Finite dimensional realizations for Heath, Jarrow and Morton type forward interest rate models with jumps

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Abstract

The problem of finite dimensional realizations for Heath, Jarrow and Morton type interest rate models is well discussed in the literature for the purely Wiener driven case and for certain jump diffusion cases, e.g. when the coefficients are deterministic (cf. [B-G]). The general jump diffusion case, however, requires a completely different geometry to be solved. Therefore certain techniques of geometric measure theory are applied. The existence of such a finite dimensional realizations can be shown under some assumptions on the Lie algebra generated by the coefficients. Moreover it can be characterized by approximating tangent spaces in a similar fashion as finite dimensional realizations can be characterized by invariant tangential manifolds in the purely Wiener driven case.

Keywords: HJM interest rate model with jumps, finite dimensional realizations, market completeness, equivalent martingale measure, forward curve manifold, point processes, approximating tangent spaces, geometric measure theory

AMS-Classification: 60H10; 60H15; 60G55, 60G57, 60H05, 49Q15, 91B28

1 Introduction

The problem of finite dimensional realizations for Heath, Jarrow and Morton type interest rate models is well discussed in the literature. For example the purely Wiener driven case has been studied in [Bj-S], [B-C] and in a completely general Hilbert space setting in [L1], [Fi] and [Fi-T]. Also certain situations with jumps have been discussed, e.g. in [B-DM-K-R],[B-K-R] and [L2]. The existence of these finite dimensional realizations depends strongly on the Lie algebra generated by drift, diffusion and jump volatility. In the purely Wiener driven case one obtains a result of the form: there exist finite dimensional realizations for the stochastic differential equation of the forward rate process at a certain point if and only if there exists a finite dimensional invariant tangential manifold at the same point. Although one can derive conditions under which finite dimensional realizations exist in the general jump diffusion case, it is not possible to characterize them by tangential manifolds, since one cannot apply the standard methods of differential geometry as in the purely Wiener driven case because of the presence of the jumps. There is no obvious way of defining tangential manifolds at the jump times and thus the theory used in [Bj-S] and [L1] cannot be applied to the general setting with jumps. However, there are certain situations where the same methodology can be applied to derive finite dimensional realizations. This is e.g. the case when the interest rate curve can be described by a stochastic differential equation with deterministic coefficients (cf. [B-G] for details) or by a linear stochastic differential equation as in [L2]. In the general case we use the notion of approximating tangent spaces and methods from geometric measure theory to solve the problem. In such a setting it is then possible to characterize finite dimensional realizations via approximating tangent spaces. We then derive similar results as in the continuous case, that is, conditions on the Lie algebra generated by the coefficients are exhibited which yield the existence of finite dimensional realizations and permit the representation of these by approximating tangent spaces. We discuss this general setting for interest rates characterized by stochastic differential equations with values in $\mathbb{R}^n$ for simplicity. Thus we should actually speak about *lower dimensional realizations* in this case. An extension to the general Hilbert space or even Fréchet space setting is left for future work. For this also a more general geometric measure theory is needed. There are already some results on geometric measure theory on general metric spaces (see for example [AmKi], [Ki]) and we are sure that we can extend our theory to a general Hilbert space setting using these results.

2 Some Results from Geometric Measure Theory

In the following we will provide a brief introduction to approximating tangent spaces which we will use in the next section to characterize finite dimensional realizations.

Definition 2.1 Let X be a metric space and $A \subset X$ with

$$A \subset \bigcup_{j=1}^{\infty} C_j, \quad \text{for } C_j \subset X \text{ with } \operatorname{diam} C_j < \delta, \tag{1}$$

for some $\delta > 0$. Define

$$\mathcal{H}^n_\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} C_j}{2} \right)^n \right\}, \tag{2}$$

where ω_n are certain weights for $n \in \mathbb{N}$.
The *n-dimensional Hausdorff measure* $\mathcal{H}^n$ of A is then defined by

$$\mathcal{H}^n(A) := \lim_{\delta \downarrow 0} \mathcal{H}^n_\delta(A). \tag{3}$$

The weights ω_n in the (2) are given by the volume of the unit ball in $\mathbb{R}^n$ if $n \in \mathbb{N}$ and they are defined as any convenient positive constants otherwise. The infimum in (2) is taken over all countable collections $(C_j)_{j\in\mathbb{N}}$ of subset of X such that (1) holds. The limit in equation (3) always exists (although it may be $+\infty$), because $\mathcal{H}^n_\delta(A)$ is a decreasing function of δ. Thus

$$\mathcal{H}^n(A) = \sup_{\delta>0} \mathcal{H}^n_\delta(A).$$

We have $\mathcal{H}^n_\delta(A) < +\infty$ for all $\delta > 0, m \geq 0,$ if A is a totally bounded subset of X.
One can show (see for example [Sim]) that on $\mathbb{R}^n$ the Hausdorff measure $\mathcal{H}^n$ and the n-dimensional Lebesgue measure which we will always denote by $d\lambda^n$ coincide. We will make use of this property later on, when we want to show Hausdorff measurability of a set M.

Definition 2.2 A set $M \subseteq \mathbb{R}^{n+m}$ is called *countable n-rectifiable* if

$$M \subseteq M_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n),$$

where $\mathcal{H}^n(M_0) = 0$ and $F_j : \mathbb{R}^n \to \mathbb{R}^{n+m}$ is Lipschitz.

Remark 2.3 Countable n-rectifiable sets can also be characterized as follows:
An $\mathcal{H}^n$-measurable set $M \subseteq \mathbb{R}^{n+m}$ is countable n-rectifiable if and only if

$$M \subset \bigcup_{j=0}^{\infty} N_j$$

with $\mathcal{H}^n(N_0) = 0$ and where each set N_j for $j \geq 1$ is an embedded n-dimensional C^1-submanifold of $\mathbb{R}^{n+m}$ (cf. [Sim] Lemma 11.1.). This characterization is derived by approximating the Lipschitz functions F_j by C^1-functions.

If M is $\mathcal{H}^n$-measurable, countable n-rectifiable, then we can also write M as the disjoint union

$$M = \bigcup_{j=0}^{\infty} M_j,$$

where $\mathcal{H}^n(M_0) = 0$, $M_j \subset N_j$ for $j \geq 1$, M_j are $\mathcal{H}^n$-measurable and N_j are embedded n-dimensional C^1-submanifolds of $\mathbb{R}^{n+m}$. We simply define the M_j's inductively by

$$M_j := (M \cap N_j) \setminus \bigcup_{i=0}^{j-1} M_j,$$

for $j \geq 1$, with $M_0 := M \setminus \bigcup_{j=0}^{\infty} N_j$, having $\mathcal{H}^n$-measure zero.

Let $G(n, n+m)$ denoted the space of all n-dimensional non orientated subspaces of $\mathbb{R}^{n+m}$.

Definition 2.4 (i) Let $M \subseteq \mathbb{R}^{n+m}$ be $\mathcal{H}^n$-measurable with $\mathcal{H}^n(M \cap K) < \infty$ for compact K. Hence $\mathcal{H}^n \lfloor M$ is a Radon measure, where the symbol $\lfloor$ denotes that we consider the measure $\mathcal{H}^n$ only on the set M. $P \in G(n, n+m)$ is called an *approximating tangent space in a given point* $x \in \mathbb{R}^{n+m}$, if

$$\lim_{\rho \downarrow 0} \int_{\zeta_{x,\rho}(M)} \phi d\mathcal{H}^n = \int_P \phi d\mathcal{H}^n \text{ for all test functions } \phi \in C_0^0(\mathbb{R}^{n+m})$$

or, equivalently, if

$$\mathcal{H}^n \lfloor \zeta_{x,\rho}(M) \to \mathcal{H}^n \lfloor P \text{ weak* in } C_0^0(B_R(0))^* \text{ for } R > 0.$$

Then we write $T_x M = P$. In this definition $\zeta_{x,\rho}(M)$ denotes a *blow-up function* $\zeta_{x,\rho}(M) : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ defined by

$$\zeta_{x,\rho}(y) := \rho^{-1}(y - x)$$

for positive ρ and $x, y \in \mathbb{R}^{n+m}$.
(ii) Let $M \subseteq \mathbb{R}^{n+m}$ be $\mathcal{H}^n$-measurable and let θ be a positive locally $\mathcal{H}^n$-integrable function on M, i.e. $\theta \lfloor M \in L^1_{loc}(\mathcal{H}^n)$ or respectively $\theta \cdot \mathcal{H}^n \lfloor M$ is a Radon measure. An n-dimensional subspace $P \in G(n, n+m)$ is called an *approximating tangent space of* M *with respect to* θ *in a fixed point* $x \in \mathbb{R}^{n+m}$, if

$$\lim_{\rho \downarrow 0} \int_{\zeta_{x,\rho}(M)} \phi(y)\theta(x + \rho y) d\mathcal{H}^n(y) = \theta(x) \int_P \phi d\mathcal{H}^n \text{ for all test functions } \phi \in C_0^0(\mathbb{R}^{n+m})$$

or, equivalently, if

$$\rho^{-n} \zeta_{x,\rho}(\theta \cdot \mathcal{H}^n \lfloor M) \to \theta(x) \cdot P \text{ weak* in } C_0^0(B_R(0))^* \text{ for } R > 0,$$

where the image measure is generally defined by $\zeta_{x,\rho}(\mu)(A) = \mu(x + \rho A)$ for $A \subseteq \mathbb{R}^{n+m}$ and where μ is a measure on $\mathbb{R}^{n+m}$.

It is often more convenient to show the existence of a positive locally $\mathcal{H}^n$-integrable function θ on M, which is evidently equivalent to the fact that M can be expressed as the countable union of $\mathcal{H}^n$-measurable sets with locally finite $\mathcal{H}^n$-measure. This motivates definition (2.4) (ii), since the condition $\mathcal{H}^n(M \cap K) < \infty$ for all compact K in definition 2.4 (i) can then be relaxed. If $\mu := \mathcal{H}^n \lfloor \theta$ and if $M_\eta := \{x \in M : \theta(x) > \eta\}$, then $\mathcal{H}^n(M_\eta \cap K) < \infty$ for each compact K and

$$\limsup_{\rho \downarrow 0} \frac{\mu(M \sim M_\eta \cap B_\rho(x))}{\omega_n \rho^n} = 0,$$

where $B_\rho(x)$ denotes the closed ball of radius ρ around x and $\omega_n \rho^n$ denotes the volume of B_ρ in $\mathbb{R}^n$. This is the *n-dimensional upper density of μ* (see [Sim].) Hence the approximating tangent space for M with respect to θ coincides with $T_x M_\eta$ as defined in (2.4) (i) if the latter exists. Moreover the approximating tangent spaces of M with respect to two different positive $\mathcal{H}^n$-integrable functions $\theta, \tilde{\theta}$ coincide $\mathcal{H}^n$-a.e. in M. Furthermore, one can show (see [Sim], Lemma 17.11) that under certain conditions the approximating tangent space coincides with the "classical" tangent plane.

The following theorem gives the important characterization of countably n-rectifiable sets in terms of existence of approximating tangent spaces.

Theorem 2.5 *A Radon measure μ on $\mathbb{R}^{n+m}$ is of the form*

$$\mu = \theta \cdot \mathcal{H}^n \lfloor M$$

with countable n-rectifiable set M and positive, local $\mathcal{H}^n$-integrable function θ on M if and only if there exists an approximating tangent space of M with density $\theta(x) > 0$ for μ-almost all $x \in \mathbb{R}^{n+m}$.

In other words: suppose M is $\mathcal{H}^n$-measurable. Then M is countable n-rectifiable if and only if there is a positive locally $\mathcal{H}^n$-integrable function θ on M with respect to which the approximating tangent space $T_x M$ exists for $\mathcal{H}^n$-a.e. $x \in M$.
The following corollary is a direct consequence of the above theorem.

Corollary 2.6 *Let $M \subseteq \mathbb{R}^{n+m}$ be a countable n-rectifiable set and let θ be a positive local $\mathcal{H}^n$-integrable function on M. Then there exists an approximating tangent space of M with respect to θ for $\mathcal{H}^n$-a.a. $x \in M$.*

3 Finite Dimensional Realizations for General Interest Rate Models

Let r_t be an interest rate process at time $t \in [0, T]$ where T is the maturity time. r_t can be viewed as a process of the time-to-maturity $x := T - t$. In most situations it is convenient to use Musiela parametrization which is a parametrization in terms of *time-to-maturity* rather than *time-of-maturity.* Hence we will also work with this notation. Assume that $r_t \in H$ where H is a separable Hilbert space satisfying the following conditions.

Assumption 3.1 The Hilbert space H of forward curves is assumed to satisfy the following properties:

(1) Assume that the Hilbert space H can be continuously embedded in the space $L^2(\mathbb{R}_+)$ (that is, for every $x \in \mathbb{R}_+$ the pointwise evaluation $\mathrm{ev}_x : h \mapsto h(x)$ is a continuous linear functional on H).

(2) Assume that H is separable.

Remark 3.2 One can show that the family of right-shifts $(S_t f)(x) = f(t+x)$ forms a strongly continuous semigroup S on the space $L^2(\mathbb{R}_+)$ and is generated by the operator $A = \frac{d}{dx}$ with domain $D(A) = H^{1,2}(\mathbb{R}_+)$.

This general Hilbert space of forward curves includes several important Heath, Jarrow and Morton interest rate models. A similar setting for continuous forward rate models was introduced by Filipovic and Teichmann in [Fi-T] and we also used the above setting in [L2] where finite dimensional realizations for linear interest rate models with jumps were studied. In the real world, however, one is interested in computing the interest rate r_t explicitly. Since the Hilbert space setting is too complex to be used for numerical computations, one has to discretize the problem. Since the Hilbert space is separable, there exists a complete orthonormal basis of H. Usually one considers a finite dimensional subspace spanned by n basis vectors such that the finite dimensional subspaces converge to the infinite dimensional space as $n \to \infty$. Since every finite dimensional Hilbert space is isomorphic to an $\mathbb{R}^n$ for some $n \in \mathbb{N}$, we can consider the problem for $r_t \in \mathbb{R}^n$. For n sufficiently large this will produce a good approximation of the real situation. The problem we consider in this paper is that even such a finite dimensional setting it is often too high dimensional. Hence we are interested in lower dimensional representations of such a model. This will be explained in more detail below.

Consider a general interest rate model of the form

$$
\left\{ \begin{array}{rcl} dr_t & = & \alpha(t, r_t)dt + \beta(t, r_t) \circ dW_t + \int_E \delta(t, r_t, y)\bar{\mu}(dt, dy) \\ r_s & = & r^0, \end{array} \right\} \tag{4}
$$

where $\circ$ denotes the Stratonovich integral (see for example [Kar] or [IW]) and $r^0 \in \mathbb{R}^n$. The coefficients are defined as follows

$$
\begin{array}{l} \alpha : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ \beta : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times d} \\ \delta : [0,T] \times \mathbb{R}^n \times E \longrightarrow \mathbb{R}^n \end{array}
$$

and $W = (W_1, \ldots, W_d)$ is a d-dimensional Wiener process and $\bar{\mu} := \mu - \nu$ is the local martingale corresponding to a marked point process μ with compensator $\nu(dt, dy)$ on a Blackwell space $(E, \mathcal{E})$ (see for example [De-Me] or [Ge] for the definition and properties of such a space) The compensator of a marked point process always exists when μ is an optional $\tilde{\mathcal{P}} - \sigma$-finite marked point process, which we will assume here. For the definition of an optional $\tilde{\mathcal{P}} - \sigma$-finite marked point process and properties see for example [J]. Moreover we assume that there exists an intensity measure λ for μ such that

$$
\nu(dt, dy) = \lambda_t(dy)dt.
$$

We now define lower dimensional realizations as follows.

Definition 3.3 The stochastic differential equation (4) has a local *m-dimensional realization* at the initial curve r^0 for $m < n$ if there exist a point $z_0 \in V$, smooth vector fields $a : V \to \mathbb{R}^n$, $b : V \to \mathbb{R}^{n\times d}$ and $c : V \times E \to \mathbb{R}^n$ on an open m-dimensional subset V of $\mathbb{R}^n$ and a smooth mapping $g : V \to \mathbb{R}^n$ with $g(z^0) = r^0$, such that r has a local representation

$$r_t = g(Z_t), \tag{5}$$

where Z is the solution of the m-dimensional Stratonovich stochastic differential equation:

$$\left\{ \begin{array}{rcl} dZ(t) &=& a(Z(t))dt + b(Z(t)) \circ dW_t + \displaystyle\int_E c(Z(t), y)\bar{\mu}(dt, dy) \\ Z_s &=& z^0. \end{array} \right\} \tag{6}$$

Here "local" means that the representation holds for all times t with $s \leq t < \tau(r^0, s)$ P-a.s., where $\tau(r^0, s)$ is a strictly positive stopping time for every $(r^0, s) \in \mathbb{R}^n \times \mathbb{R}_+$ with $\tau(r^0, s) > s$.

However there are some difficulties since there are jumps at certain times and hence we will not be able to describe finite dimensional realizations in terms of invariant tangential submanifolds as in [Bj-S],[B-C], [L1], since whenever there is a jump invariance will get lost and moreover tangency can no longer be defined in the usual way. Here *invariance* is meant in the sense that a submanifold $G \subset \mathbb{R}^n$ is invariant under the stochastic differential equation (4) if $r_t \in G$ for all $t \geq s$ and for all initial conditions $(r^0, s) \in G \times \mathbb{R}_+$. Furthermore, we will have to change the initial curve r^0 after every jump. Thus, if we assume that there exists a finite dimensional realization to equation (4) near r^0 for an initial time s before the first jump occurs, we have to add the jump to the initial curve at the jump time to obtain the new initial curve for the next time interval, i.e. up to the next jump. However this will change the whole solution of the stochastic differential equation (4). This is why it is not possible to describe finite dimensional realization for the stochastic differential equation (4) via invariant tangential manifolds when the coefficients are arbitrary functions of t and the forward rate r_t itself. However, if the coefficients would be linear in r the solution r of the stochastic differential equation (4) for the new initial point after the first jump would just be the solution of the stochastic differential equation before the jump without any jumps plus the solution of the same stochastic differential equation but with initial condition given by the value obtained just after the first jump. This idea is explained in more detail in [L2]. In our general model, however, it is not possible to describe finite dimensional realizations via invariant tangential submanifolds. As already mentioned aboe the problem occurs at the jump times $s_n := T_n(\omega)$ for a fixed $\omega \in \Omega$ defined by

$$\begin{array}{rcl} T_0(\omega) &:=& s \in \mathbb{R}_+ \\ T_n(\omega) &:=& \min\{t : \bar{\mu}_{t+}(\omega) - \bar{\mu}_{t-}(\omega) > 0 \text{ and } t > T_{n-1}(\omega)\}, \ n \geq 1. \end{array}$$

Even if it is possible to find finite dimensional realizations in small neighborhoods of initial points r_n^0, corresponding to initial times t_n, in which there are no jumps at all and if these finite dimensional realizations can be characterized via finite dimensional tangential manifolds G_n, the limit of these manifolds as t_n tends to a jump time $T_n(\omega)$ does not have to be a manifold any longer since there could be some singularities. This is why we introduced approximating tangent spaces in the former section 2.

Consider the set

$$M := \Big\{\alpha(t, r_t), \beta_1(t, r_t), \ldots, \beta_d(t, r_t), \delta(t, r_t, y) : \ y \in E, \ |t - s| < \epsilon, \ r_t \in \mathbb{R}^n \text{ near } r^0\Big\}.$$

This is a subset of $\mathbb{R}^n$. Our aim is now to give conditions for the existence of an approximating tangent space for M in an initial point r^0. Here we proceed as follows:

(1) We want to show now that M is $\mathcal{H}^m$-measurable for some integer $m < n$ and that $\mathcal{H}^m(M \cap K) < \infty$ for compact K. Then $\mathcal{H}^m \lfloor M$ is a Radon measure. Since the Hausdorff measure equals the Lebesgue measure on $\mathbb{R}^n$ we only have to show Lebesgue measurability.

(2) Since we want to apply corollary 2.6 we have to show that M is m-rectifiable. Then the lemma states that the limit in definition 2.4 (i) exists and equals the integral over an m-dimensional subspace P of $\mathbb{R}^n$.

We first show countable m-rectification. We assumed α, β and $\delta(y)$ for fixed $y \in E$ to be smooth vector fields from $[0,T] \times \mathbb{R}^n$ in $\mathbb{R}^n$. To show m-rectification we have to find Lipschitz functions $F_j : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that

$$M \subseteq M_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^m)$$

where $\mathcal{H}^m(M_0) = 0$. For arbitrary coefficients this is very unlikely to hold. Hence we have to impose some assumption on the coefficients α, β and δ. Therefore we suppose that

$$\overline{M} := \operatorname{span}\Big(\Big\{\alpha(t,r_t), \beta_1(t,r_t), \ldots, \beta_d(t,r_t), \delta(t,r_t,y) : \\ y \in E,\ |t-s| < \epsilon,\ r_t \in \mathbb{R}^n \text{ near } r^0\Big\}\Big).$$

is m-dimensional for some $m < n$. Then we can choose a basis $(e_1, \ldots, e_m)$ of the span $\overline{M}$ above. $\overline{M}$ is a linear subspace of $\mathbb{R}^n$ and $\overline{M}$ can be mapped to $\mathbb{R}^m$ via a rotation $O \in O(\mathbb{R}^n)$, where $O(\mathbb{R}^n)$ is the set of orthogonal matrices on $\mathbb{R}^n$. Without loss of generality we can then renumber the coordinates so that $(e_1, \ldots, e_m)$ are the first m coordinates. We choose $F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ as

$$F(x_1, \ldots, x_m) = O^{-1}(x_1, \ldots, x_m, 0, \ldots, 0). \tag{7}$$

Since F is a linear mapping and $\|F\| = \|O\|$ is bounded, F is Lipschitz and we obtain

$$M \subseteq F(\mathbb{R}^m).$$

Hence M is m-rectifiable under the assumption that

$$\operatorname{span}\Big(\Big\{\alpha(t,r_t), \beta_1(t,r_t), \ldots, \beta_d(t,r_t), \delta(t,r_t,y) :\ y \in E,\ |t-s| < \epsilon,\ r_t \in \mathbb{R}^n \text{ near } r^0\Big\}\Big)$$

is m-dimensional.

Now we have to show Lebesgue integrability of M under the condition that the space $\overline{M}$ is m-dimensional. With the rotation O above we can achieve that the image of M under the mapping O satisfies $O(M) \subset \mathbb{R}^m \times \{0\}^{n-m}$ and we can view M as a subset of $\mathbb{R}^m$. Hence the n-dimensional Lebesgue measure of M vanishes. Since M is the graph of integrable functions α, β and δ, M is itself integrable with respect to the m-dimensional Lebesgue measure. Thus we know that M is Hausdorff measurable. In particular, we also know that $\mathcal{H}^m(M \cap K) < \infty$ for all compact K by the same reason.

Now we define θ in corollary 2.6 as the function identically 1. Then all assumptions of the lemma are fulfilled and we obtain that there exists an approximating tangent space of M. We will denote this tangent space by P.

The next question is whether it is possible to find an m-dimensional stochastic process Z_t living in the approximating tangent space P and a mapping $g : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that

$$r_t = g(Z_t).$$

To ensure the existence of an approximating tangent space for M we will always assume the following, which was the only condition needed above to show m-rectification and $\mathcal{H}^m$-measurability.

Assumption 3.4 *Suppose that*

$$\overline{M} := \operatorname{span}\Big(\Big\{\alpha(t,r_t), \beta_1(t,r_t), \ldots, \beta_d(t,r_t), \delta(t,r_t,y) : \\ y \in E,\ \ |t-s| < \epsilon,\ \ r_t \in \mathbb{R}^n \ \textit{near}\ r^0\Big\}\Big)$$

is m-dimensional for some $m < n$.

Moreover we will need some conditions for the marked point process so that there are only finitely many jumps in every finite time interval.

Assumption 3.5 *(1) Suppose that μ has only finitely many jumps in every finite time interval.*

(2) Assume that the marked point process is cádlág.

Idea 3.6 Consider the stochastic differential equation (4) at an initial curve r^0 for an initial time s, which is not a jump time. Then by assumption (3.5) there exists a neighborhood of s such that r_t has only finitely many jumps and if we choose the neighborhood of s small enough we can achieve that r_t has no jumps at all for all t near s. Thus we obtain from previous observations (see e.g. the discussion of the continuous case in [L2]) that there exists an m-dimensional realization of (4) near r^0 if and only if the Lie algebra generated by the drift and both volatilities is m-dimensional. For this to make sense the Lie algebra obviously has to be well-defined which was achieved in [L2] and [Fi-T] by assuming the initial point r^0 to be in the Fréchet space $D(A_0^\infty)$ where A is the operator $\frac{\partial}{\partial x}$ which appears in the Heath, Jarrow and Morton drift condition. Then the Lie algebra of the coefficients, which were assumed to be smooth vector fields, is well defined. Thus we have to ensure here as well that the Lie algebra is well defined. Therefore we assume the coefficients to be smooth vector fields. Hence we have an m-dimensional process Z and a mapping $g : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that

$$r_t = g(Z_t)$$

for all t near the initial time s. Now we let the initial time s tend to the next jump time s_n. Lets say we have a sequence of initial times $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i \to s_n$ for some $n \in \mathbb{N}$. Suppose for each such initial time t_i we have an m_i-dimensional realization of the stochastic differential equation (4) near corresponding initial curves r_i^0. By assumption (3.5) there exists an integer $N \in \mathbb{N}$, namely the number of jumps in the finite interval $[s, s_n]$, such that the solution processes r_t^i for $i > N$ have no jumps in the time interval $[t_i, s_n]$. We now want to show that the limit of the corresponding tangential manifolds G_i as i tends to ∞ is contained in the approximating tangent space $T_{r^0}M$ of M at the initial curve r^0 for the initial time s_n and that the dimensions m_i of the tangential manifolds G_i tend to m. Then we can define the process Z as the limit process of the processes Z_i as i tends to ∞ and the mapping g as the limit of the mappings g_i.

Lemma 3.7 *Suppose assumptions (3.5) and (3.4) hold.*
The limit of the tangential manifolds G_i *as* i *tends to* ∞ *is contained in the approximating tangent space* P *of* M *at* r^0.

Proof 3.8 *We have to consider the sequence of initial curves* r_i^0 *for initial times* t_i *which converges to* r^0 *as* t_i *converge to the jump time* s_n. *Moreover, we have a sequence of tangential manifolds* G_i *and the approximating tangent space* P *of* M *at* r^0, *which exists by corollary 2.6 and the observations above. However, at the initial curves* r_i^0 *for times* t_i *where we have tangential manifolds* G_i, *we also have approximating tangent spaces* P_i. *Obviously the tangential manifold* G_i *is contained in the approximating tangent space* P_i *and the dimension of* P_i *equals the dimension* m_i *of the tangential manifold* G_i. *Hence it is enough to investigate whether the limit of the approximating tangent spaces* P_i *as* i *tends to* ∞ *is contained in or equals the approximating tangent space* P *of* M *at* r^0. *Thus we have to consider*

$$\lim_{i\to\infty}\lim_{\rho\downarrow 0}\int_{\zeta_{r_i^0,\rho}(M)} \phi d\mathcal{H}^{m_i} \text{ for all test functions } \phi\in C_0^0(\mathbb{R}^n).$$

This limit equals

$$\int_P \phi d\mathcal{H}^m \text{ for all test functions } \phi\in C_0^0(\mathbb{R}^n)$$

since from elementary functional analysis it follows that

$$\begin{aligned}
&\left|\left\langle \phi\cdot\chi_{\zeta_{r^0,\rho}(M)}, \mathcal{H}^m \llcorner P\right\rangle - \left\langle \phi\cdot\chi_{\zeta_{r_i^0,\rho}(M)}, \mathcal{H}^{m_i}\llcorner \zeta_{r_i^0,\rho}(M)\right\rangle\right| \\
\leq\ & \left|\left\langle \phi\cdot\chi_{\zeta_{r^0,\rho}(M)}, \left(\mathcal{H}^m\llcorner P - \mathcal{H}^{m_i}\llcorner\zeta_{r_i^0,\rho}(M)\right)\right\rangle\right| \\
&+\left\|\phi\cdot\chi_{\zeta_{r^0,\rho}(M)} - \phi\cdot\chi_{\zeta_{r_i^0,\rho}(M)}\right\|\cdot\left\|\mathcal{H}^{m_i}\llcorner\zeta_{r_i^0,\rho}(M)\right\|
\end{aligned}$$

for all test functions $\phi\in C_0^0(\mathbb{R}^n)$ *and weak* convergent sequences are bounded. Here* $\langle\cdot,\cdot\rangle$ *denotes the duality product*

$$\langle x, x'\rangle = x'(x)$$

for x' *in the dual space* X' *and* $x\in X$. *This means that*

$$\begin{aligned}
\lim_{i\to\infty}\lim_{\rho\downarrow 0}\int_{\zeta_{r_i^0,\rho}(M)}\phi d\mathcal{H}^{m_i} &= \lim_{\rho\downarrow 0}\lim_{i\to\infty}\int_{\zeta_{r_i^0,\rho}(M)}\phi d\mathcal{H}^{m_i} \\
&= \lim_{\rho\downarrow 0}\int_{\zeta_{r^0,\rho}(M)}\phi d\mathcal{H}^m = \int_P \phi d\mathcal{H}^m
\end{aligned}$$

for all test functions $\phi\in C_0^0(\mathbb{R}^n)$. □

Now we can proceed as described in (3.6). For every initial curve r_i^0 for an initial time t_i we have an m_i-dimensional realization of the stochastic differential equation (4) if and only if the Lie algebra generated by drift and volatilities is m_i-dimensional at r_i^0. Then we also have an m_i-dimensional tangential manifold G_i at r_i^0 defined via a mapping g_i by $G_i = \mathrm{Im}[g_i]$. Moreover, we have an m_i-dimensional process Z^i defined by

$$\begin{aligned}
dZ_t^i &= a^i(Z_t^i)dt + b^i(Z_t^i)\circ dW_t + \int_E c^i(Z_t^i,y)\bar{\mu}(dt,dy) \\
Z_s^i &= z_0^i
\end{aligned}$$

such that $r_t = g_i(Z_t^i)$ for $|t_i - t|$ small. Now we define a process Z by

$$\begin{aligned} dZ_t &= \lim_{i\to\infty} a^i(Z_t^i)dt + \lim_{i\to\infty} b^i(Z_t^i) \circ dW_t + \int_E \lim_{i\to\infty} c^i(Z_t^i, y)\bar{\mu}(dt, dy) \\ Z_s &= \lim_{i\to\infty} z_0^i \end{aligned}$$

pointwise. This process is m-dimensional since the dimension m_i tends to m as i tends to ∞ and the coefficients a^i, b^i and c^i are assumed to be $\mathbb{R}^{m_i}$-valued ($\mathbb{R}^{m_i\times d}$-valued respectively). Furthermore, we define a mapping g by

$$g := \lim_{i\to\infty} g_i.$$

The image of this mapping does not in general define a tangential manifold any longer via $G := \mathrm{Im}[g]$ since there might be some singularities but it is a suitable mapping to define m-dimensional realizations for the stochastic differential equation (4) at r^0 and we know that $G \subset P = T_{r^0}M$ since

$$g_i(Z_t^i) \in \mathrm{Im}[g_i] = G_i \subset P_i \longrightarrow P$$

and $g_i(Z_t^i) \longrightarrow g(Z_t)$ pointwise as $i \to \infty$. Hence we obtain the following result

Theorem 3.9 *We assume assumptions (3.4) and (3.5). The Lie algebra generated by drift and volatilities is m dimensional in a neighborhood of r^0 and the dimension of $\overline{M}$ is also m if and only if there exists an m-dimensional realization of the stochastic differential equation (4) at r^0.*

Proof 3.10 $\Rightarrow$ *Under the given assumptions we can apply the above construction which defines an m-dimensional process Z and a mapping g such that $r_t = g(Z_t)$ near r^0. Moreover $G := \mathrm{span}(\mathrm{Im}[g])$ defines the approximating tangent space P, the existence of which follows from the condition that the dimension of $\overline{M}$ is m.*
$\Leftarrow$ *If there exists an m-dimensional realization of the stochastic differential equation (4) at r^0 then there also exists a small neighborhood of r^0 such that there exist m-dimensional realizations for (4) for all initial curves r_i^0 near r^0 for initial times t_i near the initial time s for r^0. Hence there are also some m-dimensional realizations of (4) at some r_i^0 for a non-jump time t_i. In that case, we can apply the standard theory which states that the Lie algebra generated by drift and volatilities is m-dimensional in a neighborhood of r_i^0. Since one can choose r_i^0 and t_i in such a way that $t_i \to s$ for $i \to \infty$ and $r_i^0 \to r^0$ as i tends to ∞, we obtain that the approximating tangent space P for M at r^0 exists and is m-dimensional. Theorem 2.5 then states that M is m-rectifiable and hence that $\overline{M}$ is m-dimensional, which proves the theorem.* □

The next result is a stability result for stochastic differential equations with jumps. Therefore we will need in addition to assumption (3.5) also the following.

Assumption 3.11 *Suppose that all coefficients α, β and δ are Lipschitz continuous in the r-variable with Lipschitz constants C_1, C_2 and C_3 respectively.*

Theorem 3.12 (Stability of Stochastic Differential Equations) *Consider the SDE (4) with initial time s equal to a jump time $T_n(\omega)$ and initial curve r^0. Let $\{t_i\}_{i\in\mathbb{N}}$ be a sequence of initial times such that $t_i > s$ and such that t_i tends to s as $i \to \infty$. Moreover let r_i^0 be a sequence of initial curves corresponding to the initial times t_i such that r_i^0 tends to r^0 as $t_i \to s$. Under the assumptions (3.5) and (3.11) there exists an* *integer N such that the solution r_t^i of the stochastic differential equation (4) with initial time t_i and initial curve r_i^0 for $i > N$ converges in L^p to the solution of the same stochastic differential equation (4) with initial curve r^0 for initial time $s = T_n(\omega)$ as $i \to \infty$ for any $p \geq 2$.*

Proof 3.13 *We first show right-continuous dependence on initial data for fixed initial time s in a small neighborhood of s. Hence we want to show that the solution r_t^n of the stochastic differential equation (4) with initial curve r_n^0 for initial time s converges in L^p to the solution r_t of the same stochastic differential equation with initial curve r^0 for the same initial time s when t is near s. By assumption (3.5) the marked point process μ has only finitely many jumps in finite time intervals. Hence we can define stopping times T^n and T which should denote the next jump time for the process r_t^n and r_t, respectively, after time s, which is then strictly greater than s. Without loss of generality we assume that $T(\omega) < T^n(\omega)$. Hence we can consider a small time interval $(s, T(\omega))$ in which the processes r_t^n and r_t are continuous. Since we have assumed that the process μ has only finitely many jumps in every finite time interval, we can assume that $\tau = \inf_{\omega\in\Omega} T(\omega) > s$. Then we can compute for $s < t < \tau$*

$$
\begin{aligned}
E\left[|r_t^n - r_t|^p\right] \leq\; & E\left[|r_n^0 - r^0|^p\right] + E\left[\int_s^t |\alpha(r_u^n) - \alpha(r_u)|^p du\right] \\
& + E\left[\left|\int_s^t (\beta(r_u^n) - \beta(r_u))dW_u\right|^p\right] \\
& + E\left[\left|\int_s^t \int_E (\delta(r_u^n, y) - \delta(r_u, y))\bar{\mu}(du, dy)\right|^p\right].
\end{aligned}
$$

Since $t < \tau$ we can apply the assumed Lipschitz-continuity of the coefficients α, β and δ with Lipschitz-constants C_1, C_2 and C_3, respectively, for $t < \tau$ and Burkholder's inequality with constants C_B^1 and C_B^2, and obtain that the right hand side is less or equal

$$
\begin{aligned}
& E\left[|r_n^0 - r^0|^p\right] + C_1^p \cdot E\left[\int_s^t |r_u^n - r_u|^p du\right] \\
& + C_2^p \cdot C_B^1 \cdot E\left[\int_s^t |r_u^n - r_u|^p du\right] + C_3^p \cdot C_B^2 \cdot E\left[\int_s^t |r_u^n - r_u|^p \bar{\mu}(du, E)\right] \\
= \; & E\left[|r_n^0 - r^0|^p\right] + C_1^p \cdot E\left[\int_s^t |r_u^n - r_u|^p du\right] \\
& + C_2^p \cdot C_B^1 \cdot E\left[\int_s^t |r_u^n - r_u|^p du\right] + C_3^p \cdot C_B^2 \cdot E\left[\int_s^t |r_u^n - r_u|^p \lambda_u(E) du\right] \\
= \; & E\left[|r_n^0 - r^0|^p\right] + \left(C_1^p + C_2^p \cdot C_B^1 + C_3^p \cdot C_B^2 \lambda(E)\right) \cdot E\left[\int_s^t |r_u^n - r_u|^p du\right]
\end{aligned}
$$

Let $C := C_1^p + C_2^p \cdot C_B^1 + C_3^p \cdot C_B^2 \lambda(E)$ and $a_n := E\left[|r_n^0 - r^0|^p\right]$. Since the processes r_t^n and r_t are continuous in the time interval (s, T) and thus also $|r_t^n - r_t|^p$ is continuous in (s, T) we can apply Gronwall's lemma and obtain

$$
\begin{aligned}
E\left[|r_t^n - r_t|^p\right] & \leq a_n + C \cdot E\left[\int_s^t a_n e^{C(t-u)} du\right] \\
& = a_n + C \cdot a_n 1/C \cdot (1 - e^{C(t-s)}) \\
& = a_n + a_n \cdot (1 - e^{C(t-s)}).
\end{aligned}
$$

This, however, converges to 0 as $n \to \infty$ since $r_n^0 \to r^0$ by construction and hence $a_n \to 0$ as $n \to \infty$ and the last term converges to zero as $t \to s$. Hence the solution r_t^n converges to the solution r_t for $t \to s$.

We now reduce the problem such that only the initial curves vary and the initial time is fixed. Then the claim follows from the above. Therefore let T be a fixed stopping time greater than t_m and such that the integrals below are finite (i.e. T is not arbitrary large), and $m > n$.

$$
\begin{aligned}
E[|r_t^n - r_T^m|^p] \leq\ & E\left[\left|r_n^0 - r_{t_n}^m\right|^p\right] + E\left[\int_{t_n}^T |\alpha(r_t^n) - \alpha(r_t^m)|^p\, dt\right] \\
& + E\left[\left|\int_{t_n}^T (\beta(r_t^n) - \beta(r_t^m)) dW_t\right|^p\right] \\
& + E\left[\left|\int_{t_n}^T \int_E (\delta(r_t^n, y) - \delta(r_t^m, y))\bar{\mu}(dt, dy)\right|^p\right]
\end{aligned}
$$

Since we assumed α, β and δ to be Lipschitz continuous with Lipschitz-constants C_1, C_2 and C_3 respectively, and by applying Burkholder's inequality with constants C_B^1 and C_B^2 this is smaller or equal to

$$
\begin{aligned}
\leq\ & E\left[\left|r_n^0 - r_{t_n}^m\right|^p\right] + C_1^p \cdot E\left[\int_{t_n}^T |r_t^n - r_t^m|^p\, dt\right] \\
& + C_B^1 \cdot E\left[\int_{t_n}^T |\beta(r_t^n) - \beta(r_t^m)|^p\, dt\right] \\
& + C_B^2 \cdot E\left[\int_{t_n}^T \int_E |\delta(r_t^n, y) - \delta(r_t^m, y)|^p\, \bar{\mu}(dt, dy)\right] \\
\leq\ & E\left[\left|r_n^- r_{t_n}^m\right|^p\right] + C_1^p \cdot E\left[\int_{t_n}^T |r_t^n - r_t^m|^p\, dt\right] \\
& + C_B^1 \cdot C_2^p \cdot E\left[\int_{t_n}^T |r_t^n - r_t^m|^p\, dt\right] + C_B^2 \cdot C_3^p \cdot E\left[\int_{t_n}^T |r_t^n - r_t^m|^p\, \bar{\mu}(dt, E)\right]
\end{aligned}
$$

This, however, converges to 0 since $E\left[\left|r_n^0 - r_{t_n}^m\right|^p\right]$ converges to 0 because $r_n^0 \to r^0$ and $r_{t_n}^m \to r_s = r^0$ as $n, m \to \infty$ and $|r_t^n - r_t^m|^p$ converges to 0 as shown above. □

Remark 3.14 This result also holds for a general Hilbert space setting, i.e. for the case where r_t is an element of an infinite dimensional Hilbert space.

For the following we need some more assumptions.

Assumption 3.15 *Let g' denote the derivative of g with respect to the z variable. We assume the following:*
The mapping $z \mapsto g(z)$ is injective and the derivative $g'(z)$ is injective for all $z \in V$.

Theorem 3.16 (Main Result) *Let r^0 be an initial curve for an initial time s equal to a jump time. Then by assumption (3.5) there exists a neighborhood of s with only one jump time, namely the initial time s. Let $\{t_i\}_{i \in \mathbb{N}}$ be a sequence in this neighborhood such that $t_i > s$ for all $i \in \mathbb{N}$ and consider initial curves r_i^0 near r^0 for these initial times t_i, $i \in \mathbb{N}$. Under assumption (3.11) the following statements are equivalent whenever we can assume that there exists a solution of the stochastic differential equation (4):*

(i) There exist local m-dimensional realizations at initial curves r_i^0 for initial times t_i, respectively, for the stochastic differential equation (4) given via mappings g_i and m_i-dimensional processes Z^i such that

$$r_t = g_i(Z_t^i)$$

and the mappings g_i satisfy assumption (3.15). Moreover there exists an m-dimensional realization of the stochastic differential equation (4) at r^0 given via a mapping g and an m-dimensional process Z such that $r_t = g(Z_t)$.

(ii) *There exist m-dimensional tangential manifolds G_i of M near r_i^0 given as*

$$G_i = \mathrm{Im}[g_i]$$

with $r_i^0 \in G_i$ and there exists an m-dimensional approximating tangent space denoted $P = T_{r^0}M$ of M at r^0 given as $P = \mathrm{span}[G]$ for some $G = \mathrm{Im}[g]$ such that g is the limit of g_i as i tends to ∞.

Proof 3.17 (i) $\Rightarrow$ (ii) *This follows from theorem 3.9 and corollary 2.6 and previous observations for the initial times t_i which are non-jump times.*
(ii) $\Rightarrow$ (i) *Since there are initial points r_i^0 near r^0 for initial times t_i which are non-jump times, the approximating tangent spaces P_i contain the tangential manifolds G_i for the stochastic differential equation (4) at r_i^0. Hence by applying standard theory. i.e. the theory of finite dimensional realizations for continuous stochastic differential equations developed in ([L2]), we know that there exist finite dimensional realizations of (4) at r_i^0 given as*

$$r_t^i = g_i(Z_t^i)$$

with mappings g_i such that $G_i = \mathrm{Im}[g_i]$ and m-dimensional processes Z^i. By theorem (3.12) the solutions r_t^i of the stochastic differential equations (4) with initial curves r_i^0 for initial times t_i converge pointwise to the solution of the same stochastic differential equation with initial curve r^0 for initial time s. Since $\overline{M}$ is m-dimensional at r^0 and $P = T_{r^0}M$ exists and is m-dimensional at r^0, the limits of Z^i and G_i also exist and define an m-dimensional realization of the stochastic differential equation (4) at r^0. Thus we obtain

$$r_t = \lim_{i\to\infty} r_t^i = \lim_{i\to\infty} g_i(Z_t^i) = g(Z_t).$$

□

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