phi(x,y)\right] \\
&= \int_\Lambda zm(dx) \int_\Gamma \mu(d\gamma) \prod_{y \in \gamma} (1 + (\exp[2(s-1)\phi(x,y)] - 1)) \\
&= \int_\Lambda zm(dx) \int_\Gamma \mu(d\gamma) \biggl(1 + \sum_{n=1}^\infty \sum_{\{y_1,\dots,y_n\} \subset \gamma} \prod_{i=1}^n (\exp[2(s-1)\phi(x,y_i)] - 1)\biggr) \\
&= \int_\Lambda zm(dx) \biggl(1 + \sum_{n=1}^\infty \int_{X^n} \prod_{i=1}^n (\exp[2(s-1)\phi(x,y_i)] - 1) \\
&\qquad \times k_\mu^{(n)}(y_1, \dots, y_n) \, m(dy_1) \cdots m(dy_n)\biggr).
\end{aligned} \tag{4.2}$$

By (2.4), (I), and the boundedness of ϕ from below, we have:

$$\begin{aligned}
&\int_\Lambda zm(dx) \biggl(1 + \sum_{n=1}^\infty \int_{X^n} \prod_{i=1}^n |\exp[2(s-1)\phi(x,y_i)] - 1| \\
&\qquad \times k_\mu^{(n)}(y_1, \dots, y_n) \, m(dy_1) \cdots m(dy_n)\biggr) \\
&\le \int_\Lambda zm(dx) \exp\left[\xi \int_X |\exp[2(s-1)\phi(x,y)] - 1| \, m(dy)\right] \\
&\le \int_\Lambda zm(dx) \exp\left[\xi \int_X |\exp[-2\phi(x,y)] - 1| \, m(dy)\right] \\
&\le \int_\Lambda zm(dx) \exp\biggl[\xi \int_X |\exp[-\phi(x,y)] - 1| \, m(dy) \\
&\qquad + \int_X \exp[-\phi(x,y)] |\exp[-\phi(x,y)] - 1| \, m(dy)\biggr] < \infty.
\end{aligned} \tag{4.3}$$

Thus, by (4.2) and (4.3), condition (3.14) is satisfied.

2) By (2.1), we have:

$$\int_\Gamma \left(\int_\Lambda \gamma(dx) d_s(x,\gamma) \right)^2 \mu(d\gamma)$$

$$= \int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx) d_s(x,\gamma)^2 + \int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X (\gamma \setminus x)(dy) d_s(x,\gamma) d_s(y,\gamma)$$

$$= \int_\Gamma \mu(d\gamma) \int_\Lambda zm(dx) \exp\left[(2s-1) \sum_{y\in\gamma} \phi(x,y) \right]$$

$$+ \int_\Gamma \mu(d\gamma) \int_\Lambda zm(dx) \int_\Lambda zm(dy) \exp\left[(s-1) \sum_{x'\in\gamma} \phi(x,x') \right.$$

$$\left. + (s-1) \sum_{y'\in\gamma} \phi(y,y') - \phi(x,y) \right]$$

$$= \int_\Lambda zm(dx) \int_\Gamma \mu(d\gamma) \exp\left[(2s-1) \sum_{y\in\gamma} \phi(x,y) \right]$$

$$+ \int_\Lambda zm(dx) \int_\Lambda zm(dy) \exp[-\phi(x,y)] \int_\Gamma \mu(d\gamma)$$

$$\times \exp\left[\sum_{u\in\gamma} ((s-1)\phi(x,u) + (s-1)\phi(y,u)) \right]. \tag{4.4}$$

Analogously to (4.2), (4.3), the first summand in (4.4) is finite. Indeed,

$$\sup_{x\in\Lambda} \int_X |\exp[(2s-1)\phi(x,y)] - 1|\, m(dy) < \infty$$

for $s \in [0,1/2]$ by (I), and for $s \in (1/2,1]$ by (4.1).

To show the finiteness of the second summand in (4.4), we first note that

$$\int_\Lambda zm(dx) \int_\Lambda zm(dy) \exp[-\phi(x,y)]$$

$$\leq \int_\Lambda zm(dx) \int_X zm(dy) |\exp[-\phi(x,y)] - 1| + m(\Lambda)^2 < \infty$$

by (I). Therefore, it suffices to show that

$$\sup_{x\in\Lambda} \sup_{y\in\Lambda} \int_X |\exp[(s-1)\phi(x,u) + (s-1)\phi(y,u)] - 1|\, m(du) < \infty.$$

But this follows from (I), the boundedness of ϕ from below, and the estimate

$$\int_X |\exp[(s-1)\phi(x,u) + (s-1)\phi(y,u)] - 1|\, m(du)$$

$$\leq \int_X |\exp[(s-1)\phi(x,u)] - 1|\, m(du)$$
$$+ \int_X \exp[(s-1)\phi(x,u)] |\exp[(s-1)\phi(y,u)] - 1|\, m(du).$$

Finally, the second condition in (3.24) trivially follows from the above estimates. □

Proposition 4.2 *Assume that* $d(x,\gamma)$ *is* $\gamma(dx)\mu(d\gamma)$*-a.s. bounded and* $b(x,\gamma)$ *is* $m\otimes\mu$*-a.s. bounded. Then* (3.14) *and* (3.24) *are satisfied.*

Proof. (3.14) follows from the estimates (4.2), (4.3) with $s = 0$. The condition (3.24) is now trivially satisfied. □

As an example of a Glauber dynamics with bounded coefficients, one can consider the following one: for $s \in [0,1]$

$$d(x,y) = d_s(x,y) := \frac{s}{1+\exp[-E(x,\gamma\setminus x)]} + \frac{(1-s)\exp[E(x,\gamma\setminus x)]}{1+\exp[E(x,\gamma\setminus x)]},$$
$$b(x,y) = b_s(x,y) := \frac{s\exp[-E(x,\gamma)]}{1+\exp[-E(x,\gamma)]} + \frac{1-s}{1+\exp[E(x,\gamma)]}.$$

4.2 Kawasaki dynamics

Let $a : X^2 \to \mathbb{R}$ be a symmetric measurable function which satisfies (3.16).

Remark 4.1 In the case $X = \mathbb{R}^d$, it is natural to suppose that the function a is translation invariant, i.e., $a(x,y) = A(x-y)$ for some $A : X \to \mathbb{R}$, $A(-x) = A(x)$, $x \in \mathbb{R}^d$, in which case (3.16) is equivalent to the integrability of A.

For $s \in [0,1]$ we define

$$\varkappa(x,y,\gamma) = \varkappa_s(x,y,\gamma) := \exp[sE(x,\gamma\setminus x) - (1-s)E(y,\gamma)],$$

and

$$c(x,y,\gamma) = c_s(x,y,\gamma) := a(x,y)\varkappa_s(x,y,\gamma).$$

We evidently have $\tilde{c}_s(x,y,\gamma) = c_s(x,y,\gamma)$.

Proposition 4.3 1) *For each* $s \in [0,1]$, a_s *and* $\varkappa_s$ *satisfy* (3.17).

2) *Assume that the function* a *is bounded. Then, for each* $s \in [0,1/2]$, *the coefficient* c_s *satisfies* (3.28). *Furthermore, for each* $s \in (1/2,1]$, (3.28) *is satisfied if additionally*

$$\sup_{x\in X} \int_X |\exp[(2s-1)\phi(x,y)] - 1|\, m(dy) < \infty. \tag{4.5}$$

Proof. 1) We have

$$\int_\Gamma \mu(d\gamma)\int_X zm(dx)\int_Z zm(dy)a(x,y)$$
$$\times\exp[-2E(x,\gamma)+2sE(x,\gamma)-2(1-s)E(y,\gamma\cup x)](\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))$$
$$=\int_X zm(dx)\int_X zm(dy)a(x,y)\exp[-2(1-s)\phi(x,y)](\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))$$
$$\times\int_\Gamma \mu(d\gamma)\exp\Big[-2(1-s)\sum_{u\in\gamma}(\phi(x,u)+\phi(y,u))\Big]. \tag{4.6}$$

Analogously to (4.2) and (4.3), by using (3.16), (I), and the boundedness of ϕ from below, we can now easily show that the expression in (4.6) is finite.

2) Analogously to (4.2), (4.3) and using (4.5), we have:

$$\int_\Gamma \mu(d\gamma)\Big(\int_X \gamma(dx)\int_X zm(dy)c_s(x,y,\gamma)(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))\Big)^2$$
$$=\int_\Gamma \mu(d\gamma)\int_X \gamma(dx)\int_X zm(dy)\int_X zm(dy')c_s(x,y,\gamma)c_s(x,y',\gamma)$$
$$\times(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y'))$$
$$+\int_\Gamma \mu(d\gamma)\int_X \gamma(dx)\int_X (\gamma\setminus x)(dx')\int_X zm(dy)\int_X zm(dy')$$
$$\times c_s(x,y,\gamma)c_s(x',y',\gamma)(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x')+\mathbf{1}_\Lambda(y'))$$
$$=\int_\Gamma \mu(d\gamma)\int_X zm(dx)\int_X zm(dy)\int_X zm(dy')\exp[-E(x,\gamma)]$$
$$\times c_s(x,y,\gamma\cup x)c_s(x,y',\gamma\cup x)(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y'))$$
$$+\int_\Gamma \mu(d\gamma)\int_X zm(dx)\int_X zm(dx')\int_X zm(dy)\int_X zm(dy')$$
$$\times\exp[-E(x,\gamma)-E(x',\gamma)-\phi(x,x')]$$
$$\times c_s(x,y,\gamma\cup x\cup x')c_s(x',y',\gamma\cup x\cup x')(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x')+\mathbf{1}_\Lambda(y'))$$
$$=\int_X zm(dx)\int_X zm(dy)\int_X zm(dy')a(x,y)a(x,y')(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y'))$$
$$\times\exp[-(1-s)\phi(x,y)-(1-s)\phi(x,y')]$$
$$\times\int_\Gamma \mu(d\gamma)\exp\Big[\sum_{u\in\gamma}\big((2s-1)\phi(x,u)-(1-s)\phi(y,u)-(1-s)\phi(y',u)\big)\Big]$$
$$+\int_X zm(dx)\int_X zm(dx')\int_X zm(dy)\int_X zm(dy')a(x,y)a(x',y')$$
$$\times(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x')+\mathbf{1}_\Lambda(y'))$$
$$\times\exp[(2s-1)\phi(x,x')-(1-s)\phi(x,y)-(1-s)\phi(x',y)$$

$$
\begin{aligned}
&\qquad - (1-s)\phi(x,y') - (1-s)\phi(x',y')] \\
&\quad \times \int_\Gamma \mu(d\gamma)\exp\left[\sum_{u\in\gamma} -(1-s)(\phi(x,u)+\phi(x',u)+\phi(y,u)+\phi(y',u))\right] \\
&\le C_6\bigg(\int_X zm(dx)\int_X zm(dy)\int_X zm(dy')a(x,y)a(x,y')(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y')) \\
&\qquad \times \exp\bigg[\xi \sup_{x\in X}\sup_{y\in X}\sup_{y'\in X}\int_X |\exp[(2s-1)\phi(x,u) \\
&\qquad - (1-s)\phi(y,u) - (1-s)\phi(y',u)] - 1|\, m(du)\bigg] \\
&\quad + \int_X zm(dx)\int_X zm(dx')\int_X zm(dy)\int_X zm(dy')a(x,y)a(x',y') \\
&\quad \times (\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x')+\mathbf{1}_\Lambda(y'))\exp[(2s-1)\phi(x,x')] \\
&\quad \times \exp\bigg[\xi \sup_{x\in X}\sup_{x'\in X}\sup_{y\in X}\sup_{y'\in X}\int_X |\exp[-(1-s)(\phi(x,u)+\phi(x',u) \\
&\qquad + \phi(y,u)+\phi(y',u))] - 1|\, m(du)\bigg]\bigg) \\
&\le C_7\bigg(\int_X zm(dx)\int_X zm(dy)\int_X zm(dy')a(x,y)a(x,y')(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y')) \\
&\quad + \int_X zm(dx)\int_X zm(dx')\int_X zm(dy)\int_X zm(dy')a(x,y)a(x',y') \\
&\quad \times (\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))(\mathbf{1}_\Lambda(x')+\mathbf{1}_\Lambda(y'))|\exp[(2s-1)\phi(x,x')] - 1| \\
&\quad + \left(\int_X zm(dx)\int_X zm(dy)a(x,y)(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y))\right)^2\bigg), \qquad (4.7)
\end{aligned}
$$

where $C_6, C_7 > 0$. Using (I), (3.16), (4.5), and the boundedness of a, we easily conclude that the expression in (4.7) is finite. Indeed, for example, we have:

$$
\begin{aligned}
&\int_X zm(dx)\int_X zm(dx')\int_\Lambda zm(dy)\int_\Lambda zm(dy')a(x,y)a(x',y')|\exp[(2s-1)\phi(x,x')] - 1| \\
&\quad \le \left(\sup_{(u,v)\in X^2} a(u,v)\right)\int_\Lambda zm(dy)\int_\Lambda zm(dy') \\
&\qquad \times \int_X zm(dx)a(x,y)\int_X zm(dx')|\exp[(2s-1)\phi(x,x')] - 1| < \infty.
\end{aligned}
$$

Thus, the proposition is proved. □

Let us now present a straightforward generalization of the above result. Let now $a : X^2 \to \mathbb{R}$ be a measurable function which satisfies (3.16) (and which is not necessarily symmetric). For $u, v \in [0,1]$, we define

$$\varkappa(x, y, \gamma) = \varkappa_{u,v}(x, y, \gamma) := \exp[uE(x, \gamma \setminus x) - (1 - v)E(y, \gamma)],$$

and

$$c(x, y, \gamma) = c_{u,v}(x, y, \gamma) := a(x, y)\varkappa_{u,v}(x, y, \gamma).$$

In particular, for $u = v$, we get the previous example of a Kawasaki dynamics. Note also that, for $u = 0$ and $v = 1$, we get

$$c_{0,1}(x, y, \gamma) = a(x, y).$$

By (3.23), we have

$$\begin{aligned}\tilde{c}_{u,v}(x, y, \gamma) = \frac{1}{2}\big(a(x, y)\exp[uE(x, \gamma \setminus x) - (1 - v)E(y, \gamma)] \\ + a(y, x)\exp[vE(x, \gamma \setminus x) - (1 - u)E(y, \gamma)]\big).\end{aligned}$$

Absolutely analogously to Proposition 4.3, one can prove its following generalization.

Proposition 4.4 1) *For each* $u, v \in [0, 1]$, $a_{u,v}$ *and* $\varkappa_{u,v}$ *satisfy* (3.17).

2) *Assume that the function* a *is bounded. Then,* (3.28) *is satisfied if*

$$\sup_{x \in X} \int_X |\exp[(2(u \vee v) - 1)\phi(x, y)] - 1|\, m(dy) < \infty.$$

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