Hervé–Harnack's Inequalities for the generalized Ginzburg–Landau Equation

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0 Introduction

This paper is devoted to a study of the continuous solutions of the following generalized Ginzburg-Landau equation :

(1) $$Lu - u(|u|^{2\alpha} - 1) = 0 \text{ in the distributional sense on } \mathbb{R}^d.$$

Where $\alpha > 0$ and L is a strongly elliptic operator with bounded , uniformly Hölder continuous coefficients and admitting an adjoint L^* in the distributional sense.

The equation $\Delta u - u(|u|^2 - 1) = 0$ was recently investigated on $\mathbb{R}^2$ and for complex valued solutions by F.Bethuel/H.Brezis/ F.Helein/F.Merle and T.Rivière [BeBrH1] [BeBrH2] [BrMT] for variational methods and R.M.Hervé/M.Hervé [HH94] [HH96] by methods of analytic functions.

In this paper we intend to show that in fact semilinear perturbations of partial differential equations leads in a very simple and natural way to results for the equation (1) known for the equation $\Delta u - u(|u|^2 - 1) = 0$ on $\mathbb{R}^2$ by methods of analytic functions and others . We shall obtain Hervés and Hervé-Harnack inequalities.We shall discuss the solvability of the Dirichlet problem for real and complex valued solutions and give more results about the scheaf of solutions of the equation (1) .

Our paper is organized as follows : For the convenience of the reader who is not familiar with linear and semilinear potential theory we shall devote section 1 and section 2 to a short presentation of the definitions, notations, nonlinear perturbations and related results necessary for the investigation of the generalized Ginzburg-Landau equation. In section 3 we consider on $\mathbb{R}^d$, $d \geq 2$, a strongly

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elliptic operator with bounded and locally Hölder continuous coefficients in the following form :

$$Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i}(x).$$

For every open set U,let

$$\mathcal{H}_L(U) := \{u \in \mathcal{C}^2(U) : Lu = 0\}.$$

and

$$\mathcal{H}(U) := \left\{ u \in \mathcal{C}(U) : \left(u + \int {}^{L}G_t^V u(t) \left(|u(t)|^{2\alpha} - 1 \right) dt \right) \in \mathcal{H}_L(V) \text{ for } V \subset \overline{V} \subset U \right\}$$

where ${}^{L}G^{V}$ is the Green function for L on V and $\mathcal{C}(U)$ is the set of real continuous functions on U .We set $K_1 = 4(\frac{1}{\alpha}(\frac{1}{\alpha}+1))M$ and $K_2 = \frac{2}{\alpha}B$ where $M = \sup\{M(x), x \in \mathbb{R}^d\}$, $B = \sup\{\sum_{i=1}^{d} |b_i(x)|, x \in \mathbb{R}^d\}$ and $M(x)$ the biggest eigenvalue of the symmetric real Matrix $(a_{ij}(x))$. We prove for the solutions of the generalized Ginzburg-Landau equation the following Hervé̀s inequality :
For every $x \in \mathbb{R}^d$, every $R > 0$ and $u \in \mathcal{H}(B(x,R))$ we have $|u(x)| \leq \sigma(R)$. where $\sigma(R) = [1 + \frac{K_1}{R^2} + \frac{K_2}{R}]^{1/2\alpha}$. Moreover we get a Hervé-Harnack inequality as follows :
For every open set U and every compact set K of U we have $|u(x)| \leq \sigma(d(K, \complement U))$ for every $x \in K$ and $u \in \mathcal{H}(U)$. This inequality yields $\mathcal{H}(U)$ compact for the local uniform convergence.We finish this section by a comparison in the case $d = 2$ and $\alpha = 1$ between $\sigma(R)$ and $\sigma_0(R) = (\frac{1}{2} + \sup\left(\frac{12}{R^2}, \sqrt{\frac{1}{4} + \frac{48}{R^4}}\right))^{1/2}$ obtained by Hervé [HH96] for complex valued solutions by different methods.Among others we get the following : $\sigma(R) \leq \sigma_0(R)$ if and only if $R \leq 2\sqrt{2}$. In section 4 , we investigate existence and unicity of the following Dirichlet problem :
Let $f \in \mathcal{C}(\partial U, \mathbb{C})$ be a continuous complex valued function at the boundary of U. We look for a continuous complex valued solution $u \in \mathcal{C}(U, \mathbb{C})$ of the following system

$$(*) \begin{cases} Lu + u(1 - |u|^{2\alpha}) = 0 & \text{on U in the distributional sense,} \\ \lim_{x \longmapsto y} u(x) = f(y) & \text{for every regular point y in } \partial U \end{cases}$$

where $|u|^2 = (Re\, u)^2 + (Im\, u)^2$.
We prove without any assumption on the regularity of the boundary and differentiability of f that $(*)$ admits a solution on every open set U satisfying : $\delta(U) := \sup\{\int G_t^U(x)dt \;\; x \in U\} \leq 1$. The unicity is treated in the following way : For every $K > 1$, there exists a basis ${}_K$ such that for $V \in \mathcal{V}_K$ and $f \in \mathcal{C}(\partial U, \mathbb{C})$ with $\|f\|_\infty \leq K$ there exists on U a unique solution of $(*)$.

For $L = \Delta$ on $\mathbb{R}^d$, $\alpha = 1$, B a ball with radius R we have the following interesting results :

$$\begin{aligned} \delta(B) < 1 \text{ if and only if } R &< \left(\frac{4d\pi^{d/2}}{\Gamma(d/2)}\right)^{1/2} \\ B \in \mathcal{V}_K \text{ if and only if } R &< \left(\frac{4d\pi^{d/2}}{\Gamma(d/2)(1+3K^2)}\right)^{1/2} \end{aligned}$$

In section 5 we prove, for complex valued solutions of the Ginzburg-Landau equation and as in the real case, a Hervé inequality with a bound σ having in the case of $\mathbb{R}^2$ and for $\alpha = 1$ the same behaviour (as R tends to infinty) as the bound σ_0 obtained by Hervé [HH96], we proove that $\sigma_0 \leq \sigma$. Moreover we obtain for the general case $|u| \leq 1$ for every $u \in \mathcal{H}(\mathbb{R}^d, \mathbb{C})$, for every non empty open set U in $\mathbb{R}^d$, $\mathcal{H}(U, \mathbb{C})$ is compact for the local uniform convergence.

In the last section 6, we consider $\alpha \geq 0$ and for every open set U in $\mathbb{R}^d$

$$\mathcal{H}_1(U, \mathbb{C}) = \{u \in \mathcal{C}(U, \mathbb{C}) : Lu = u(|u|^{2\alpha} - 1) \text{ in } D.S \text{ on } U \text{ with } |u| \leq 1\}.$$

We prove a generalisation of the Hervé-Harnack inequality, obtained by Hervé in the case of $\mathbb{R}^2$ and for $\alpha = 1$ [HH96], in the following form :

For an open domain U in $\mathbb{R}^2$, and $K \subset U$ compact, there exists $C_K \geq 1$ such that $(1 - |u(x)|) \leq C_K(1 - |u(y)|)$ for every $x, y \in K$ and every $u \in \mathcal{H}_1(U, \mathbb{C})$.

For an open domain U in $\mathbb{R}^d$, $d \geq 3$, $q > \frac{d}{2}$ and $K \subset U$ compact, there exists C_K such that $(1 - |u(x)|) \leq C_K(1 - |u(y)|)^{1/q}$ for every $x, y \in K$ and every $u \in \mathcal{H}_1(U, \mathbb{C})$.

The previous inequalities yields the following interesting convergence criterion: Let U be a domain in $\mathbb{R}^d$, $d \geq 2$, $(u_n)_n \subset \mathcal{H}_1(U, \mathbb{C})$ and $\beta \in \mathbb{C}$ with $|\beta| = 1$ then the following properties are equivalent.

1) $(u_n)_n$ converges locally uniformly to β on U.

2) There exists $x \in U$ such that $u_n(x)$ converges to β.

Furthermore we have analogous results as bevore if we consider for $m \geq 0$ the equation $Lu - u(|u|^{2\alpha} - m^{2\alpha}) = 0$ in the distributional sense on $\mathbb{R}^d$.

Solutions u of the semilinear equations considered in this work can be interpreeeted as particle concentrations in physical and biological sciences. Such interpretations can be found in our paper [BS], where we investigate nonlinear semigroups with evolutionary law governed by weakly an autonomous system of partial differential equations of parabolic type. Let us allready now that our methods are applicable to a broader class of weakly coupled system of elliptic parabolic operators of second order in the following form : for an open subset

U in $\mathbb{R}^d$, $d \geq 2$, and an integer $n \geq 2$ we denote by $u \in \mathcal{C}(U, \mathbb{R}^n)$ the set of continous functions from to $\mathbb{R}^n$ and consider the following generalisation of the Ginzburg-Landau equation:We look for $v = (u_1, u_2, ..., u_n) \in \mathcal{C}(U, \mathbb{R}^n)$ satisfying:

For every $i \in \{1, 2, ..., n\}$, $Lu_i - u_i(|v|^{2\alpha} - 1) = 0$ in the distributional sence on U, where $|v|^2 = {u_1}^2 + {u_2}^2 + ... + {u_n}^2$.

1 Definitions and Notations

Let X be a locally compact space with countable base. For every open subset U of X, let $\mathcal{C}(U)$($\mathcal{B}(U)$ resp.)be the set of all continuous real (Borel measurable numerical resp.) functions on U.Given any set $\mathcal{A}$ of numerical functions $\mathcal{A}_b$ ($\mathcal{A}_+$ resp.) will denote the set of all bounded (positive resp.) function in $\mathcal{A}$.

Let $(X, \mathcal{G})$ be a linear harmonic Bauer space in the sense of [CC]. For every relatively compact subset U of X , H_U is the harmonic kernel defined by $H_U(x, .) = \mu_x^U$ for every $x \in U$ and $H_U(x, .) = \varepsilon_x$ for every $x \in X \setminus U$. μ_x^U is the harmonic measure associated with U and x by the Perron-Wiener-Brelot method.Further we will denote by ${}^*\mathcal{G}(U)$ the set of hyperharmonic functions,by $\mathcal{S}(U)$ the set of superharmonic functions and by $\mathcal{P}(U)$ the potentials on U (see [CC] or [BHH]).$\mathcal{U}(\mathcal{G})$ denote the set of all relatively compact open subsets U of X for which the closure $\overline{U}$ is contained in some P–set(i.e. an open set V on which there exists a strictly positive potential $p \in \mathcal{P}(V)$).

A family $M = (M_U)_{U \in \mathcal{U}(\mathcal{G})}$ is called a positive section of continuous potentials if $M_U \in \mathcal{P}(U)$ for all $U \in \mathcal{U}(\mathcal{G})$ and $M_U - M_V$ is harmonic on $U \cap V$ for all U, V in $\mathcal{U}(\mathcal{G})$ (see[BHH]).We will denote by $\mathcal{M}$ the set of all such sections.

The symbole $\prec$ denote the specific order on $\mathcal{P}(U)$ and $\bullet$ is the specific multiplication.

In what follows we fix $M \in \mathcal{M}$ and we recall from [BBM] the following notions:

A Borel measurable function f from X to $\overline{\mathbb{R}}$ is in the local Kato-class relatively to M,denoted by K_{loc}^M, if the specific product $|f| \bullet M$ is again a positive section of positive and real potentials.

We recall that if $\mathcal{G}$ is the scheaf of the classical harmonic functions given by the solutions of the Laplace equation on $\mathbb{R}^d (d \geq 1)$ and M is given by the Lebesgue measure,then K_M^{loc} is the Kato-class K_{loc}^n introduced by Aisenman/simon [AS].

Let now φ be a Borel measurable function from $X \times \mathbb{R}$ to $\overline{\mathbb{R}}$. From [BBM] or [BM] we recall the following :

a)φ is called locally Kato-bounded relatively to M , if for every $c \in \mathbb{R}_+^*$,there exists $p^c \in \mathcal{M}$ such that : $|\varphi(., g)| \bullet M_U \prec p_U^c$ for every $U \in \mathcal{U}(\mathcal{G})$ and $g \in \mathcal{B}_b(X)$ with $\|g\|_\infty \leq c$.

b)φ is called Kato-bounded relatively to M , if there exists $p \in \mathcal{M}$ such that : $|\varphi(., g)| \bullet M_U \prec p_U$ for every $U \in \mathcal{U}(\mathcal{G})$ and $g \in \mathcal{B}_b(X)$.

c)φ is called locally Kato-Lipschitzian relatively to M if for every $c \in \mathbb{R}_+^*$, there exists $p^c \in \mathcal{M}$ such that : $|\varphi(., u) - \varphi(., v)| \bullet M_U \prec |u - v| \bullet p_U^c$ for every $U \in$

$\mathcal{U}(\mathcal{G})$ and $u, v \in \mathcal{B}_b(X)$ with $\|u\|_\infty \leq c$ and $\|v\|_\infty \leq c$.
It is then easy to see that a function φ satisfying the previous conditions need not to be (locally) bounded or (locally) lipschitzian.

2 Nonlinear perturbation of harmonic spaces

In this section we recall some notations and known facts about the nonlinear perturbation of linear harmonic spaces (see among others [vG1], [BBM], [B1], [B2], [Ba]).Let $(X, \mathcal{G})$ be a linear Bauer space in the sense of [CC] . Fix M a positive section M of continuous and real potential and consider for every $U \in \mathcal{U}(\mathcal{G})$ the potential kernel $K_U^M = K_{M_U}$ associated with M on U. Let φ be a Borel measurable function from $X \times \mathbb{R}$ to $\overline{\mathbb{R}}$ with the following conditions:

1) For every $x \in X$, $t \longmapsto t\varphi(x,t)$ is increasing.

2) For every $x \in X$ $\varphi(x, \cdot)$ is continuous.

3) φ is locally Kato–bounded and φ^- is Kato–bounded relatively to M. (Here $\varphi^-(x,t) = \sup(-\varphi(x,t), 0)$).

For every open subset U of X, we set

$$\mathcal{H}(U) = \{u \in \mathcal{C}(U) : (u + (u\varphi(\cdot, u)) \bullet M_V) \in \mathcal{G}(V) \text{ for every} V \in \mathcal{U}(\mathcal{G}) \text{ with } \overline{V} \subset U\}.$$

By [BBM], $(X, \mathcal{H})$is a nonlinear harmonic Bauer space in the sense of [B1]. Let U be an open set of X. A function u from U to $\overline{\mathbb{R}}$ lower semicontinuous and locally lower bounded is termed hyperharmonic on U, if for every regular subset V in X with $\overline{V} \subset U$ we have $H_V u \leq u$ on V. A function u from U to $\overline{\mathbb{R}}$ upper semicontinuous and locally upper bounded is said to be hypoharmonic on U, if for every regular subset V in $(X, \mathcal{H})$ with $\overline{V} \subset U$, we have $H_V u \geq u$ on V. We will denote by ${}^*\mathcal{H}(U)$ (resp. ${}_*\mathcal{H}(U)$) the set of hyperharmonic (resp. hypoharmonic) functions on U. An easy proof gives the following:

$$\begin{aligned} {}^*\mathcal{H}(U) \cap B_b(U) &= \{u \in B_b(U) : (u + (u\varphi(\cdot, u)) \bullet M_V) \in^* \mathcal{G},\ \ \overline{V} \subset U \text{ with } V \in \mathcal{U}(\mathcal{G})\} \\ &\text{and} \\ {}_*\mathcal{H}(U) \cap B_b(U) &= \{u \in B_b(U) : (u + (u\varphi(\cdot, u)) \bullet M_V) \in^* \mathcal{G},\ \ \overline{V} \subset U \text{ with } V \in \mathcal{U}(\mathcal{G})\}. \end{aligned}$$

We then obtain that ${}^*\mathcal{H}$ and ${}_*\mathcal{H}$ are nonlinear sheaves.
Let

$$\mathcal{V} = \{V \in X \quad \text{-regular such that } ||M_V|| < 1\}.$$

then $\mathcal{V}$ is a basis.

For every $V \in \mathcal{U}(\mathcal{G})$ we set ${}^-M_V = (I - M_V)^{-1} M_V$,

$$^{-}\mathcal{G}(U) = \{u \in \mathcal{C}(U) : (u - u \bullet M_V) \in \mathcal{G}(V), \quad V \in \mathcal{U}(\mathcal{G}) \text{ with } \overline{V} \subset U\},$$

and

$$\mathcal{H}(U) = \{u \in \mathcal{C}(U) : (u + (|u|^{2\alpha} - 1) \bullet M_V) \in \mathcal{G}(V), \quad V \in \mathcal{U}(\mathcal{G}) \text{ with } \overline{V} \subset U\},$$

we have by[BHH]and[B1]

Proposition 2.1.

$$\mathcal{H}(U) = \{u \in \mathcal{C}(U) : (u + (|u|^{2\alpha}) \bullet^{-} M_V) \in^{-} \mathcal{G}(V), \quad V \in \mathcal{U}(\mathcal{G}) \textit{ with } \overline{V} \subset U\}$$

and hence $(X, \mathcal{H})$ *is a harmonic Bauer space.*

From the previous proposition and [BHH] [BBM] we have:

Proposition 2.2. $^{*}\mathcal{H}$ *and* $_{*}\mathcal{H}$ *are scheaves and*

$$\begin{aligned} ^{*}\mathcal{H}(U) \cap B_b(U) &= \{u \in B_b(U) : (u + (u(|u|^{2\alpha} - 1)) \bullet M_V) \in^{*} \mathcal{H}(V), \quad V \in \mathcal{U}(\mathcal{G}) \textit{ with } \overline{V} \subset U\} \\ _{*}\mathcal{H}(U) \cap B_b(U) &= \{u \in B_b(U) : (u + (u(|u|^{2\alpha} - 1)) \bullet M_V) \in_{*} \mathcal{G}(V), \quad V \in \mathcal{U}(\mathcal{G}) \textit{ with } \overline{V} \subset U\} \end{aligned}$$

The harmonic space $(X, \mathcal{H})$ will play an important role for the investigation of the generalized Ginzburg–Landau equation.

Let $(X, \widetilde{\mathcal{H}})$ be a general Bauer space (linear or nonlinear) and U an open set in X. We shall say that U is an MP–set in $(X, \widetilde{\mathcal{H}})$, if the following comparison principle is satisfied: Let $u \in^{*} \widetilde{\mathcal{H}}(U)$, $v \in_{*} \widetilde{\mathcal{H}}(U)$ such that $\liminf_{x \longmapsto z} u(x) \geq \limsup_{x \longmapsto z} v(x)$ for every $z \in \partial U$ and if both sides of the inequality are not simultaneously $+\infty$ and $-\infty$, then $u \geq v$ on U.
In the sequel, we fix $\alpha > 0$, φ from $X \times \mathbb{R}$ to $\mathbb{R}$ defined by $\varphi(x,t) = |t|^{2\alpha} - 1$ and the harmonic space $(X, \mathcal{H})$.
It is easy to see (e.g. by the investigation of the classical Ginzburg–Landau equation over [2]) that $(X, \mathcal{H})$ and $(X, \mathcal{G})$ do not have the same MP–sets. We have the following useful results.

Proposition 2.3. *Let* U *be an* MP*–set in* $(X, \mathcal{G})$, $v \in {}_{*}\mathcal{H}(U)$ *and* $u \in {}^{*}\mathcal{H}(U) \cap B_b(U)$. $\liminf_{x \longmapsto x} u(x) \geq \limsup_{x \longmapsto z} v(x)$ *for every* $z \in \partial U$ *and* $v(1 - |v|^{2\alpha}) \leq u(1 - |u|^{2\alpha})$ *on* U*, then* $u \geq v$ *on* U.

Proof. Let $V \subset \overline{V} \subset U$, V regular in $(X, \mathcal{H})$, hence V is regular in $(X, \mathcal{G})$. Let $u_1 = u + (u(|u|^{2\alpha} - 1)) \bullet M_V$ and $v_1 = v + (v(|v|^{2\alpha} - 1)) \bullet M_V$, from the above characterization of $^{*}\mathcal{H}$ and $_{*}\mathcal{H}$ 2.1 , we get $u_1 \in {}^{*}\mathcal{G}(U)$ and $v_1 \in {}_{*}\mathcal{G}(U)$. Since $u - v = u_1 - v_1 + (u(1 - |u|^{2\alpha}) - v(1 - |v|^{2\alpha})) \bullet M_V$ we have $u - v \in {}^{*}\mathcal{G}(U)$, since U is an MP–set in $(X, \mathcal{G})$ and $\liminf_{x \longmapsto z} u(x) \geq \limsup_{x \longmapsto z} v(x)$ for every $z \in \partial U$, we thus obtain $u \geq v$ on U. □

The following corollary is important for the proof of the Hervé's inequalities for the (generalized) Ginzburg–Landau equation in real and complex case.

Corollary 2.4. *Let U be an MP–set in $(X,\mathcal{G})$, $u \in^* \mathcal{H}(U)$, $v \in_* \mathcal{H}(U)\mathcal{C}_b(U)$ and u, v continuous on U such that*

$$\liminf_{x\longmapsto z} u(x) \geq \limsup_{x\longmapsto z} v(x) \quad \text{for all} \quad z \in \partial U, \ u \geq 1 \quad \text{on} \quad U$$

then $u \geq v$ on U.

Proof. Let $\Omega = \{x \in U : u(x) < v(x)\}$, Ω is then open and since $\Omega \subset U$, Ω is even an MP–set in $(X,\mathcal{G})$ (see [CC]). Moreover, we have $\liminf_{x\longmapsto z}(u(x) - v(x)) \geq 0$ for every $z \in \partial\Omega$, an easy calculation gives $v(1 - |v|^{2\alpha}) \leq u(1 - |u|^{2\alpha})$ on Ω and the statement follows from proposition 2.3. □

Corollary 2.5. *Assume $1 \in^* \mathcal{H}$, then for every $\mathcal{H}$-regular set U in X and $f \in (\partial U)$ we have $|H_V f| \leq \sup(\|f\|_\infty, 1)$, where $\|f\|_\infty = \sup\{|f(x)|, x \in \partial U\}$.*

3 Generalized Ginzburg–Landau equation on $\mathbb{R}^d$, $d \geq 1$, the real case.

We consider on $\mathbb{R}^d$, $d \geq 2$, a partial differential operator L in the following form:

$$Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i}(x).$$

We assume that L satisfies the following conditions:

1) (Strong ellipticity). There exists constant $\gamma > 0$ such that $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^d \xi_i^2$ for every $x \in \mathbb{R}^d$.

2) (Boundedness and uniform Hölder continuity). a_{ij}, b_j are bounded on $\mathbb{R}^d$, (a_{ij}) is symmetric and there exist $A > 0$, $s \in]0,1[$ such that

$$\sum_{i,j=1}^{d} |a_{ij}(x) - a_{ij}(y)| + \sum_{i=1}^{d} |b_i(x) - b_i(y)| \leq A|x-y|^s.$$

For every open set U in X, we set

$$\mathcal{H}_L(U) = \{u \in \mathcal{C}^2(U) : Lu = 0\}.$$

It is well known (e.g. by [RMH]) that $(\mathbb{R}^d, \mathcal{H}_L)$ is a Brelot space and the L–regular sets are the same as for Laplacian. By [HS84], every $C^{1,1}$ domain V on

$\mathbb{R}^d$ admits a Green function ${}^LG^V$ which is comparable to the Green function G^V of the Laplacian.

Let $\alpha > 0$. We define the (generalized) Ginzburg–Landau operator L_1 by $L_1 u = Lu + u(1-|u|^{2\alpha})$.

For every open set U, we set

$$\mathcal{H}(U) = \left\{ u \in (U) : \left(u + \int^L G_t^V u(t) \left(|u(t)|^{2\alpha - 1} \right) dt \right) \in \mathcal{H}_L(V) \text{ for } V \subset \overline{V} \subset U \right\}$$

where ${}^LG^V$ is the Green function for L on V.

We denote by $M(x)$ the biggest eigenvalue of the symmetric real Matrix $(a_{ij}(x))$, by $B(x) = \sum_{i=1}^d |b_i(x)|$. By the hypothesis 2) on L, $M(\cdot)$ and $B(\cdot)$ are bounded, we set M and B respectively their lower bound.

Proposition 3.1. *Let $x_0 \in \mathbb{R}^d$, $R > 0$ and $B = B(x_0, R)$. Then there exists on B, $v \in^\infty (B)$ positive such that $L_1 v \leq 0$ and $\lim_{x \longmapsto z} v(x) = +\infty$ for every $z \in \partial B(x_0, R)$.*

Proof. We consider $v_\lambda(x) = \frac{\lambda}{(R^2 - r^2(x))^{1/\alpha}}$, $r(x) = \|x - x_0\|^2 < R^2$, an easy calculation shows that for $\lambda \geq \lambda_0(R) = [4C_1 M R^2 + C_2 B R^3 + R^4]^{1/2\alpha}$ with $C_1 = \frac{1}{\alpha}\left(\frac{1}{\alpha} + 1\right)$, $C_2 = \frac{2}{\alpha}$, v_λ satisfies the required statement of the proposition. □

In the following we set $v := \frac{\lambda_0(R)}{(R^2 - r^2)^{1/\alpha}}$

Remark 3.2. *We have $v(x) \geq \frac{\lambda_0(R)}{(R^2)^{1/\alpha}} \geq \frac{(R^4)^{1/2\alpha}}{(R^2)^{1/\alpha}} = 1$ for every $x \in B(x_0, R)$.*

Proposition 3.3. *$v \in^* \mathcal{H}(B)$.*

Proof. Same as in [B2], Theorem 4.5. □

Theorem 3.4 (Hervé's Inequality). *Let $x_0 \in \mathbb{R}^d$, $R > 0$, $B(x_0, R)$, $K_1 = 4(\frac{1}{\alpha}(\frac{1}{\alpha} + 1))M$ and $K_2 = \frac{2}{\alpha}B$. Then $|u(x_0)| \leq \left[1 + \frac{K_1}{R^2} + \frac{K_2}{R}\right]^{1/2\alpha}$ for every $u \in \mathcal{H}(B)$.*

Proof. Let $0 < s < R$ and $u \in \mathcal{H}(B)$ then $u \in \mathcal{H}(B(x_0, s))$ and u is bounded in $B(x_0, s)$. Let $g \in \mathcal{C}^2(B(x_0, s))$ given by $g(x) = \frac{\lambda_0(s)}{(s^2 - r^2)^{1/\alpha}}$, then $g \in {}^*\mathcal{H}(B)$ and $\lim_{x \longmapsto z} g(x) = +\infty \geq \limsup_{x \longmapsto z} u(x) = u(z)$ for every $z \in \partial B(x_0, s)$. Remark 3.2 yields $g \geq 1$ and then corollary 2.4 implies $u \leq g$. We therefore have $u(x_0) \leq g(x_0) = \frac{\lambda_0(s)}{(s)^{1/2\alpha}}$. Since $-u$ is again in $\mathcal{H}(B)$, we also have $-u(x_0) \leq \frac{\lambda_0(s)}{(s)^{1/2\alpha}}$. Thus $|u(x_0)| \leq \left[1 + \frac{K_1}{s^2} + \frac{K_2}{s}\right]^{1/2\alpha}$ for every $s \in [0, R[$ and $|u(x_0)| \leq \left[1 + \frac{K_1}{R^2} + \frac{K_2}{R}\right]^{1/2\alpha}$. In the sequel we set $\sigma(R) = \left[1 + \frac{K_1}{R^2} + \frac{K_2}{R}\right]^{1/2\alpha}$.

□

Corollary 3.5. *For every $u \in \mathcal{H}(\mathbb{R}^d)$ we have $|u(x)| \leq 1$ for every $x \in \mathbb{R}^d$.*

Corollary 3.6 (Generalized Hervé-Harnack inequality, see [B1]). *For every open set U and every compact subset K of Ω there exists $C > 0$ such that $|u(x)| \leq C$ for every $x \in K$ and $u \in \mathcal{H}(U)$.*

Proof. We have $|u(x)| \leq \sigma(d(K, \complement U))$, where $d(K, \complement U) = \inf\{\|x - y\|, x \in K, y \in \complement U\}$. $\square$

Corollary 3.7. *For every open set U, $\mathcal{H}(U)$ is compact for the local uniform convergence.*

Proof. Let $(u_n)_n \subset \mathcal{H}(U)$ and $V \subset \overline{V} \subset U$, (u_n) is bounded in V and $g_n = u_n + \int G_t^V u_n(t)(|u_n(t)|^{2\alpha} - 1)dt$ is L–harmonic on V and bounded on $\overline{V}$ since $(\mathbb{R}^d, \mathcal{H}_L)$ is a harmonic space, by [CC, Theorem 11.1.1], (g_n) has a convergent subsequence, without loss of generality we assume that (g_n) converges to g locally uniformly on V. Since (u_n) bounded, by [H1], the set $\left\{\int G_t^V u(t)\,(|u_n(t)|^{2\alpha} - 1)\,dt, n \in \mathbb{N}\right\}$ is equicontinuous and therefore has a locally uniformly convergent subsequence, so (u_n), by the Lebesgue convergence theorem, we get $g = u + \int G_t^V u(t)(|u(t)|^{2\alpha} - 1)dt$. Since $g \in \mathcal{H}_L(V)$, we hence obtain $u \in \mathcal{H}(V)$. Choosing an exhaustion of U by relatively compact open sets and a diagonal procedure, we obtain the desired result. $\square$

Applications 3.8. *Let $L = \Delta$, $\alpha = 1$ and $d = 2$. We obtain here the real solutions of the classical Ginzburg–Landau equation: $\Delta u = u(|u|^2 - 2)$. In [HH96], M. Hervé and R.M.Hervé obtained in ^{2} for complex valued solutions u on $B(x_0, R)$ the following inequality*

$$|u(x_0)|^2 \leq \frac{1}{2} + \sup\left(\frac{12}{R^2}, \sqrt{\frac{1}{4} + \frac{48}{R^4}}\right) = \sigma_0^2(R).$$

We have $\sigma(R) \leq \sigma_0(R)$ if and only if $R \leq 2\sqrt{2}$, and for real solutions $\sigma_0(R)$ is not the best majorizing constant in the Hervés Inequality . We set $\widetilde{\sigma}(R) = \sigma(R)$ for $R \leq 2\sqrt{2}$, $\widetilde{\sigma}(R) = \left(\frac{1}{2} + \frac{12}{R^2}\right)^{1/2}$ for $2\sqrt{2} \leq R \leq (2^7 3)^{1/4}$ and $\sigma(R) = \left(\frac{1}{2} + \left(\frac{1}{4} + \frac{48}{R^4}\right)^{1/2}\right)^{1/2}$ for $R \geq (2^7 3)^{1/4}$. We therefore have $|u(x_0)| \leq \widetilde{\sigma}(R)$ for every real continuous solution of $\Delta u = u(|u|^2 - 1)$ on $B(x_0, R)$ in the distributional sense. However, for complex valued solutions of the Ginzburg–Landau in ^{2}, we will see that the $\sigma_0(R)$ obtained by Hervé is until now the best bound.

4 Generalized Ginzburg–Landau equation in $\mathbb{R}^d$,$d \geq 2$. The complex case

For every $A \subset \mathbb{R}^d$, we shall add $\mathbb{C}$ for functions from A to $\mathbb{C}$, e.g. $\mathcal{C}(A, \mathbb{C})$, $\mathcal{B}(A, \mathbb{C})$, $\mathcal{H}(A, \mathbb{C})$. We will consider the same operator L as in the previous section with the following additional condition. The coefficients of L are sufficiently smooth so that L admits an adjoint L^* in the distributional sense and the solutions in the distributional sense on an open set U are $\mathcal{C}^2(U)$. We recall that the L-regularity of the boundary points of an open set U is (e.g. by [RMH]) the same as the classical regularity for the Laplacian on $\mathbb{R}^d$. We shall say regular instead of L-regular.

We are interested among others in the following Dirichlet problem. Let $f \in \mathcal{C}(\partial U, \mathbb{C})$ be a continuous complex valued function at the boundary of U. We look for a continuous complex valued solution $u \in (U,)$ of the following system

$$(*) \begin{cases} Lu + u(1 - |u|^{2\alpha}) &= 0 \quad \text{on U in the distributional sense,} \\ \lim_{x \longmapsto y} u(x) &= f(y) \quad \text{for every y regular in } \partial U \end{cases}$$

where $|u|^2 = (Re\, u)^2 + (Im\, u)^2$.

In what follows, we will prove in contrast to many other proofs and without any assumption on the regularity of the boundary, but unfortunately for "small" open regular subset, that the problem $(*)$ has a solution. The unicity will be treated in the following way: For every $K > 0$, there exist $\delta(K)$ such that for every relatively compact open set U which diameter smaller than $\delta(K)$ and every $f \in \mathcal{C}(\partial U, \mathbb{C})$ with $\|f\|_\infty \le K$, there exists on U a unique solution of $(*)$.

Let $\mathcal{V} = \{$regular open sets U such that $\sup_{x \in \mathcal{U}} \int {}^L G^U_t(x) \lambda(dt) < 1\}$. Where ${}^L G^U$ is the Green function for L and U. It is then easy to see that $\mathcal{V}$ is a basis of regular open sets in X. Let $V \in \mathcal{V}$ and $f \in \mathcal{C}(\partial V, \mathbb{C})$. Then $f = f_1 + i f_2$ with $f_i \in \mathcal{C}(\partial V)$. We set $H_V f := {}^L H_V f_1 + i {}^L H_V f_2$ where ${}^L H_V f_i$ are the solution of the Dirichlet problem associated V and f_i for $i \in \{1, 2\}$. Let $K > 0$ such that $\|f\| + K \sup_{x \in V} \int G^V_t(x) dt \le K$, where $\|f\| = \|f_1\|_\infty + \|f_2\|_\infty$. Let $E = \{v \in \mathcal{C}_b(U, \mathbb{C}), \|v\| \le K\}$, $\|v\| = \|v_1\|_\infty + \|v_2 1\|_\infty$ whenever $v = v_1 + i v_2$, $\|v_i\|_\infty = \sup\{|v_i(x)|, x \in U\}$. For $v \in E$, we denote by $T(v)$ the following function:

$$T(v) := (I + K_v)^{-1}(H_V f + \int G^V_t v(t) d\lambda(t)),$$

where $K_v f = \int G^V_t |v(t)|^{2\alpha} f(t) dt$, $|v|^2 = (Rev)^2 + (Imv)^2$. It is well known that $(I + K_v)$ invertible (see among others [Me68], [BHH]). Moreover, we have: $\|T(v)\| \le \|f\| + \|v\| \int G^V_t d\lambda(t)$ which yields $\|T(v)\| \le \|f\| + K \int G^U_t d\lambda(t) \le K$. We hence obtain that $T(E) \subset E$. A fix point of T gives a solution of $(*)$ on V.

Proposition 4.1. *$T(E)$ is a compact subspace of E.*

Proof. We use an idea similar to [H2]. Let U be a relatively compact subset of $\mathbb{R}^d$ with $V \subset \overline{V} \subset U$ and $v_n \in E$. We set for $g \in B(V)$: $\widetilde{g} = g$ on V and $\widetilde{g} = 0$ on $U \setminus V$,

$$h_n := \int G^U_t \widetilde{Re\, T(v_n)} |\widetilde{v_n}|^{2\alpha}(t) \text{ and } h'_n = \int G^U_t (\widetilde{Re\, v_n})(t) dt$$

hence (h_n) and (h'_n) are relatively compact for the local uniform convergence and since $Re(T(v_n)) = H_V f_1 + \int G^V_t Re(v_n)(t) dt - \int G^V_t Re(T(v_n)) |v_n|^{2\alpha}(t) dt$,

$Re(T(v_n))$ is then compact for the uniform convergence on V. The same proof is valid for $Im(T(v_n))$. □

Proposition 4.2. *T is continuous on E for the uniform convergence.*

Proof. Let $(v_n)_n$ and v in E such that (v_n) converges uniformly on V to v. Since T is compact for the uniform convergence on E, there exists a subsequence $T(v_{\rho(n)})$ which is uniformly convergent to g on V. Since

$$H_U f + \int G^V{}_t v_{\rho(n)}(t) d\lambda(t) = T(v_{\rho(n)}) + \int G_t^V T(v_{\rho(n)}) |v_{\rho(n)}(t)|^{2\alpha} d\lambda(t)$$

we get

$$\begin{aligned} H_U f + \int G_t^V v(t) d\lambda(t) &= g + \int G_t^V g(t)|v(t)|2\alpha d\lambda(t) \\ &= T(v) + \int G_t^V T(v)|v(t)|2\alpha d\lambda(t). \end{aligned}$$

Therefore $g = T(v)$ and thus $T(v_n)$ converges uniformly to $T(v)$ and we have the desired result. □

Theorem 4.3. *There exists $u \in \mathcal{C}_b(V, \mathbb{C})$ satisfying $(*)$.*

Proof. By the fixpoint theorem of Leray-Schauder, T admits on E a fixpoint u. Therefore, u fulfills the desired result.

□

In the sequel we shall discuss the unicity of solution of $(*)$. Let U be an open subset of $\mathbb{R}^d$, $h \in \mathcal{H}(U, \mathbb{C})$ and $v = |h|^2$. The assumptions on L yield by an easy verification the following useful result:

Proposition 4.4. *We have $\frac{1}{2} Lv \geq v(v^\alpha - 1)$ in the distributional sense on U.*

Corollary 4.5. *Let $V \in \mathcal{V}$, $f \in \mathcal{C}(\partial V, \mathbb{C})$ $\|f\|_\infty = \sup\{|f(x)|, x \in \partial V\}$ and h a solution of $(*)$ corresponding to V and f. Then we have $|h| \leq \max(\|f\|_\infty, 1)$. Let $c = \max(\|f\|_\infty, 1)$ then $c \geq 1$ and $\frac{1}{2} Lc = 0 \leq c(c^\alpha - 1)$. Corollary 2.4 yields the desired result.*

The following lemma is very useful for the investigation of the unicity of solutions of the problem $(*)$.

Lemma 4.6. *Let $z, z' \in$ and $\alpha > 0$. Then*

$$\left| z|z|^{2\alpha} - z'|z'|^{2\alpha} \right| \leq |z - z'|(1 + 2\alpha) \max(|z|^{2\alpha}, |z'|^{2\alpha}).$$

Theorem 4.7. *a) let* $\widetilde{\mathcal{V}} := \{V \in \mathcal{V} : \sup_{x\in V} \int G_t^V(x)\lambda(dt) < \frac{1}{2+2\alpha}\}$*, then for every* $V \in \widetilde{\mathcal{V}}$ *and* $f \in \mathcal{C}(\partial V, \mathbb{C})$ *with* $\|f\|_\infty \leq 1$*, there exists a unique solution of* $(*)$ *associated with* V *and* f*.*
b) For every $K > 1$ *let* $\mathcal{V}_K := \{V \in \mathcal{V} : \sup_{x\in V} \int G_t^V(x)\lambda(dt) < \frac{1}{1+(1+2\alpha)K^{2\alpha}}\}$ *, then for every* $V \in \mathcal{V}_K$ *and every* $f \in \mathcal{C}(\partial V, \mathbb{C})$ *with* $\|f\|_\infty \leq K$*, there exists a unique solution of* $(*)$ *associated with* f *and* V*.*

Proof. Let $V \in \mathcal{V}$, $f \in \mathcal{C}(\partial V, \mathbb{C})$ u and u' be solutions of $(*)$ associated with V and f. Let $\beta = \sup_{x\in V} \int G_t^V(x)\lambda(dt)$, the definition of the solutions u and u' gives that

$$u - u' = (I - \int G_t^V)^{-1}(\int G_t^V(u'(t)|u'(t)|^{2\alpha} - u(t)|u(t)|^{2\alpha})\lambda(dt)).$$

By the lemma 4.6 and corollary 4.5 we have

$$|u - u'| \leq (I - \int G_t^V)^{-1}(\int G_t^V|(u-u')(t)|(1+2\alpha)(\max(\|f\|_\infty, 1)^{2\alpha})),$$

hence

$$\|u - u'\|_\infty \leq (1+2\alpha)(\max(\|f\|_\infty, 1))^{2\alpha}\|u-u'\|_\infty \frac{\beta}{1-\beta}.$$

If $(1+2\alpha)(\max(\|f\|_\infty, 1))^{2\alpha}\frac{\beta}{1-\beta} < 1$, we then have the unicity.
An easy calculation yields the desired results. □

Applications 4.8. *Let* $L = \Delta$ *on* $\mathbb{R}^d$*,* $d \geq 2$*,* $\alpha = 1$*, we then obtain the classical Ginzburg–Landau Equation.*

$$Lu = \Delta u - u(|u|^2 - 1) = 0.$$

In what follows, we give a characterization of the Balls B *with radius* R *which belong to the basis* $\mathcal{V}$ *,* $\mathcal{V} - K$ *for* $K > 1$ *, constructed in the previous theorem 4.7*

Theorem 4.9. *Let* $d > 2$*,* $R > 0$ *and* $x_0 \in \mathbb{R}^d$*. We have* $\sup_{x\in B} \int G_t^B(x)dt = \frac{\Gamma(d/2)}{4d\pi^{d/2}}R^2$*.*

Proof. Let G^B such that $\Delta G_t^B = -\varepsilon_t$ on B in the distributional sense , then G^B is the Green function for the Laplace operator on B, we have :

$$I = \int G_t^B(x)dt = \int G^B(x,t)dt = \int_0^R (\int G(x, rz)\sigma_{d-1}(dz))r^{d-1}dr$$

where $(t = rz, \|z\| = 1)$ and σ_{d-1} is the surface measure on the unit sphere on $\mathbb{R}^d$. Hence

$$
\begin{aligned}
I &= \int_0^{\|x-x_0\|}(\int G(x,rz)\sigma_{d-1}(dz))r^{d-1}dr + \int_{\|x-x_0\|}^{R}(\int G(x,rz)\sigma_{d-1}(dz))r^{d-1}dr \\
&= c_d\int_0^{\|x-x_0\|}(\frac{1}{\|x-x_0\|^{d-2}} - \frac{1}{R^{d-2}})r^{d-1}dr + c_d\int_{\|x-x_0\|}^{R}(\frac{1}{r^d} - \frac{1}{R^{d-2}})r^{d-1}dr \\
&= c_d(\frac{1}{\|x-x_0\|^{d-2}} - \frac{1}{R^{d-2}})\frac{\|x-x_0\|^d}{d} + c_d\left[\frac{r^2}{2} - \frac{1}{d\,R^{d-2}}r^d\right]_{\|x-x_0\|}^{R} \\
&= \frac{d-2}{2d}c_d\left[R^2 - \|x-x_0\|^2\right].
\end{aligned}
$$

Since $c_d = \frac{\Gamma(d/2)}{2(d-2)\pi^{d/2}}$ we get $I = \frac{\Gamma(d/2)}{4d\pi^{d/2}}\left[R^2 - \|x-x_0\|\right]$. Thus $\sup_{x\in B}\int G_t^B(x)dt = \frac{\Gamma(d/2)}{4d\pi^{d/2}}R^2$. ☐

Let $\widetilde{\mathcal{V}}$, $\mathcal{V}_K$ be the base determined in theorem 4.7 and a ball B in $\mathbb{R}^d$ with $R>0$.We have the following results:

Corollary 4.10. *a) $B\in\mathcal{V}$ if and only if $R < \left(\frac{4d\pi^{d/2}}{\Gamma(d/2)}\right)^{1/2}$*

b) $B\in\widetilde{\mathcal{V}}$ if and only if $R < \left(\frac{d\pi^{d/2}}{\Gamma(d/2)}\right)^{1/2}$.

c) For $K>1$, $B\in\mathcal{V}_K$ if and only if $R < \left(\frac{4d\pi^{d/2}}{\Gamma(d/2)(1+3K^2)}\right)^{1/2}$.

Theorem 4.11. *Let $d=2$, $R>0$ and $x_0\in^2$ and B a ball with radius R and center x_0 . We have $\sup_{x\in B}\int G_t^B(x)dt = \frac{1}{8\pi}R^2$.*

Proof. Let G^B be the Green function for the Laplace operator on B such that $\Delta G^B(\cdot,t) = -\varepsilon_t$ in the distributional sense. Let $I=\int G_t^B dt$ then

$$
\begin{aligned}
I &= \int_0^R\left(\int G(x,rz)\sigma_2(dz)\right)r\,dr \\
&= \int_0^{\|x-x_0\|}\frac{1}{2\pi}\left(\log\frac{R}{\|x-x_0\|}\right)r\,dr + \int_{\|x-x_0\|}^{R}\frac{1}{2\pi}\left(\log\frac{R}{r}\right)r\,dr
\end{aligned}
$$

and an elementary calculus yields $I = \frac{1}{8\pi}[R^2 - \|x-x_0\|^2]$ and the desired result. ☐

Corollary 4.12. *Let $d=2$, $R>0$ and a ball B with radius R , then*

a) $B\in\mathcal{V}$ if and only if $R < 2(2\pi)^{1/2}$.

b) $B \in \widetilde{\mathcal{V}}$ *if and only if* $R < (2\pi)^{1/2}$.

c) *For* $K > 1$: $B \in \mathcal{V}_K$ *if and only if* $R < \left(\frac{8\pi}{1+3K^2}\right)^{1/2}$.

Proposition 4.13. *Let U be an open regular set (for the Lapace equation) in $\mathbb{R}^d$ and $f \in \mathcal{C}(\partial U)$ real. Then there exists a real solution u of the problem $(*)$ associated with U and f.*

Proof. Let $K = \|f\|_\infty$. Without loss of generality we assume that $K \geq 1$. It is easy to show that there exists a compact set $C \subset U$ such that $U \setminus C$ regular and $U \setminus C \in \mathcal{V}_K$, furthermore there exists a recoverment of C by a finite family of regular sets $\mathcal{V}_1, \ldots, V_p$ such that $C \subset U_{i=1}^p V_i \subset \mathcal{V}$. Let $V_0 = U \setminus C$ and $(U_n) = (V_0, V_0, V_1, V_0, V_1, V_2, V_0, V_1, V_3,$
$V_0, V_1, V_3, \ldots)$. The proof is then the same as in [BHH] which uses only the minimum principle for V_i and the sheaf properties valid in $(X, \mathcal{H})$ by the first section, here $\mathcal{H}$ is the sheaf of real solutions of $(*)$.

□

5 Hervé's Inequality for complex solutions of the generalized Ginzburg–Landau equation on $\mathbb{R}^d (d \geq 2)$.

We consider here an operator L over $\mathbb{R}^d (d \geq 2)$ with the same assumptions as in the previous section. We will keep here the same notations as before.

Theorem 5.1 (Hervé's inequality). *Let* $R > 0$, $x_0 \in \mathbb{R}^d$, $K_1 = \frac{4}{\alpha}\left(\frac{2}{\alpha} + 1\right) M$, $K_2 = \frac{2}{\alpha} B$. *Then* $|u(x_0)| \leq \left[1 + \frac{K_1}{R^2} + \frac{K_2}{R}\right]^{1/2\alpha}$ *for every* $u \in \mathcal{H}(B(x_0, R), \mathbb{C})$.

Proof. Let $g = |u|^2 = (Re\, u^2 + Im\, u^2)$, by proposition 4.4 we then have $\frac{1}{2} Lg \geq g(g^\alpha - 1)$ in the distributional sense. For every open set U we set
$\widetilde{\mathcal{H}}(U) = \{h \in \mathcal{C}(U) \quad h + \int \frac{1}{2}{}^L G_t\, h(t)(|h(t)|^\alpha - 1) dt \in \mathcal{H}_{\frac{1}{2}L}(V)$ for every open set $V \subset \overline{V} \subset U\}$. Since $\frac{1}{2}L$ satisfies the same assumptions as L, $(X, \widetilde{\mathcal{H}})$ is a Bauer space and by section 1, $g \in {}_*\widetilde{\mathcal{H}}(B(x_0, R))$. Let $s < R$ and $v = \frac{\lambda_0\left(\frac{\alpha}{2}, s\right)}{(s^2 - r^2)^{1/\alpha}}$, $r = \|x - x_0\| < s$, $\lambda_0(\frac{\alpha}{2}, s) = [2C_1 M R^2 + \frac{1}{2} C_2 B R^3 + R^4]^{1/\alpha}$, $C_1 = \frac{2}{\alpha}(\frac{2}{\alpha} + 1)$, $C_2 = \frac{4}{\alpha}$. The same proof as in proposition 3.1 yields $v \in^* \widetilde{\mathcal{H}}(B(x_0, r))$. By the minimum principle in corollary 2.4 we get

$$\begin{aligned} g &\leq v \text{ on } B(x_0, s) \text{ and hence} \\ g(x_0) &\leq [1 + \frac{K_1}{s^2} + \frac{K_2}{s}]^{1/\alpha}. \text{ Since } g(x_0) = |u(x_0)|^2, \text{ we get} \\ |u(x_0)| &\leq [1 + \frac{K_1}{s^2} + \frac{K_2}{s}]^{\frac{1}{2\alpha}} \text{ for every } s < R, \text{ this yields the desired inequality.} \end{aligned}$$

□

In what follows we set $\widetilde{\sigma}(r) := \left[1 + \frac{K_1}{R^2} + \frac{K_2}{R}\right]^{\frac{1}{2\alpha}}$

Corollary 5.2. *Classical Ginzburg–Landau equation. Let $L = \Delta$, $\alpha = 1$, we then have for every $R > 0$, $x_0 \in \mathbb{R}^d$ and $u \in \mathcal{H}(B(x_0, R),)$ $|u(x_0)| \leq [1 + \frac{12}{R^2}]^{1/2}$.*

Proof. We have $M = 1$, $B = 0$ and then $K_2 = 0$ and $K_1 = 4(2 + 1) = 12$. The previous theorem yields $|u(x_0)| \leq [1 + \frac{12}{R^2}]^{1/2}$ for every $u \in \mathcal{H}(B(x_0, R), \mathbb{C})$.

□

Remark 5.3. *a) We have $\sigma_0(R) \leq (1+\frac{12}{R^2})^{1/2}$ where $\sigma_0(R) = \frac{1}{2}+\sup(\frac{12}{R^2}, \sqrt{\frac{1}{4} + \frac{48}{R^4}})$ is the bound obtained for $d = 2$ by Hervé in [HH96] by other methods.*
b) By the same proofs as in corollaries 3.5, 3.6 and 3.7 we obtain the following results for continuous complex solutions of the generalized Ginzburg–Landau equation $Lu = u(|u|^{2\alpha} - 1)$ in the distributional sense, indeed we have:

i) For every $u \in \mathcal{H}(\mathbb{R}^d, \mathbb{C})$ $|u(x)| \leq 1$ for every $x \in \mathbb{R}^d$.

ii) For every open set U in $\mathbb{R}^d$ and every compact subset $K \subset U$ there exists $C > 0$ such that $|u(x)| \leq C$ for every $x \in K$. C is every constant $\geq \widetilde{\sigma}(d(K, \complement U))$.

iii For every open set U, $(U,)$ is compact for the local uniform convergence.

6 Hervé–Harnack Inequality for complex valued solutions of the generalized Ginzburg–Landau Equation.

Let L be a differential operator with the same form and assumptions as in section 4. Let $\alpha > 0$ and for every open set U in $\mathbb{R}^d$ we set:

$$\mathcal{H}_1(U, \mathbb{C}) = \{u \in \mathcal{C}(U, \mathbb{C}) : Lu = u(|u|^{2\alpha} - 1) \text{ in } D.S \text{ on } U \text{ with } |u| \leq 1\}.$$

The aim of this section is the proof of the following result:
Let U be a domain in $\mathbb{R}^d$, $(u_n)_n \subset \mathcal{H}_1(U, \mathbb{C})$ and $a \in \mathbb{C}$ with $|a| = 1$. Then the following properties are equivalent:

1) u_n converges locally uniformly to the constant function a on U.

2) There exists $x \in U$ such that $(u_n(x))$ converges to a.

For this purpose we shall prove for $d = 2$ a similar inequality as by Hervé in [HH96] and for $d \geq 3$ an other inequality which yields the desired convergence results.

Proposition 6.1. *Let $x \in \mathbb{R}^d$, $R > 0$ and $B_R = B(x,R)$, then there exists $K > 0$ such that for every $r < \inf(R,1)$ we have*

$$\int_{S(x,r)} (1-|u(y)|)\sigma(dy) \leq K(1-|u(x)|) \text{ for every } u \in \mathcal{H}_1(B,\mathbb{C})$$

where $S(x,r) = \{y \in \mathbb{R}^d \cdot \|x-y\| = r\}$, $\sigma = \frac{\sigma^r_{d-1}}{\sigma^r_{d-1}(1)}$ and σ^r_{d-1} is the surface measure on the sphere $S(x,r)$.

Proof. Let $u \in \mathcal{H}_1(B,\mathbb{C})$ and $v = |u|^2$, we have by proposition 4.4, $\frac{1}{2}Lv - v^{\alpha+1} + v \geq 0$. Since $L1 = 0$, we get $\frac{1}{2}L(1-v) \leq v - v^{\alpha+1} \leq \alpha(1-v)$. Hence $1-v$ is a superharmonic function on B for the linear harmonic structure given by $\mathcal{H}_{L-2\alpha}$ where for every open set U in X, $\mathcal{H}_{L-2\alpha}(U) = \{u \in \mathcal{C}^2(U) : Lu - 2\alpha u = 0\}$. It is well known (see e.g. [Se], [A] or [HS83]) that there exists $C \geq 1$ such that for every $x \in \mathbb{R}^d$, $\rho \in]0,1[$ and $f \in \mathcal{C}(\partial B(x,\rho))$ we have $\frac{1}{C}{}^{\Delta}H_{B_\rho}f \leq {}^{\alpha}H_{B_\rho}f \leq C^{\Delta}H_{B_\rho}f$, where ${}^{\Delta}H_{B_\rho}$ and ${}^{\alpha}H_{B_\rho}$ are respectively the harmonic kernels associated with B_ρ, Δ and $L-2\alpha$. For $\rho = r$ and since $1-v$ is superharmonic on $B(x,R)$ we have

$${}^{\Delta}H_{B_r}(1-v)(x) = \int_{S(x,r)} (1-v(y))\sigma(dy) \leq C\ {}^{\alpha}H_{B_r}(1-v)(x) \leq C(1-v)(x)$$

for every r with $r < \inf(R,1)$. Since

$$\begin{aligned} 1-|u(y)| \leq (1-|u(y)|^2) &\leq 2(1-|u(y)|). \\ \int_{S(x,r)} (1-|u(y)|)\sigma(dy) &\leq 2C(1-|u(x)|), \end{aligned}$$

$K = 2C$ yields then the desired inequality.

□

In the following we assume $d = 2$.

Theorem 6.2. *Let $x \in^2$, $R > 0$ and $B = B(x,R)$, then for every $r \in]0,\inf(R,1)[$, there exists $K_r = K(r)$ such that $(1-|u(y)|) \leq K_r[1-|u(x)|]$ for every $y \in B(x,r)$ and every $u \in \mathcal{H}_1(B,\mathbb{C})$.*

Proof. Let $x \in \mathbb{R}^2$ and $u \in \mathcal{H}_1(B,\mathbb{C})$ with $u(x) \in \mathbb{R}_+$. Let $v = |u|^2$. By an easy calculation we have $(1-v^\alpha) \leq \sup(1,\alpha)(1-v)$. Let $r \in]0,\inf(R,1)[$, $r_0 = \frac{r+\inf(R,1)}{2}$, $B_0 = B(x,r_0)$ and ${}^{L}G^{B_0}$ be the Green function for L and B_0, by [HS83], there exists $C > 1$ such that $\frac{1}{C}{}^{\Delta}G \leq {}^{L}G^{B_0} \leq C^{\Delta}G$.

Let $h := u - \int {}^{L}G^{B_0}(\cdot,y)u(y)(1-|u(y)|^{2\alpha})dy$, then h, $Re\, h$, $Im h$ are L–harmonic and by the maximum principle their modulus is smaller than 1. Let

$$(I) \qquad I := I(y) =| \int {}^{L}G^{B_0}(y,z)u(z)[1-|u(z)|^{2\alpha}]dz.$$

We set $G = {}^{\Delta}G^{B_0}$ The comparison of the green functions on B_0 yields $I \leq C\sup(1,\alpha)\int_{B_0} G(y,z)(1-v(z))dz = C_1\int_0^{r_0}(\int_{S(x,t)} G(y,z)(1-v(z))\sigma(dz)t\,dt)$ where $C_1 = C\sup(1,\alpha)$. Hence

$$I \leq C_1 \int_0^{r_0} \sup_{z\in S(x,t)} G(y,z)(\int_{S(x,t)}(1-v(z))\sigma(dz)t\,dt).$$

By the previous proposition we obtain:

$$I \leq C_1(\int_0^{r_0} \sup_{z\in S(x,t)} G(y,z)t\,dt)K(1-v(x)).$$

Since $d = 2$, an easy calculation yields

$$\sup_{z\in B_0}(\int_0^{r_0} \sup_{z\in S(x,t)} G(y,z)t\,dt) = C_2 < +\infty.$$

Therefore, $I \leq C_1C_2K\,[\,1-|v(x)|\,] \leq 2C_1C_2K\,[\,1-|u(x)|\,]$. On the other hand since $Re\,h$ is L_1–harmonic on $B(x_0,r_0)$ and $r < r_0$, by the Harnack inequality for $(\mathbb{R}^2,\mathcal{H}_L)$ there exists $C_3 > 0$ such that $(1-Re\,h(y)) \leq C_3(1-Re\,h(x))$ for every $y \in B(x,r)$. By (I) we have:

$$\begin{aligned}(1-|u(y)|) &\leq (1-|h(y)|)+I(y)\\ &\leq (1-Re\,h(y))+I(y)\\ &\leq C_3(1-Re\,h(x))+I(y).\end{aligned}$$

Again, from (I) we have

$$\begin{aligned}1-Re\,h(x) &\leq 1-Re\,u(x)+I(x),\ \text{hence}\\ (1-|u(y)|) &\leq C_3(1-Re\,u(x))+C_3I(x)+I(z)\\ &\leq [\,C_3+2C_1C_2K(1+C_2)\,](1-Re\,u(x)).\end{aligned}$$

Since $u(x) \in \mathbb{R}_+$ we have $Reu(x) = |u(x)|$ and therefore $(1-|u(y)|) \leq K_r(1-|u(x)|)$ with $K_r = C_3+2C_1C_2K(1+C_2)$. Let $u \in \mathcal{H}_1(B,\mathbb{C})$ and $u(x) \neq 0$ we set $v(y) = \frac{|u(x)|}{u(x)}u(y)$. Then $v \in \mathcal{H}_1(B,\mathbb{C})$ with $v(x) = |u(x)| \in \mathbb{R}_+$ then $(1-|v(y)|) \leq K_r(1-|v(x)|)$ which yields the required statement since $|v| = |u|$. □

Remark 6.3. *The previous proof is not valid for higher dimension $d \geq 3$ since $\sup_{y\in B_0}\int_0^{r_0}\sup_{z\in S(x,t)} {}^{\Delta}G^{B_0}(y,z)t\,dt = +\infty$.*

Corollary 6.4. *Let $x_0 \in \mathbb{R}^d$ and $R > 0$ then for every $0 < R' < R$ there exists $C = C(R',R)$ such that*

$(1-|u(x)|) \leq C[\,1-|u(x_0)|\,]$ *for every $x \in B(x_0,R')$ and every $u \in \mathcal{H}_1(B(x_0,R),\mathbb{C})$.*

Corollary 6.5. *Let U be an open domain and $K \subset U$ a compact subset of U, then there exists $C_K \geq 1$ such that*

$$(1-|u(x)|) \leq C_K[\,1-|u(y)|\,] \text{ for every } x,y \in K \text{ and every } u \in \mathcal{H}_1(U,\mathbb{C}).$$

In the following we consider $d \geq 3$ and $q > \frac{d}{2}$.

Theorem 6.6. *Let $x \in \mathbb{R}^d$, $R > 0$ and $B = B(x,R)$, then for every $r \in]0,\inf(R,1)[$, there exists $K_r = K(r,p,L,\alpha,d)$ such that $(1-|u(y)|) \leq K_r[\,1-|u(x)|\,]^{\frac{1}{q}}$ for every $y \in B(x,r)$ and every $u \in \mathcal{H}_1(B,\mathbb{C})$.*

Proof. Let $u \in \mathcal{H}_1(U,\mathbb{C})$, $r \in]\,0,\inf(R,1)[$, $r_0 = \frac{r+\inf(R,1)}{2}$ and $B_0 = B(x,r_0)$. Then $r_0 < \inf(R,1)$, $u \in \mathcal{H}_1(B_0,\mathbb{C})$ and $h = u - \int {}^LG^{B_0}(\cdot,t)u(t)(1-|u(z)|^{2\alpha})dt$ is in $\mathcal{H}_L(B_0)$. We have $(1-|u(y)|) \leq (1-|h(y)|) + |\int {}^LG^{B_0}(y,t)u(t)(1-|u(t)|^{2\alpha})dt|$. We set $I(y) = |\int {}^LG^{B_0}(y,t)u(t)(1-|u(t)|^{2\alpha})dt|$,it follows $I(y) \leq \int {}^LG^{B_0}(y,t)(1-|u(t)|^{2\alpha})dt$. Let $p > 1$ such that $\frac{1}{p}+\frac{1}{q} = 1$, $q > \frac{d}{2}$ implies $p < \frac{d}{d-2}$, by the Hölder inequality we get
$I(y) \leq (\int({}^LG^{B_0}(y,t))^p dt)^{1/p} \times (\int_{B_0}(1-|u(t)|^{2\alpha})^q dt)^{1/q}$. By (e.g. [HS83]) there exists $C > 1$ such that ${}^LG^{B_0} < C\,{}^{\Delta}G^{B_0}$ and since ${}^{\Delta}G^{B_0}(t,y) \leq \frac{1}{\|t-y\|^{d-2}}$ we have $(\int({}^LG^{B_0}(y,t))^p\,dt) \leq C(\int_{B_0}\frac{1}{\|t-y\|^{p(d-2)}}dt)^{1/p}$.
By an easy verification we get $C_1 := C_1(r,p) := C\sup_{y\in B_0}(\int_{B_0}\frac{1}{\|t-y\|^{p(d-2)}}dt)^{1/p} < +\infty$. Therefore, $I(y) \leq C_1(\int_B(1-|u(t)|^{2\alpha})^q dt)^{1/q}$. Let $v = |u|^2$ since $(1-v^\alpha) \leq \sup(1,\alpha)(1-v)$. It follows from $0 \leq 1-v \leq 1$ and $q > 1$ that $(1-v)^q \leq (1-v)$ and $I(y) \leq C_1\sup(1,\alpha)(\int_{B_0}(1-v(t))dt)^{1/q}$. We have $\int_{B_0}(1-v(t))dt = \int_0^{r_0}(\int_{S(x,s)}(1-v(z))\sigma(dz))s^{d-1}ds$ and by proposition 6.1 we have then $\int_{B_0}(1-v(t))dt \leq K(1-v(x))\int_0^{r_0}s^{d-1}ds \leq \frac{K}{d}(1-v(x))$. Therefore,

$$\begin{aligned} I(y) &\leq C_1\left(\frac{K}{d}\right)^{1/q}(1-v(x))^{1/q} \\ &\leq 2C_1\left(\frac{K}{d}\right)^{1/q}(1-|u(x)|)^{1/q}. \end{aligned}$$

On the other hand since $Re\,h$ is harmonic and smaller than 1, the Harnack inequality in $(\mathbb{R}^d,\mathcal{H}_L)$ yields the existence of γ_r such that $(1-|h(y)|) \leq (1-Re\,h(y)) \leq \gamma_r(1-Re\,h(x))$ for every $y \in B(x_0,r)$. Furthermore, we have $(1-Re\,h(x)) \leq (1-Reu(x)) + I(x)$. Let $C_2 = 2C_1\left(\frac{K}{d}\right)^{1/q}$ and $(1-Reh(x)) \leq (1-Re\,u(x)) + C_2(1-|u(x)|)^{1/q}$. Therefore,

$$\begin{aligned} (1-|u(y)|) &\leq (1-Re\,h(y)) + c_2(1-|u(x)|)^{1/q} \\ &\leq \gamma_r(1-Re\,h(x)) + C_2(1-|u(x)|)^{1/q} \\ &\leq \gamma_r(1-Re\,u(x)) + 2C_2(1-|u(x)|)^{1/q}. \end{aligned}$$

Let $u(x) \neq 0$ and $g(y) = \frac{|u(x)|}{u(x)} u(y)$, then g is even in $\mathcal{H}_1(B_0, \mathbb{C})$ with $|g| = |u|$, hence

$$\begin{aligned} (1 - |g(y)|) &\leq \gamma_r(1 - Reg(x)) + 2C_2(1 - |u(x)|)^{1/q} \\ &\leq \gamma_r(1 - |u(x)|) + 2C_2(1 - |u(x)|)^{1/q}. \end{aligned}$$

Since $0 < \frac{1}{q} \leq 1$ and $(1 - |u(x)|) \leq 1$ we have $(1 - |u(x)|) \leq (1 - |u(x)|)^{1/q}$. The required inequality is then given by $K_r = \gamma_r + 2C_2$. □

Corollary 6.7. *Let $x \in \mathbb{R}^d$, $d \geq 3$ and $R > 0$. Then for every $0 < R' < R$ there exists a constant $C = C(R', R, p, \alpha, L_1, d)$ such that $(1-|u(y)|) \leq C(1-|u(x)|)^{1/q}$ for every $y \in B(x, R')$ and $u \in \mathcal{H}_1(B(x, R), \mathbb{C})$.*

Corollary 6.8 (Hervé–Harnack inequality). *Let U be a domain in $\mathbb{R}^d$, $d \geq 3$ and $K \subset U$ compact, then there exists C_K such that $(1 - |u(x)|) \leq C_K(1 - |u(y)|)^{1/q}$ for every $x, y \in K$ and every $u \in \mathcal{H}_1(U, \mathbb{C})$.*

Corollary 6.9. *Let U be a domain in $\mathbb{R}^d$, $d \geq 2$, $(u_n)_n \subset \mathcal{H}_1(U, \mathbb{C})$ and $\beta \in \mathbb{C}$ with $|\beta| = 1$ then the following properties are equivalent.*

1) $(u_n)_n$ converges locally uniformly to β on U.

2) There exists $x \in U$ such that $u_n(x)$ converges to β.

Proof. We have only to prove $2 \implies 1$. By the Hervé–Harnack inequality, $|u_n|$ converges locally uniformly to 1. On the other hand,by section 5 (u_n) is relatively compact for the local uniform convergence. Let $u_{\rho(n)}$ be a subsequence of (u_n) which is locally uniformly convergent to u, then $|u| = 1$ and hence since $Lu = u(|u|^{2\alpha} - 1)$, we get $Lu = 0$. Therefore $u = \beta$. □

In the sequel we will show that the previous inequalities are also valid if we replace the constant 1 by $m \in]0 + \infty[$ in the generalized Ginzburg–Landau Equation, i.e., $Lu = u(|u|^{2\alpha} - m^{2\alpha})$. We set $\mathcal{H}_m(U, \mathbb{C}) = \{u \in (U,) : Lu = u(|u|^{2\alpha} - m^{2\alpha})$ in the distributional sense with $|u| \leq m\}$

Theorem 6.10. *Let $m \in]0, +\infty[$ and $\alpha > 0$. Let U be a domain in $\mathbb{R}^d$ and K be a compact subset of U, then we have the following:*

1) For $d = 2$ there exists C_K such that $(m - |u(x)|) \leq C_K(m - |u(y)|)$ for every $x, y \in K$ and every $u \in \mathcal{H}_m(U, \mathbb{C})$.

2) Let $d \geq 3$, $q > \frac{d}{2}$, then there exists $C_K > 1$ such that $(m - |u(y)|) \leq C_K(m - |u(x)|)^{1/q}$ for every $x, y \in K$ and every $u \in \mathcal{H}_m(U, \mathbb{C})$.

Proof. Let $u \in \mathcal{H}_m(U, \mathbb{C})$ then $Lu = u(|u|^{2\alpha} - m^{2\alpha})$ with $|u| \leq m$. Let $v = \frac{u}{m}$, then $|v| \leq 1$ and $Lv = \frac{1}{m}Lu = \frac{1}{m}u(|u|^{2\alpha} - m^{2\alpha}) = v \times m^{2\alpha}(|v|^{2\alpha} - 1)$ then v is a solution of $L_1 v = v(|v|^{2\alpha} - 1)$ where $L_1 = \frac{1}{m^{2\alpha}}L$, L_1 satisfies the same conditions as L, hence by the Hervé–Harnack inequalities (corollary 6.5 and 6.8) we have: For $d = 2$, there exists a constant $C_K > 1$ such that

$$\begin{aligned} (1 - |v(y)|) &\leq C_K[\,1 - |v(x)|\,] \text{ and hence} \\ (m - |u(x)|) &\leq C_K[\,m - |u(x)|\,] \text{ for every } x, y \in K. \end{aligned}$$

For $d \geq 3$, there exists $\lambda_K > 1$ such that

$$\begin{aligned} (1 - |v(y)|) &\leq \lambda_K(1 - |v(y)|)^{1/q} \text{ which gives} \\ (m - |u(y)|) &\leq mC_K m^{-1/q}(m - |u(y)|)^{1/q}. \end{aligned}$$

□

Theorem 6.11. *Let $m \in]0, +\infty[$, U be a domain in $\mathbb{R}^d$ and (u_n) be a sequence of continuous complex solutions in the distributional sense of $L_1 u = u(|u|^{2\alpha} - m^{2\alpha})$ with $|u_n| \leq m$ for every $n \in$. Let $\beta \in$ with $|\beta| = m$ then the following properties are equivalent:*

1) (u_n) converges locally uniformly to β on U.

2) There exists $x \in U$ such that $u_n(x)$ converges to β.

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