Hitting distributions domination and subordonate resolvents; an analytic approach

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Abstract. We give an analytic version of a well known Shih's theorem concerning the Markov processes whose hitting distributions are dominated by those of a given process. The treatment is purely analytic, completely different from Shih's arguments and improves essentially his result.

Introduction

In the paper [11] C.T. Shih has considered two Hunt processes X and X', having a common locally compact separable metric space $(E, \mathcal{T})$ as state space and he proved that under some obvious necessary conditions, if the hitting distributions of X' are dominated by those of X then there exists a process Y obtained from a random time change in a subprocess of X that is equivalent to X' (i.e. they have the same transition function). See also [7] for a different method.

In this article we extend the above result to the general case when the common state space $(E, \mathcal{T})$ is a Lusin topological space. Our approach is purely analytic and it is completely different from that developed in [11]. This new treatement extends a similar one (see [6]) considered in the particular case when there exists a reference measure for the given process X.

If $\mathcal{U}$ and $\mathcal{U}'$ are the subMarkovian resolvents associated with X and X' then the fact that the hitting distributions of X dominate those of X' means that ${}'R^A \leq R^A$ for all Borel subset A of E where R^A (resp. ${}'R^A$) is the réduite kernel on A associated with $\mathcal{U}$ (resp. $\mathcal{U}'$). The fact that there exists a process Y obtained from a random time change in a subprocess of X which is equivalent to X' means that there exists a sub-Markovian resolvent $\mathcal{W}$ which is exactly subordinate to $\mathcal{U}$ (in the sense of P.A. Meyer [8], [9]) and the set $\mathcal{E}(\mathcal{W})$ of all $\mathcal{W}$-excessive functions coincides with the set $\mathcal{E}(\mathcal{U}')$ of all $\mathcal{U}'$-excessive functions.

Our constructions give a sub-Markovian resolvent $\mathcal{W}$ with the above property and moreover it possesses the following maximality property: If $\mathcal{W}'$ is a second sub-Markovian resolvent which is related with $\mathcal{U}$ and $\mathcal{U}'$ as above then we have $W'f \leq Wf$ and $Wf - W'f \in \mathcal{E}(\mathcal{U}')$ for all $f \in p\mathcal{B}$ with $Wf < \infty$.

1 Preliminaries and exact subordination operators

In this paper $(E, \mathcal{T})$ is a Lusin topological space and $\mathcal{U} = (U_\alpha)_{\alpha \geq 0}$ a proper subMarkovian rezolvent of kernels on $(E, \mathcal{B})$ where $\mathcal{B} = \mathcal{B}(E)$ is the σ-algebra of all Borel subsets of $(E, \mathcal{T})$. We denote by $\mathcal{B}^{(u)}$ the σ-algebra of all universally $\mathcal{B}$-measurable subsets of E.

As usually we denote by $p\mathcal{B}$ (resp. $p\mathcal{B}^{(u)}$) the set of all positive $\mathcal{B}$ (resp. $\mathcal{B}^{(u)}$)-measurable functions on E. If $\mathcal{F} \subset p\mathcal{B}^{(u)}$ we denote by $b\mathcal{F}$ the set of all bounded functions from $\mathcal{F}$.

We denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}^{(u)}$ measurable functions which are $\mathcal{U}$-excessive. *We assume that the set $\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ is min-stable, contains the positive constant functions and generates $\mathcal{B}$.*

We recall that a $\mathcal{U}$-excessive measure ξ on $(E, \mathcal{B})$ is termed $\mathcal{U}$-potential if it is of the form $\xi = \mu \circ U$ where μ is a σ-finite measure on $(E, \mathcal{B})$.

The set E is called *semisaturated* with respect to $\mathcal{U}$ if any $\mathcal{U}$-excessive measure dominated by a $\mathcal{U}$-potential is also a $\mathcal{U}$-potential. *In the sequel we assume that E is semisaturated with respect to $\mathcal{U}$.*

If ξ is a $\mathcal{U}$-excessive measure on $(E,\mathcal{B})$, a subset A of E is called ξ-polar if there exists $s\in\mathcal{E}(\mathcal{U})$ such that $s=+\infty$ on A and $s<\infty\xi$-a.e. If μ is a σ-finite measure on E such that $\mu\circ U\in$ Exc$_{\mathcal{U}}$ then we say μ-polar instead of $\mu\circ U$-polar.

A subset A of E is called *nearly $\mathcal{B}$-measurable* with respect to $\mathcal{U}$ if for any finite measure μ on $(E,\mathcal{B})$ there exists $A_0\in\mathcal{B}$, $A_0\subset A$ such that $A\backslash A_0$ is μ-polar and μ-negligible.

The set of all nearly $\mathcal{B}$-measurable sets is denoted by $\mathcal{B}^{(n)}$. Obviously $\mathcal{B}^{(n)}$ is a σ-algebra and $\mathcal{B}\subset B^{(n)}\subset\mathcal{B}^{(u)}$. Is is know that any $\mathcal{U}$-excessive function is $\mathcal{B}^{(n)}$-measurable.

For any $f:E\to\bar{\mathbb{R}}_+$ we denote by Rf the function

$$Rf=\inf\{t\in\mathcal{E}(\mathcal{U})\mid t\geq f\}$$

called the *réduite of f with respect to $\mathcal{U}$.*

For any subset A of E and $s\in\mathcal{E}(\mathcal{U})$, the function $R^As=R(1_As)$ is called the *réduite of s on A*. We use the convention $0\cdot(+\infty)=(+\infty)\cdot 0=0$. Is is know that ([2], [4] ch. I) for any $A\in\mathcal{B}^{(n)}$ and $s\in\mathcal{E}(\mathcal{U})$ the function R^As is $\mathcal{B}^{(u)}$-measurable and it is $\mathcal{U}$-supermedian. In this case we denote by B^As the $\mathcal{U}$-excessive regularization of R^As, i.e.

$$B^As=\sup_\alpha \alpha U_\alpha R^As.$$

Since E is semisaturated we respect to $\mathcal{U}$ then for any $A\in\mathcal{B}^{(n)}$ and $x\in E$ there exists a positive measure denoted R^A_x (resp. B^A_x) on $(E,\mathcal{B})$ such that

$$R^A_x(s)=R^As(x)\quad (\text{resp.} B^A_x(s)=B^As(x)).$$

Moreover we denote by R^A (resp. B^A) the kernel on $(E,\mathcal{B}^{(u)})$ such that

$$R^Af(x)=R^A_x(f)\quad (\text{resp. } B^Af(x)=B^A_x(f))$$

for all $f\in p\mathcal{B}^{(u)}$ and $x\in E$.

A set $A\in\mathcal{B}^{(n)}$ is called $\mathcal{U}$ *thin at x* if there exists $s\in b\mathcal{E}(\mathcal{U})$ such that $B^As(x)<s(x)$ or equivalently $B^A_x\neq\varepsilon_x$.

The *$\mathcal{U}$-fine topology* on E is the coarsest topology on E for which any function from $\mathcal{E}(\mathcal{U})\cap p\mathcal{B}$ is continuous. It is easy to see that any $\mathcal{U}$-excessive function is $\mathcal{U}$-fine continuous. It is know that if $A\in\mathcal{B}^{(n)}$ then it will be $\mathcal{U}$-finely open if and only if the set $E\backslash A$ is $\mathcal{U}$-thin at any point of A. Also if $A\in\mathcal{B}^{(n)}$ is $\mathcal{U}$-finely closed then for any $x\in E$ the measures R^A_x, B^A_x are carried by A.

A set $A\subset E$ is called *$\mathcal{U}$-absorbent* if there exists $s\in\mathcal{E}(\mathcal{U})$ such that $A=[s=0]$. Obviously any $\mathcal{U}$-absorbent set is $\mathcal{U}$-finely open.

A set $A\in\mathcal{B}^{(n)}$ is called *$\mathcal{U}$-subbasic* if A is not thin at any point of A or equivalently $R^A=B^A$. In this case we have $B^As=s$ on A for all $s\in\mathcal{E}(\mathcal{U})$ and so $B^A(B^A)=B^A$.

A set $A\in\mathcal{B}^{(n)}$ is called *$\mathcal{U}$-basic* if A is $\mathcal{U}$-subbasic and A is $\mathcal{U}$-finely closed. If $A\in\mathcal{B}^{(n)}$ is $\mathcal{U}$-subbasic and $f_0\in p\mathcal{B}$, $0<f_0\leq 1$ is such that $p_0:=Uf_0$ is bounded then the set $[B^AUp_0=p_0]$ is the $\mathcal{U}$-fine closure of A and represent a $\mathcal{U}$-basic set.

A set $A\in\mathcal{B}^{(n)}$ is called *$\mathcal{B}$–$\mathcal{U}$-subbasic* if it is $\mathcal{U}$-subbasic and $B^As\in p\mathcal{B}\cap\mathcal{E}(\mathcal{U})$ for all $s\in p\mathcal{B}\cap\mathcal{E}(\mathcal{U})$.

A set $A\in\mathcal{B}^{(n)}$ is called *$\mathcal{B}-\mathcal{U}$-basic* if it is $\mathcal{B}-\mathcal{U}$-subbasic and $\mathcal{U}$-basic set in the same time. In this case $A\in\mathcal{B}$ and we have $A=[B^Ap_0=p_0]$ where p_0 is as above. We notice that if A is a $\mathcal{B}-\mathcal{U}$-subbasic set then B^A is a kernel on $(E,\mathcal{B})$.

In the sequel we consider a second sub-Markovian resolvent $\mathcal{U}'=(U'_\alpha)_{\alpha>0}$ on $(E,\mathcal{B})$ such that the set $\mathcal{E}(\mathcal{U}')\cap p\mathcal{B}$ is min-stable, contains the positive constant functions and generates $\mathcal{B}$. We assume that E is semisaturated with respect to $\mathcal{U}'$. Also we suppose that the following assertions hold.

1) The topology $\mathcal{T}$ is *natural* with respect to both $\mathcal{U}$ and $\mathcal{U}'$, i.e. any $G \in \mathcal{T}$ is $\mathcal{U}$-finely open and $\mathcal{U}'$-finely open.

2) For all $A \in \mathcal{B}$ and $f \in p\mathcal{B}$ we have

$$'R^A f \leq R^A f.$$

3) For any $a \in E$ such that the set $\{a\}$ is $\mathcal{U}'$-absorbent the set $\{a\}$ is $\mathcal{U}$-finely open.

Proposition 1.1. *For any $x \in E$ the set $\{x\}$ is $\mathcal{U}'$-finely open if and if it is $\mathcal{U}$-finely open.*

Proof. Assume that $\{x\}$ is not $\mathcal{U}'$-finely open. Then this means that $'R^{E\setminus\{x\}}s(x) = s(x)$ for all $s \in \mathcal{E}(\mathcal{U}')$ or equivalently $'R^{E\setminus\{x\}}f(x) = f(x)$ for all $f \in p\mathcal{B}$, i.e. $'R_x^{E\setminus\{x\}} = \varepsilon_x$.

By hypothesis 2) it follows that

$$f(x) = {}'R^{E\setminus\{x\}}f(x) \leq R^{E\setminus\{x\}}f(x) \quad \forall f \in p\mathcal{B}$$

and so $s(x) = R^{E\setminus\{x\}}s(x)$ for all $s \in \mathcal{E}(\mathcal{U})$, i.e. $\{x\}$ is not $\mathcal{U}$-finely open.

Conversely, assume that $\{x\}$ is not $\mathcal{U}$-finely open but $\{x\}$ is $\mathcal{U}'$–finely open. We get

$$R^{E\setminus\{x\}}f(x) = f(x) \quad \forall f \in p\mathcal{B}$$

and from

$$'R^{E\setminus\{x\}}f(x) \leq R^{E\setminus\{x\}}f(x) = f(x) \quad \forall f \in p\mathcal{B}$$

we deduce that there exists $0 \leq \alpha \leq 1$ such that

$$'R^{E\setminus\{x\}}f(x) = \alpha f(x) \quad \forall f \in p\mathcal{B}$$

Since $\{x\}$ is $\mathcal{U}'$-finely open it follows that the measure $'R_x^{E\setminus\{x\}}$ is carried by $E\setminus\{x\}$ and so $\alpha = 0$. Hence

$$'R^{E\setminus\{x\}}1(x) = 0$$

i.e. $\{x\}$ is absorbent with respect to $\mathcal{U}'$ and so by hypothesis 3) $\{x\}$ is $\mathcal{U}$-finely open, contradiction. □

Proposition 1.2. *For every $A \in \mathcal{B}$ and $x \in E\setminus A$ the set A will be $\mathcal{U}$-thin at x if and only if it is $\mathcal{U}'$-thin at x. Particularly if $A \in \mathcal{B}$ then A is $\mathcal{U}$-finely open if and only if A is $\mathcal{U}'$-finely open.*

Proof. Assume that A is $\mathcal{U}$-thin at x. Then there exists $s \in b\mathcal{E}(\mathcal{U})$ such that $R^A s(x) < s(x)$, i.e. the measure R_x^A is different from ε_x. Since by hypotheses 2) we have $'R_x^A \leq R_x^A$ it follows that $'R_x^A(s) \leq R_x^A(s) < s(x)$, i.e. $'R_x^A \neq \varepsilon_x$ and consequently A is $\mathcal{U}'$-thin at x.

Suppose now that A is $\mathcal{U}'$-thin at x, i.e. $'R_x^A \neq \varepsilon_x$ or equivalently the $\mathcal{U}'$-fine closure of A does not contains x and so $'R_x^A$ does not charge $\{x\}$.

Assume now that A is not $\mathcal{U}$-thin at x, i.e. $R_x^A = \varepsilon_x$. Since $'R_x^A \leq R_x^A$ it follows that there exists $\theta \in [0,1]$ such that $'R_x^A = \theta\varepsilon_x$ and so, because R_x^A does not charge $\{x\}$, we get $\theta = 0$. Hence $\{x\}$ is $\mathcal{U}'$-absorbent and therefore by hypothesis 3) $\{x\}$ is $\mathcal{U}$-finely open and consequently A is $\mathcal{U}$-thin at x, which leads to a contradiction. □

Corollary 1.3. *We have $\mathcal{E}(\mathcal{U}) \subset \mathcal{E}(\mathcal{U}')$ and any nearly $\mathcal{B}$-measurable set $A \subset E$ with respect to $\mathcal{U}$ is also nearly $\mathcal{B}$-measurable set with respect to $\mathcal{U}'$.*

Proof. Let $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$. Hence s is $\mathcal{U}$-finely continuous and therefore s is $\mathcal{U}'$-finely continuous. Hence for any $\mathcal{U}'$-finely open set G we have $'B^G s \le B^G s \le s$ and so s is $\mathcal{U}'$-excessive.

Let A be a nearly $\mathcal{B}$-measurable set with respect to $\mathcal{U}$. Then for any finite measure μ on $(E, \mathcal{B})$ there exists $A_0 \in \mathcal{B}$, $A_0 \subset A$, such that $A \setminus A_0$ is μ-polar (with respect to $\mathcal{U}$) and μ-negligible. From the above considerations it follows that $A \setminus A_0$ is μ-polar with respect to $\mathcal{U}'$. Hence A is nearly $\mathcal{B}$-measurable with respect to $\mathcal{U}'$. □

Proposition 1.4. *For any $A \in \mathcal{B}$ and $f \in p\mathcal{B}$ we have*

$$'B^A f \le B^A f.$$

Proof. Let f of the form $f = u - v$ where $u, v \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$, $v \le u < \infty$. By hypothesis 2) we have $'R^A u - {}'R^A v \le R^A u - R^A v$. On the other hand we have $\mathcal{E}(\mathcal{U}) \subset \mathcal{E}(\mathcal{U}')$ and for any $s \in b\mathcal{E}(\mathcal{U})$, $'B^A s$ (resp. $B^A s$) is the lower semicontinuous regularization of $'R^A s$ (resp. $R^A s$) with respect to the $\mathcal{U}$-fine topology. Hence we get $'B^A u - {}'B^A v \le B^A u - B^A v$. □

Proposition 1.5. *For any point $x \in E$ we have: $\{x\}$ is $\mathcal{U}$-thin at x if and only if $\{x\}$ is $\mathcal{U}'$-thin at x.*

Proof. Assume that $\{x\}$ is $\mathcal{U}$-thin at x. Since $'B_x^{\{x\}} \le B_x^{\{x\}}$ and from the fact that the measure $B_x^{\{x\}}$ is carred by $\{x\}$, we get $B_x^{\{x\}} = \alpha \varepsilon_x$ with $\alpha < 1$. Hence $'B^{\{x\}} = \beta \varepsilon_x$ with $\beta \le \alpha < 1$ and so $\{x\}$ is $\mathcal{U}'$-thin at x.

Assume now that $\{x\}$ is $\mathcal{U}'$-thin at x, i.e. $'B_x^{\{x\}} = \alpha \varepsilon_x$ with $\alpha < 1$. Obviously $\{x\}$ is not $\mathcal{U}'$-finely open. Let further $(V_n)_n$ be a decreasing sequence of open sets in E with $\cap_n V_n = \{x\}$. If $\{x\}$ is not $\mathcal{U}$-thin at x it follows that $B_x^{\{x\}} = B_x^{\{x\} \cup (E \setminus V_n)} = \varepsilon_x$. On the other hand we have $'B_x^{\{x\} \cup (E \setminus V_n)} 1 \nearrow 1$ and $'B_x^{\{x\} \cup (E \setminus V_n)} \le ({}'B_x^{\{x\}} + {}'B_x^{E \setminus V_n})$, $'B_x^{\{x\} \cup (E \setminus V_n)} \le B_x^{\{x\} \cup (E \setminus V_n)} = \varepsilon_x$. It follows that

$$\theta_n := B_x^{\prime \{x\} \cup (E \setminus V_n)}(1_{\{x\}}) \le {}'B_x^{\{x\}}(1_{\{x\}}) + {}'B_x^{E \setminus V_n}(1_{\{x\}}) = {}'B_x^{\{x\}}(1_{\{x\}})$$

$$'B_x^{\{x\} \cup (E \setminus V_n)} \le \varepsilon_x, {}'B_x^{\{x\} \cup (E \setminus V_n)} = \theta_n \cdot \varepsilon_x$$

and so

$$1 = \sup_n {}'B_x^{\{x\} \cup (E \setminus V_n)} 1 = \sup_n {}'B_x^{\{x\} \cup (E \setminus V_n)}(1_{\{x\}}) = \sup_n \theta_n \le {}'B_x^{\{x\}}(1_{\{x\}}) = \alpha,$$

leading to a contradiction. □

Corollary 1.6. *For any $A \in \mathcal{B}$ and $x \in E$ we have: A is $\mathcal{U}$-thin at x if and only if A is $\mathcal{U}'$-thin at x.*

Proof. From Proposition 1.5 we may assume that $\{x\}$ is $\mathcal{U}$-thin and $\mathcal{U}'$ thin at x and so we may suppose that $x \notin A$. In this case the assertion follows from Proposition 1.2. □

Notation. *Whenever $\mathcal{V}$ is a proper sub-Markovian resolvent of kernels on $(E, \mathcal{B})$, we shall denote by $\preccurlyeq_{\mathcal{E}(\mathcal{V})}$ the specific order with respect to $\mathcal{V}$, i.e. $u \preccurlyeq_{\mathcal{E}(\mathcal{V})} v$ means that there exists $s \in \mathcal{E}(\mathcal{V})$ such that $v = u + s$.*

Theorem 1.7. *For any finite families $(f_i)_{i \in I}$, $(A_i)_{i \in I}$ where $f_i \in bp\mathcal{B}$, $A_i \in \mathcal{B}$ such that A_i is $\mathcal{B} - \mathcal{U}$-basic and $\mathcal{B} - \mathcal{U}'$-basic for all $i \in I$ and any $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ with*

$$\sum_{i \in I} B^{A_i} f_i \le s$$

we have

$$0 \le \sum_{i\in I} (B^{A_i} f_i - {}'B^{A_i} f_i) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s.$$

Proof. Firstly we assume that I has a unique element. Let $f \in pb\mathcal{B}$, $A \in \mathcal{B}$ $\mathcal{B}-\mathcal{U}$-basic and $\mathcal{B}-\mathcal{U}'$-basic, and $s \in b\mathcal{E}(\mathcal{U}) \cap pb\mathcal{B}$ such that $f \le s$. We put

$$u := s - (B^A f - {}'B^A f) - {}'B^A s.$$

We have $u = s - {}'B^A s - (B^A s - {}'B^A s) + B^A(s-f) - {}'B^A(s-f)$ and so $u \ge 0$.

Let furthere $\mathcal{W} = (W_\alpha)_{\alpha\ge 0}$ be the sub-Markovian resolvent on E having as initial kernel W, where $Wf = U'f - {}'B^A U'f$ for all $f \in p\mathcal{B}$ with $U'f < \infty$. Let now $T \in \mathcal{B}$, $T \subset E\backslash A$, be a $\mathcal{W}$-basic set. Then $A \cup T$ is $\mathcal{U}'$-basic set and we have

$$\begin{aligned} {}^{\mathcal{W}}B^T(u) = {}'B^{A\cup T}(u) &= {}'B^{A\cup T}(s - {}'B^A s) + {}'B^{A\cup T}(B^A(s-f)) - {}'B^A(s-f) \\ &\le B^{A\cup T}(s - {}'B^A s) + B^A(s-f) - {}'B^A(s-f) \\ = B^{A\cup T}s - B^A s + B^A(s-f) - {}'B^A(s-f) &= B^{A\cup T}s - (B^A f - {}'B^A f) - B^A s \le u \end{aligned}$$

and so $u \in \mathcal{E}(\mathcal{W})$. Since $u \le s - {}'B^A s$, there exists $t \in \mathcal{E}(\mathcal{U}')$ such that $u = t - {}'B^A t \le s - {}'B^A s$ and so $u + {}'B^A s = t - {}'B^A t + {}'B^A s \in \mathcal{E}(\mathcal{U}')$,

$$s - (B^A f - {}'B^A f) \in \mathcal{E}(\mathcal{U}'), \quad B^A f - {}'B^A f \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s.$$

We consider the general case and let us denote by n_I the cardinal of I. From the previous considerations it follows that the assertion is true for $n_I = 1$. Assume now that the assertion is true for $n_I = n$ and let I with $n_I = n+1$. For any $i \in I$ we put

$$u_i := s - \sum_{j\in I\backslash\{i\}} (B^{A_j} f_j - {}'B^{A_j} f_j)$$

and let $g = s - \sum_{i\in I}(B^{A_i} f_i - {}'B^{A_i} f_i)$, $u = \inf_{i\in I} u_i$. Obviously we have $g \le u$ and $u_i \in \mathcal{E}(\mathcal{U}')$. We want to show that $g \in \mathcal{E}(\mathcal{U}')$. Since g is $\mathcal{U}'$-finely continuous and $g \in p\mathcal{B}$ it follows that ${}'R(g)$ is $\mathcal{U}'$-supermedian and a majorant of g and so it belongs to $\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$.

Let $\alpha \in (0,1)$ and let us consider the $\mathcal{U}$-finely open set $A \in \mathcal{B}$, given by $A := [\alpha {}'R(g) < g]$. We have ${}'B^A({}'R(g)) = {}'R(g)$. Indeed, if $t \in \mathcal{E}(\mathcal{U}')$, $t \ge {}'R(g)$ on A, then we get $(1-\alpha)t + \alpha {}'R(g) \ge g$ on E and so $(1-\alpha)t + \alpha {}'R(g) \ge {}'R(g)$, $t \ge {}'R(g)$. On the other hand let $A_0 := \bar{A}^f \cup (\bigcup_{i\in I} A_i)$, where $\bar{A}^f$ denotes the $\mathcal{U}'$-fine closure of A. We have

$$\begin{aligned} {}'B^{A_0} g &= {}'B^{A_0}\left(\sum_{i\in I} {}'B^{A_i} f_i\right) + {}'B^{A_0}\left(s - \sum_{i\in I} B^{A_i} f_i\right) \\ = \sum_{i\in I} {}'B^{A_i} f_i + {}'B^{A_0}\left(s - \sum_{i\in I} B^{A_i} f_i\right) &\le \sum_{i\in I} B^{A_i} f_i + B^{A_0}\left(s - \sum_{i\in I} B^{A_i} f_i\right) \\ &= \sum_{i\in I} B^{A_i} f_i + B^{A_0} s - \sum_{i\in I} B^{A_i} f_i \le g. \end{aligned}$$

Since $g = u_i$ on A_i and $g \le u$ it follows ${}'Rg \le u \le u_i$ for all $i \in I$ and so $\alpha {}'Rg \le \alpha u_i \le g$ on A_i for all $i \in I$. Hence $\alpha {}'Rg \le g$ on A_0 and so ${}'B^{A_0}(\alpha {}'Rg) \le {}'B^{A_0} g$. From the above considerations we deduce

$${}'Rg = {}'B^{A_0}({}'Rg) \le \frac{1}{\alpha} {}'B^{A_0}(g) \le \frac{1}{\alpha} g, \quad \alpha {}'Rg \le g \quad \text{on } E.$$

The number $\alpha \in (0,1)$ begin arbitrary we get $g = {}'Rg$, $g \in \mathcal{E}(\mathcal{U}')$, completing the proof. ☐

Let P be a kernel on $(E, \mathcal{B})$ such that $P(\mathcal{E}(\mathcal{U})) \subset \mathcal{E}(\mathcal{U})$ and such that $Ps \leq s$ for all $s \in \mathcal{E}(\mathcal{U})$. Then it is known that (cf [10], [4, ch. II], [3]) for any $s \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$ there exists $s' \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$ such that $s' - Ps' = s - Ps$ and moreover if $t \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$ is such that $s' - Ps' \leq t - Pt$ then $s' \leq t$.

A kernel P on $(E, \mathcal{B})$ is called exact subordination operator with respect to $\mathcal{U}$ provided that

a) $P(\mathcal{E}(\mathcal{U})) \subset \mathcal{E}(\mathcal{U})$, $Ps \leq s$ for all $s \in \mathcal{E}(\mathcal{U})$.

b) $\inf(s, Ps + t - Pt + Pf) \in \mathcal{E}(\mathcal{U})$ for all $s, t \in \mathcal{E}(\mathcal{U})$, $t < \infty$, and $f \in p\mathcal{B}$

We recall the following result (cf. [10], [4, ch. V]):

Theorem 1.8. (G. Mokobodzki) *If P is an exact subordination operator with respect to $\mathcal{U}$ there exists a subMarkovian rezolvent of kernels $\mathcal{W} = (W_\alpha)_{\alpha > 0}$ on $(E, \mathcal{B})$ such that*

$$W_\alpha \leq U_\alpha \quad \forall\, \alpha \geq 0 \;\text{ and }\; Wf = Uf - PUf \;\forall\, f \in p\mathcal{B}, Uf < \infty.$$

Moreover if

$$E_P := \{x \in E \mid \exists s \in \mathcal{E}(\mathcal{U}), Ps(x) < s(x)\}$$

then $W_\alpha(1_{E \setminus E_P}) = 0$ and the sub-Markovian resolvent $\mathcal{W}$ considered on E_P is such that $\mathcal{E}(\mathcal{W}) \cap p\mathcal{B}$ is minstable, contains the positive constant functions and generates $\mathcal{B}|_{E_P}$. Further the set $\{s - Ps \mid s \in b\mathcal{E}(\mathcal{U})\}$ is solid in $b\mathcal{E}(\mathcal{W})$ with respect to the natural order.

A sub-Markovian resolvent $\mathcal{W} = (W_\alpha)_{\alpha \geq 0}$ of kernels on $(E, \mathcal{B})$ is called *exactly subordinate* to $\mathcal{U}$ provided that

$$W_\alpha f \leq U_\alpha f \quad \forall \alpha > 0, f \in p\mathcal{B}, Uf < \infty.$$

and

$$Uf - Wf \in \mathcal{E}(\mathcal{U}) \quad \forall\, f \in p\mathcal{B}, Uf < \infty.$$

From Theorem 1.8 it follows that if P is an exact subordination operator with respect to $\mathcal{U}$ then the sub-Markovian resolvent $\mathcal{W} = (W_\alpha)_{\alpha \geq 0}$ associated with P by

$$Wf = Uf - PUf \quad \forall\, f \in p\mathcal{B}, Uf < \infty,$$

is exactly subordinate to $\mathcal{U}$. The following result ([8], [4, ch. V]) represents a converse one.

Theorem 1.9. (P.A. Meyer). *Let $\mathcal{W} = (W_\alpha)_{\alpha \geq 0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ which is exactly subordinate to $\mathcal{U}$. Then there exists an exact subordination operator P with respect to $\mathcal{U}$ such that*

$$Wf = Uf - PUf \quad \forall\, f \in p\mathcal{B}, Uf < \infty.$$

Theorem 1.10. *Let P be a kernel on $(E, \mathcal{B})$ and let $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that $\mathcal{E}(\mathcal{V}) \cap p\mathcal{B}$ is min-stable, $\mathcal{E}(\mathcal{U}) \subset \mathcal{E}(\mathcal{V})$ and*

$$s, t \in b\mathcal{E}(\mathcal{U}), s \leq t \Rightarrow Ps \preccurlyeq_{\mathcal{E}(\mathcal{V})} Pt \preccurlyeq_{\mathcal{E}(\mathcal{V})} t.$$

Then the following assertions are equivalent.

1) P is an exact subordination operator with respect to $\mathcal{U}$ and the set $\{s - Ps \mid s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}\}$ is solid in $\mathcal{E}(\mathcal{V})$ with respect to the natural order.

2) For any $u \in b\mathcal{E}(\mathcal{V}) \cap p\mathcal{B}$ such that there exists $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ with

$$Ps \preccurlyeq_{\mathcal{E}(\mathcal{V})} u \leq s$$

we have $u \in b\mathcal{E}(\mathcal{U})$.

If P satisfies 1) then any subset A of $E \setminus E_P$, $A \in \mathcal{B}$ is absorbent with respect to $\mathcal{V}$.

Proof. 1) $\Rightarrow$ 2). Let $u \in b\mathcal{E}(\mathcal{V}) \cap p\mathcal{B}$ such that there exists $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ with $Ps \preccurlyeq_{\mathcal{E}(\mathcal{V})} u \leq s$. Since $u - Ps \in b\mathcal{E}(\mathcal{V}) \cap p\mathcal{B}$ and $u - Ps \leq s - Ps$, there exists $s' \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ such that $u - Ps = s' - Ps'$. From $u = \inf(s, Ps + s' - Ps')$ it follows that $u \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$.

2) $\Rightarrow$ 1). Let $s_1, s_2 \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ such that $s_2 < \infty$ and $f \in bp\mathcal{B}$. We consider the function u on E given by

$$u = \inf(s_1, Ps_1 + s_2 - Ps_2 + Pf).$$

Since by hypothesis $Pf \in b\mathcal{E}(\mathcal{V}) \cap p\mathcal{B}$ for all $f \in bp\mathcal{B}$ of the form $f = s - t$, where $s, t \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ it follows that $Pf \in b\mathcal{E}(\mathcal{V}) \cap p\mathcal{B}$ for all $f \in bp\mathcal{B}$ and so $Pf \in b\mathcal{E}(\mathcal{V}) \cap p\mathcal{B}$ for all $f \in p\mathcal{B}$. Hence $u \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ and $Ps_1 \preccurlyeq_{\mathcal{E}(\mathcal{V})} u \leq s_1$. From 2) we deduce that $u \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ and therefore P is an exact subordination operator with respect to $\mathcal{U}$.

Let further $v \in b\mathcal{E}(\mathcal{V}) \cap p\mathcal{B}$ and $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ be such that $v \leq s - Ps$. Using 2) we deduce that $v + Ps \in \mathcal{E}(\mathcal{U})$. We consider $\mathcal{W} = (W_\alpha)_{\alpha \geq 0}$ the sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that $Wf = Uf - PUf$ for all $f \in p\mathcal{B}, Uf < \infty$. We denote by w the réduite of v with respect to $\mathcal{W}$. We have $w \in b\mathcal{E}(\mathcal{W}) \cap p\mathcal{B}$, $w \leq s - Ps$ and there exists an increasing sequence $(s_n)_n$ in $b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ such that $s_n \leq s$ for all $n \in \mathbb{N}$ and $s_n - Ps_n \nearrow w$. If we consider the function $s' := \sup_n s_n$ then we have $s' \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$, $v \leq w = s' - Ps'$. We have $v + Ps' \in b\mathcal{E}(\mathcal{U})$, $Ps' \preccurlyeq_{\mathcal{E}(\mathcal{V})} v + Ps' \leq s'$ and so the function s'_0 given by $s'_0 = v + Ps'$ is $\mathcal{U}$-excessive, $s'_0 \leq s$ and $v = s'_0 - Ps' \leq s'_0 - Ps'_0$. Hence $s'_0 - Ps'_0 \geq w = s' - Ps'$, $s'_0 \geq s'$, and so $s'_0 = s'$, $v = s' - Ps'$

Suppose that P satisfies 1). Then we have

$$E_P = \{x \in E \mid Uf_0(x) > PUf_0(x)\}$$

where $f_0 \in p\mathcal{B}$, $0 < f_0 \leq 1$, Uf_0 bounded. Hence $Pf(x) = f(x)$ on $E \backslash E_P$ for all $f \in p\mathcal{B}$. If $A \in \mathcal{B}$, $A \subset E \backslash E_P$ we get $P(1_{E\backslash A}) \in \mathcal{E}(\mathcal{V})$ and $A = [f_0 + P(1_{E\backslash A}) = 0]$ where $Wf_0 = Uf_0 - PUf_0$ and so A is $\mathcal{V}$-absorbent. $\square$

Theorem 1.11. *Assume that $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ is a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ which is exact subordinate to $\mathcal{U}$ and let P the exact subordination operator with respect to $\mathcal{U}$ such that*

$$Vf = Uf - PUf \quad \forall\, f \in p\mathcal{B}, Uf < \infty.$$

We assume that $E_P = E$. Then for any $A \in \mathcal{B}$ we have

$$R^A s - {}^{\mathcal{V}}R^A s = P(R^A s) - {}^{\mathcal{V}}R^A(R^A s))$$

for all $s \in b\mathcal{E}(\mathcal{U})$, where ${}^{\mathcal{V}}R^A s$ denotes the réduite of s on A with respect to $\mathcal{V}$. Particularly for all $A \in \mathcal{B}$ we have

$${}^{\mathcal{V}}R^A \leq R^A.$$

Proof. Since $E_P = E$ it follows by Theorem 1.8 that the fine topologies with respect to $\mathcal{U}$ and $\mathcal{V}$ are the same. Assume firstly that A is finely open with respect to $\mathcal{U}$. Then for all $s \in b\mathcal{E}(\mathcal{U})$ we have $R^A s - P(R^A s) \in \mathcal{E}(\mathcal{V})$, ${}^{\mathcal{V}}R^A(R^A s - PR^A s) \leq R^A s - PR^A s$ and there exists $s' \in b\mathcal{E}(\mathcal{U})$, $s' \leq s$ such that ${}^{\mathcal{V}}R^A(R^A s - PR^A s) = s' - Ps'$. From $s' - Ps' \leq R^A s - PR^A s$ it follows that $u := s' - Ps' + PR^A s \in \mathcal{E}(\mathcal{U})$, $u \leq R^A s$. On the other hand we have $u - {}^{\mathcal{V}}R^A(R^A s - PR^A s) + PR^A s$ $= (R^A s - PR^A s) + PR^A s = R^A s = s$ on A and so $u \geq R^A s, \quad u = R^A s$. Hence

$$R^A s - {}^{\mathcal{V}}R^A s = P(R^A s) - {}^{\mathcal{V}}R^A(P(R^A s))\ \forall s \in b\mathcal{E}(\mathcal{U}).$$

Let now $A \in \mathcal{B}$. For any finite measure μ on $(E, \mathcal{B})$ we have (cf. [4])

$$\mu(R^A s) = \inf\{\mu(R^G s) \mid G \text{ finely open}, G \supset A\}$$
$$\mu({}^{\mathcal{V}}R^A s) = \inf\{\mu({}^{\mathcal{V}}R^G s) \mid G \text{ finely open}, G \supset A\}.$$

On the other hand if G is finely open $G \supset A$ and G_1 is finely open with $A \subset G_1 \subset G$ we get

$$R^G s + {}^{\mathcal{V}}R^G P R^G s = {}^{\mathcal{V}}R^G s + P(R^G s)$$

and so, taking $R^{G_1} s$ instead of s,

$$R^{G_1} s + {}^{\mathcal{V}}R^G (P R^{G_1} s) = {}^{\mathcal{V}}R^{G_1} s + P(R^{G_1} s).$$

We deduce that $R^A s + {}^{\mathcal{V}}R^G (P(R^A s)) = {}^{\mathcal{V}}R^A s + P(R^A s)$ for all finely open set G with $G \supset A$. Since $P(R^A s) \in \mathcal{E}(\mathcal{U})$ we get $R^A s + {}^{\mathcal{V}}R^A (P(R^A s)) = {}^{\mathcal{V}}R^A s + P(R^A s)$. $\square$

Corollary 1.12. *Let $\mathcal{V}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ as in Theorem 1.10. Then for any $A \in \mathcal{B}$ and $x \in E$, A is $\mathcal{V}$-thin at x if and only if A is $\mathcal{U}$-thin at x.*

Proof. The assertion follows from Proposition 1.5 since the sub-Markovian resolvent $\mathcal{V}$ satisfies the two conditions 1), 2) of the resolvent $\mathcal{U}'$ given at the begining of this section. $\square$

2 The techniques for the construction of special exact subordination operators

A σ-balayage with respect to $\mathcal{U}$ is a map $B : \mathcal{E}(\mathcal{U}) \to \mathcal{E}(\mathcal{U})$ such that it is additive, increasing and σ-continuous from below, contractive (i.e. $Bs \leq s$, for all $s \in \mathcal{E}(\mathcal{U})$) and idempotent (i.e. $B^2 = B$). A σ-balayage with respect to $\mathcal{U}$ is called $\mathcal{B} - \sigma$-balayage (with respect to $\mathcal{U}$) if moreover

$$B(\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}) \subset \mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$$

If A is a $\mathcal{U}$-basic set (resp. $\mathcal{B} - \mathcal{U}$-basic set) then the map

$$s \to B^A s$$

is a σ-balayage (resp. $\mathcal{B} - \sigma$-balayage) with respect to $\mathcal{U}$.

Conversely since E is semisaturated with respect to $\mathcal{U}$ then for every σ-balayage (resp. $\mathcal{B} - \sigma$-balayage) B with respect to $\mathcal{U}$ there exists a unique $\mathcal{U}$-basic set (resp. $\mathcal{B} - \mathcal{U}$-basic set) $A := b(B)$ such that

$$Bs = B^A s \quad \forall s \in \mathcal{E}(\mathcal{U}).$$

The set A is called the *base* of B. We denote by $'B$ the σ-balayage with respect to $\mathcal{U}'$ having the same base as B.

Let B be a $\mathcal{B} - \sigma$-balayage with respect to $\mathcal{U}$ such that $'B$ is also a $\mathcal{B} - \sigma$-balayage with respect to $\mathcal{U}'$. We denote by P_B the map

$$P_B : b\mathcal{E}(\mathcal{U}) \cap \mathcal{B} \to b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$$

given by

$$P_{\mathcal{B}} s := Bs - Bs \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} {}'Bs.$$

In the sequel we say simply that B is a σ-balayage if B and $'B$ are simultaneously $\mathcal{B}-\sigma$-balayages.

Proposition 2.1. *The following assertions hold:*

1) $P_B(s+t) = P_Bs + P_Bt$.

2) $s \geq t \Rightarrow P_Bs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_Bt \preccurlyeq_{\mathcal{E}(\mathcal{U}')} t$.

3) $P_B(Bs) = P_Bs$.

4) $s_n \nearrow s \Rightarrow P_Bs_n \nearrow P_Bs$.

5) $Bs - 'Bs = P_Bs - 'BP_Bs$.

Proof. If $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ we have $P_Bs = Bs - Bs \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bs$, $'B(Bs \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bs) = Bs \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bs$, $'BBs = 'Bs$, and so $'B(P_Bs) = 'Bs - Bs \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bs$, $P_Bs - 'B(P_Bs) = Bs - 'Bs$. Analogously if $t \in b\mathcal{E}(\mathcal{U}) \cap pB$ we have $Bt - 'Bt = P_Bt - 'BP_Bt$, $B(s+t) - 'B(s+t) = P_B(s+t) - 'BP_B(s+t)$ and so $P_Bs + P_Bt - 'B(P_Bs + P_Bt) = P_B(s+t) - 'BP_B(s+t)$. Since $P_Bs \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bu = P_Bt \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bu = P_B(s+t) \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bu = 0$ for all $u \in b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$ it follows that $P_B(s+t) = P_Bs + P_Bt$. From the definition of P_B it follows directly $P_B(Bs) = P_Bs$ for all $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$.

Let now $s, t \in b\mathcal{E}(\mathcal{U}) \cap pB$ such that $s \leq t$. We have $P_Bs - 'BP_Bs = Bs - 'Bs$, $P_Bt - 'BP_Bt = Bt - 'Bt$ and so $P_Bt - P_Bs - 'B(P_Bt - P_Bs) = B(t-s) - B'(t-s)$. Since $t - s \in pb\mathcal{B}$ then by Theorem 1.7 there exists $u \in b(\mathcal{E}(\mathcal{U}')$ such that $B(t-s) - 'B(t-s) = u - 'Bu$ and moreover $u \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bu = 0$. We have $P_Bt + 'Bu + 'BP_Bs = P_Bs + u + 'BP_Bt$ since $P_Bs \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bv = P_Bt \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bv = 0$ for all $v \in b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$, we deduce $P_Bt = P_Bs + u$, i.e. $P_Bs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_Bt$. On the other hand we get $Bt - 'Bt \preccurlyeq_{\mathcal{E}(\mathcal{U}')} t$, $P_Bt = (Bt - 'Bt) \curlyvee_{\mathcal{E}(\mathcal{U}')} 0 \preccurlyeq_{\mathcal{E}(\mathcal{U}')} t$.

Let now $(s_n)_n$ be an increasing sequence in $b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$, $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ such that $s = \sup_n s_n$. We have $P_Bs_n - 'BP_Bs_n = Bs_n - 'Bs_n$, $P_Bs_n \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_Bs_{n+1}$ for all $n \in \mathbb{N}$ and so $Bs - B's = \sup_n P_Bs_n - 'B(\sup_n P_Bs_n)$. On the other hand we have $Bs - B's = P_Bs - 'BP_Bs$, $\sup_n P_Bs_n = \curlyvee_{\mathcal{E}(\mathcal{U}')} P_Bs_n$. Since $P_Bt \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bv = 0$ for all $v \in b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$, $t \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$, it follows that $(\sup_n P_Bs_n) \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} 'Bv = 0$ for all $v \in b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$. From $\sup_n P_Bs_n - 'B(\sup_n P_Bs_n) = P_Bs - 'BP_Bs$ we deduce that $\sup_n P_Bs_n = P_Bs$. □

Proposition 2.2. *For any σ-balayage B there exists a unique kernel on $(E, \mathcal{B})$ denoted also by P_B such that*

$$P_Bs = Bs - Bs \underset{\mathcal{E}(\mathcal{U}')}{\curlywedge} B's \quad \forall\, s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}.$$

Moreover we have

i) $f \in p\mathcal{B} \Rightarrow P_Bf \in \mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$.

ii) $Bf - 'Bf = P_Bf - 'BP_Bf \ \forall\, f \in bp\mathcal{B}$.

Proof. For any $x \in E$ the map $bp\mathcal{B} \ni f \longmapsto P_BUf(x)$ is an $\mathcal{U}$-excessive measure dominated by $\varepsilon_x \circ U$ and so it is $\mathcal{U}$-potential, i.e. there exists a measure $P_{B,x}$ on $(E, \mathcal{B})$ such that

$$P_{B,x}s = P_Bs(x) \quad \forall\, s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}.$$

Since for any $f \in (b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B} - b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B})_+$ the function $x \longmapsto P_{B,x}f$ is $\mathcal{B}$-measurable it follows that the function P_Bf given $P_Bf(x) = P_{B,x}(f)$ is $\mathcal{B}$-measurable and so the map $p\mathcal{B} \ni f \longmapsto P_Bf \in p\mathcal{B}$ is a kernel on $(E, \mathcal{B})$. Since for all $f \in (b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B} - b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B})_+$ we have $P_Bf \in \mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$ and $Bf - 'Bf = P_Bf - 'BP_Bf$ it follows that the same assertion hold for all $f \in bp\mathcal{B}$. □

Proposition 2.3. *For any $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ and any finite systems $(s_i)_{i\in I}$ in $b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ and $(B_i)_{i\in I}$ of σ-balayages such that $\sum_{i\in I} s_i \le s$ we have*

$$\sum_{i\in I} P_{B_i} s_i \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s.$$

Proof. We have $\sum_{i\in I} B_i s_i \le s$ and therefore from Theorem 1.7 we obtain $\sum_{i\in I}(B_i s_i - {}'B_i s_i) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s$. From the definition of P_{B_i} we deduce that the relation

$$\sum_{i\in I} P_{B_i} s_i \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s$$

is equivalent with the relation

$$\sum_{i\in I} B_i s_i \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s + \sum_{i\in I} B_i s_i \curlywedge_{\mathcal{E}(\mathcal{U}')} {}'B_i s_i.$$

On the other hand we have inductively (following card I)

$$s + \sum_{i\in I} B_i s_i \curlywedge_{\mathcal{E}(\mathcal{U}')} {}'B_i s_i = \curlywedge_{\substack{\mathcal{E}(\mathcal{U}') \\ J\subset I}} (s + \sum_{j\in J} B_j s_j + \sum_{j\in I\setminus J} {}'B_j s_j).$$

Moreover we have

$$\sum_{j\in I\setminus J} B_j s_j \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s + \sum_{j\in I\setminus J} {}'B_j s_i$$

or equivalently

$$\sum_{i\in I} B_i s_i \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s + \sum_{j\in J} B_i s_i + \sum_{j\in I\setminus J} {}'B_j s_j$$

and so

$$\sum_{i\in I} B_i s_i \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \curlywedge_{\mathcal{E}(\mathcal{U}')} (s + \sum_{j\in J} B_i s_i + \sum_{j\in I\setminus J} {}'B_j s_{ij}) =$$
$$= s + \textstyle\sum_{i\in I} B_i s_i \curlywedge_{\mathcal{E}(\mathcal{U}')} {}'B_i s_i.$$

□

Proposition 2.4. *Let $f \in bp\mathcal{B}$ and B be a σ-balayage such that $P_B f \preccurlyeq_{\mathcal{E}(\mathcal{U}')} f$. Then we have $Bf \le f$. Particularly if $P_B f \preccurlyeq_{\mathcal{E}(\mathcal{U}')} f$ for all σ-balayage B then $f \in \mathcal{E}(\mathcal{U})$.*

Proof. From $P_B f \preccurlyeq_{\mathcal{E}(\mathcal{U}')} f$ there exists $u \in b\mathcal{E}(\mathcal{U}')$ such that $P_B f + u = f$. On the other hand we have $Bf - {}'Bf = P_B f - {}'BP_B f$ and so $P_B f = Bf - {}'Bf + {}'BP_B f$, $f = u + Bf - {}'Bf + {}'BP_B f$. Hence ${}'Bf = {}'Bu + {}'Bf - {}'Bf + {}'BP_B f = {}'Bu + {}'BP_B f \le u + {}'BP_B f$ and therefore $Bf \le f$.

Assume that $P_B f \preccurlyeq_{\mathcal{E}(\mathcal{U}')} f$ for all σ-balayage B. Then f is $\mathcal{U}$-finely continuous and moreover $Bf \le f$ for all σ-balayage B. Hence $f = R(f)$, $f \in \mathcal{E}(\mathcal{U})$. □

Lemma 2.5. *Let $(P_n)_{n\in\mathbb{N}^*}$ be a sequence of kernels on $(E, \mathcal{B})$ such that*

a) $P_n(bp\mathcal{B}) \subset b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$.

b) $\displaystyle\sum_{\substack{k=1 \\ s_k, s\in b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}}}^{\infty} s_k \le s \Rightarrow \sum_{k=1}^{\infty} P_k s_k \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s.$

Let further $\mathcal{F}$ be a countable subset of bpB such that $\mathcal{F}+\mathcal{F}\subset\mathcal{F}$, $Q_+,\mathcal{F}\subset\mathcal{F}$, $U(\mathcal{F})\subset b\mathcal{E}(\mathcal{U})$ and $\mathcal{S}_0$ be a countable subset of $b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}$ such that

$$U(\mathcal{F})\subset\mathcal{S}_0; Q_+\mathcal{S}_0\subset\mathcal{S}_0$$

$$s,t\in\mathcal{S}_0\Rightarrow s+t, s\curlyvee_{\mathcal{E}(\mathcal{U})}t, R(s-t)\in\mathcal{S}_0.$$

Then the map

$$P_0: U(\mathcal{F})\to b\mathcal{E}(\mathcal{U}')\cap p\mathcal{B}$$

defined by

$$P_0(Uf):=\curlyvee_{\mathcal{E}(\mathcal{U}')}\left\{\sum_{k=1}^n P_k s_k \mid (s_k)_{1\le k\le n}\subset\mathcal{S}_0, \sum_{k=1}^n s_k\le Uf\right\}$$

possesses the following properties:

i) $P_0(Uf_1+Uf_2)=P_0(Uf_1)+P_0(Uf_2)$ for all $f_1,f_2\in\mathcal{F}$.

ii) $P_n(Uf)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}P_0(Uf)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}Uf$ for all $f\in\mathcal{F}$ and $n\in\mathbb{N}^*$.

iii) $f_1,f_2\in\mathcal{F}$, $s\in b\mathcal{E}(\mathcal{U}), Uf_1\le Uf_2+s\Rightarrow P_0(Uf_1)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}P_0(Uf_2)+s$.

Proof. Since $\mathcal{S}_0$ is countable it follows that the element $P_0(Uf)$ is well defined in $\mathcal{E}(\mathcal{U}')$ for all $f\in\mathcal{F}$. We show now the property iii). Let $f_1,f_2\in\mathcal{F}$, $s\in b\mathcal{E}\cap p\mathcal{B}$ be such that $Uf_1\le Uf_2+s$. We consider a system $(s_i)_{1\le i\le n}$ in $\mathcal{S}_0$ such that $\sum_{i=1}^m s_i\le Uf_1$. From $Uf_1\le Uf_2+s$ there exists $s',s''\in\mathcal{S}_0$ such that $\sum_{i=1}^m s_i=s'+s''$, $\quad s'\le Uf_2$, $\quad s''\le s$. Hence for any $1\le i\le n$ there exists $u_i,v_i\in\mathcal{S}_0$ such that $s_i=u_i+v_i$, $\quad s'=\sum_{i=1}^m u_i$, $s''=\sum_{k=1}^m v_k$. Since $\sum_{i=1}^m u_i=s'\le U(f_2)$, $\sum_{i=1}^m v_i\le s$ we get $\sum_{i=1}^m P_i(u_i)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}P_0(Uf_2)$, $\sum_{i=1}^m P_i(v_i)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}s$, $\sum_{i=1}^m P_i s_i=\sum_{i=1}^m P_i(u_i+v_i)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}P_0(Uf_2)+s$, $P_0(Uf_1)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}P_0(Uf_2)+s$.

We show that P_0 is additive. Let $f_1,f_2\in\mathcal{F}$. We have directly

$$P_0(Uf_1)+P_0(Uf_2)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}P_0(Uf_1+Uf_2).$$

Let now $(s_i)_{1\le i\le n}$ a finite system in $\mathcal{S}_0$ such that $\sum_{i=1}^n s_i\le Uf_1+Uf_2$. From the properties of $\mathcal{S}_0$ there exists two systems $(u_i)_{1\le i\le n}$, $(v_i)_{1\le i\le n}$ in $\mathcal{S}_0$ such that $s_i=u_i+v_i\quad 1\le i\le n$ and $\sum_{i=1}^m u_i\le Uf_1$, $\quad\sum_{i=1}^n v_i\le Uf_2$. Hence we define

$$\sum_{i=1}^m P_i s_i=\sum_{i=1} P_i u_i+\sum_{i=1}^n P_i v_i\preccurlyeq_{\mathcal{E}(\mathcal{U}')}P_0(Uf_1)+P_0(Uf_2)$$

and so

$$P_0(Uf_1+Uf_2)\preccurlyeq_{\mathcal{E}(\mathcal{U}')}P_0(Uf_1)+P_0(Uf_2).$$

The property ii) follows directly from the definition of P_0 and from the property b) of the given sequence $(P_n)_n$. □

Lemma 2.6. *Let $\mathcal{F}$ be a subset of $bp\mathcal{B}$ such that*

$$\mathcal{F}+\mathcal{F}\subset\mathcal{F}, Q_+\mathcal{F}\subset\mathcal{F}$$

$$f_1,f_2\in\mathcal{F}\Rightarrow\sup(f_1,f_2),\inf(f_1,f_2)\in\mathcal{F}$$

$$f_1,f_2\in\mathcal{F}, f_1\le f_2\Rightarrow f_2-f_1\in\mathcal{F}$$

and such that the monoton class $\mathcal{M}(\mathcal{F})$ in $bp\mathcal{B}$ conincides with $bp\mathcal{B}$. Assume that U is bounded and let

$$P_0: U(\mathcal{F})\to b\mathcal{E}(\mathcal{U}')\cap p\mathcal{B}$$

be a map such that

i) $P_0(Uf_1 + Uf_2) = P_0(Uf_1) + P_0(Uf_2)$

ii) $P_0(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Uf \; \forall f \in \mathcal{F}$

iii) $f_1, f_2 \in \mathcal{F}$, $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$,

$Uf_1 \leq Uf_2 + s \quad \Rightarrow P_0(Uf_1) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} + P_0(Uf_2) + s$. *Then there a kernel* $\widetilde{P}_o$ *on* $(E, \mathcal{B})$ *uniquely determined such that*

1) $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B} \Rightarrow \widetilde{P}_0 s \in b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$

2) $f \in \mathcal{F} \Rightarrow \widetilde{P}_0(Uf) = P_0(Uf)$

3) $s, t \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$, $s \leq t \Rightarrow \widetilde{P}_0 s \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(t) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} t$.

If moreover T *is a kernel on* $(E, \mathcal{B}^{(u)})$ *such that* $T(b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}) \subset b\mathcal{E}(\mathcal{U}')$ *and*

$$T(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_0(Uf) \quad \forall\, f \in \mathcal{F}$$

then

$$Ts \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0 s, \quad \forall s \in b\mathcal{E}(\mathcal{U}).$$

Proof. We denote for any non empty subset $\mathcal{A}$ of $bp\mathcal{B}$, by $\mathcal{A}_\sigma$, $\mathcal{A}_\delta$ the sets

$$\mathcal{A}_\sigma = \{f \in bp\mathcal{B} \mid \exists (f_n) \subset \mathcal{A}, f_n \nearrow f\}$$

$$\mathcal{A}_\delta = \{f \in bp\mathcal{B} \mid \exists (f_n)_n \subset \mathcal{A}, f_n \searrow f\}.$$

We remark that if $\mathcal{A}$ possesses the properties

i) $\mathcal{A} + \mathcal{A} \subset \mathcal{A}, \quad Q_+ \mathcal{A} \subset \mathcal{A}$

ii) $f_1, f_2 \in \mathcal{A} \Rightarrow \sup(f_1, f_2), \inf(f_2, f_2) \in \mathcal{A}$

then $\mathcal{A}_\sigma$ and $\mathcal{A}_\delta$ posesses also the same properties 1), 2).

If Ω is the first uncountable ordinal number we define inductively the subsets

$$\mathcal{F}^1 = \mathcal{F}_\sigma, \mathcal{F}_1 = (\mathcal{F}^1)_\delta, \mathcal{F}^2 = (\mathcal{F}_1)_\sigma, \mathcal{F}_2 = (\mathcal{F}^2)_\delta$$

and for any ordinal number $\alpha < \Omega$

$$\mathcal{F}^{\alpha+1} = (\mathcal{F}_\alpha)_\sigma, \quad \mathcal{F}_{\alpha+1} = (\mathcal{F}^{\alpha+1})_\delta, \text{ and } \mathcal{F}^\alpha = \cup_{\beta<\alpha} \mathcal{F}_\beta, \quad \mathcal{F}_\alpha = (\mathcal{F}^\alpha)_\delta$$

if α does not possess a precedent. Obviously we have

$$\alpha < \beta \Rightarrow \mathcal{F}^\alpha \subset \mathcal{F}_\alpha \subset \mathcal{F}^\beta, \text{ and } \cup_{\alpha<\beta} \mathcal{F}^\alpha = \cup_{\alpha<\beta} \mathcal{F}_\alpha$$

if β does not possesses a precedent.

If $\mathcal{M}(\mathcal{F})$ is the monoton class in $bp\mathcal{B}$ generated by $\mathcal{F}$ (i.e. the smallest subset $\mathcal{A}$ of $bp\mathcal{B}$ with $\mathcal{F} \subset \mathcal{A}$ such that for any uniformly bounded monoton sequence $(f_n)_n$ in $\mathcal{A}$ we have $\lim_{n\to\infty} f_n \in \mathcal{A}$) then we have

$$\mathcal{M}(\mathcal{F}) = \cup_{\alpha \in \Omega} \mathcal{F}^\alpha = \cup_{\alpha \in \Omega} \mathcal{F}_\alpha.$$

Firstly two remarks:

1) If $f, g \in \mathcal{F}$ are such that $Uf \leq Ug$ then $P_0(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_0(Ug)$. This fact follows from the property iii) of P_0.

2) If $(f_n)_n$, $(g_n)_n$ are two sequence in $\mathcal{F}$ such that $(Uf_n)_n$, $(Ug_n)_n$ are increasing and

$$\sup_n Uf_n \leq \sup_n Ug_n$$

then

$$\sup_n P_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \sup_n P_0(Ug_n).$$

Particularly if $\sup_n Uf_n = \sup_n Ug_n$ we have

$$\sup_n P_0 Uf_n = \sup_n P_0(Ug_n).$$

Indeed, if we denote $r_{n,m} := R(Uf_n - Ug_m)$ then we have $r_{n,m} \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$, $Uf_n \leq Ug_n + r_{n,m}$, $\inf_m r_{n,m} = 0$ and therefore, using the property iii) of P_0, we deduce that $P_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_0(Ug_m) + r_{n,m} \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \sup_m P_0(Ug_m) + r_{n,m}$, $P_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \sup_m P_0(Ug_m)$ for all $n \in \mathbb{N}$, $\sup_n P_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \sup_n P_0(Ug_n)$.

Form the above considerations it follows that for any $f \in \mathcal{F}_\sigma$ the element $\widetilde{P}_0(Uf)$ from $\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$ defined by

$$\widetilde{P}_0(Uf) := \sup_n P_0(Uf_n)$$

where $(f_n)_n$ is an increasing sequence in $\mathcal{F}$ with $f_n \nearrow f$ depends only by Uf. Also we have

$$f_1, f_2 \in \mathcal{F}_\sigma \Rightarrow \widetilde{P}_0(Uf_1 + Uf_2) = \widetilde{P}_0(Uf_1) + \widetilde{P}_0(Uf_2).$$

If $f \in \mathcal{F}_\sigma$ and $(f_n)_n$ is an increasing sequence in $\mathcal{F}$ with $f_n \nearrow f$ then using the property ii) of P_0 we get

$$P_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Uf_n \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Uf \text{ and so } \widetilde{P}_0(Uf) = \sup_n P_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Uf.$$

Let now $f, g \in \mathcal{F}_\sigma$ and $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ be such that $Uf \leq Ug + s$. We show that

$$\widetilde{P}_0(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(Ug) + s.$$

Indeed, let $(f_n)_n$ (resp. $(g_n)_n$) be an increasing sequence in $\mathcal{F}$ such that $f_n \nearrow f$, $g_n \nearrow g$. For any $n \in \mathbb{N}$ we have $Uf_n \leq Ug_n + U(g - g_n) + s$ and therefore, from the property iii) of P_0 we get $P_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_0(Ug_n) + U(g-g_n) + s \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(Ug) + U(g - g_n) + s$. Since $\curlywedge_n U(g - g_n) = 0$ we get $\widetilde{P}_0(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(Ug) + s$. Hence the map $\widetilde{P}_0 : \mathcal{U}(\mathcal{F}_\sigma) \to b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$ is an extension of P_0 which possesses the same properties i), ii), iii), as P_0. Similarly if $f \in \mathcal{F}_\sigma$ and $(f_n)_n$ is a decreasing sequence in $\mathcal{F}$, such that $f = \inf_n f_n$ the element of $b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$ defined by

$$\widetilde{P}_0(Uf) = \inf_n P_0(Uf_n)$$

does not depend on the sequence $(f_n)_n$ and we have

i) $\widetilde{P}_0(Uf_1 + Uf_2) = \widetilde{P}_0(Uf_1) + \widetilde{P}_0(Uf_2)\ \forall f_1, f_2 \in \mathcal{F}_\delta$

ii) $\widetilde{P}_0(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_0(Uf)\ \forall f \in \mathcal{F}_\delta$

iii) if $f, g \in \mathcal{F}_\delta$ and $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ are such that $Uf \leq Ug + s$ then $\widetilde{P}_0(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(Ug) + s$.

From the above consideration it follows that there exists a map

$$\widetilde{P}_0 : U(\mathcal{M}(\mathcal{F})) \to b\mathcal{E}(\mathcal{U}') \cap p\mathcal{F}$$

which is an extension of the map

$$P_0 : U(\mathcal{F}) \to b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B},$$

such that

i) $\widetilde{P}_0(Uf_1 + Uf_2) = \widetilde{P}_0(Uf_1) + \widetilde{P}_0(Uf_2)\ \forall f_1, f_2 \in \mathcal{M}(\mathcal{F})$

ii) $f \in \mathcal{M}(\mathcal{F}) \Rightarrow \widetilde{P}_0(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Uf$

iii) $f_1, f_2 \in \mathcal{M}(\mathcal{F}), s \in \mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$, $Uf_1 \le Uf_2 + s \Rightarrow \hat{P}_0 Uf_1 \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \hat{P}_0 Uf_2 + s$.

Let now $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ and let $(f_n)_n$ a sequence in $\mathcal{M}(\mathcal{F})$ such that $Uf_n \nearrow s$. Obviously we have

$$\widetilde{P}_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(Uf_{n+1}) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s.$$

By the above remark 2) it follows that the element from $b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$ define by

$$\widetilde{P}_0 s := \sup_n \widetilde{P}_0 Uf_n$$

does not depend on the sequence $(Uf_n)_n$ as above. Using the above definition of $\widetilde{P}_0 s$ with $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ we have immediately

$$\widetilde{P}_0(s+t) = \widetilde{P}_0 s + \widetilde{P}_0 t, \widetilde{P}_0 s \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s$$

and

$$s, t \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}, s \le t \Rightarrow Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Pt.$$

Let now $s \in \mathcal{E}(\mathcal{U} \cap p\mathcal{B}$ and $(s_n)_n$ an increasing sequence in $b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ such that $s_n \nearrow s$. We show that $\widetilde{P}_0 s_n \nearrow \widetilde{P}_0 s$. Let $(f_n)_n$ be a sequence in $\mathcal{M}(\mathcal{F})$ such that $Uf_n \nearrow s$. From $Uf_n \le s_m + R(Uf_n - s_m)$ we deduce

$$\widetilde{P}_0(Uf_n) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(s_m) + R(Uf_n - s_m).$$

Since $\inf_m R(Uf_n - s_m) = 0$ we get $\widetilde{P}_0(Uf_1) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \sup_m \widetilde{P}_0(s_m)$ and so

$$\widetilde{P}_0 s \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \sup_m \widetilde{P}_0(s_m), \widetilde{P}_0 = \sup_n \widetilde{P}_0(s_n).$$

Since E is semisaturated with respect to $\mathcal{U}$ it follows that the map

$$\widetilde{P}_0 : b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B} \to b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$$

can be extended to a kernel on $(E, \mathcal{B})$ denoted also by $\widetilde{P}_0$. By construction $\widetilde{P}_0$ satisfies the required conditions 1)- 3). Using the property $\mathcal{M}(\mathcal{F}) = bp\mathcal{B}$ and $\widetilde{P}_0 \mid_{\mathcal{U}(\mathcal{F})} = P_0$ it is easy to see that $\widetilde{P}_0$ is uniquely determined.

Let now T be a kernel on $(E, \mathcal{B}^{(u)})$ such that $T(b\mathcal{E}(\mathcal{U})) \subset b\mathcal{E}(\mathcal{U}')$ and

$$T(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_0(Uf) \ \ \forall f \in \mathcal{F}.$$

We have $T(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(Uf)$ for all $f \in \mathcal{M}(\mathcal{F})$ and so $Ts \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \widetilde{P}_0(s)$ for all $s \in b\mathcal{E}(\mathcal{U})$. □

Theorem 2.7. *Let $\mathcal{T}_0$ be a countable base of the topology $\mathcal{T}$ such that*

$$G_1, G_2 \in \mathcal{T}_0 \Rightarrow G_1 \cup G_2, G_1 \cap G_2 \in \mathcal{T}_0,$$

let $\mathcal{F}$ be a countable subset of $pb\mathcal{B}$ such that $\mathcal{F} + \mathcal{F} \subset \mathcal{F}$, $Q_+\mathcal{F} \subset \mathcal{F}$

$$f_1, f_2 \in \mathcal{F} \Rightarrow \sup(f_1, f_2), \inf(f_1, f_2) \in \mathcal{F},$$

$$f_1, f_2 \in \mathcal{F}, f_1 \le f_2 \Rightarrow f_2 - f_1 \in \mathcal{F},$$

there exists $f_0 \in \mathcal{F}$, $0 < f_0 \leq 1$ with Uf_0 is bounded and such that the monotone classe in $bp\mathcal{B}$ generated by $\mathcal{F}$ coincides with $bp\mathcal{B}$. Let further $\mathcal{S}_0$ be a countable subset of $b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ such that $U(\mathcal{F}) \subset \mathcal{S}_0$, $Q_+\mathcal{S}_0 \subset \mathcal{S}_0$ and

$$s, t \in \mathcal{S}_0 \Rightarrow s + t, s \underset{\mathcal{E}(\mathcal{U})}{\curlyvee} t, R(s-t) \in \mathcal{S}_0.$$

Then there exists a kernel P on $(E, \mathcal{B})$ uniquely determined such that
1) $P(b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}) \subset b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$
2) $s, t \in b\mathcal{E}(\mu), s \leq t \Rightarrow Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Pt \preccurlyeq_{\mathcal{E}(\mathcal{U}')} t$
3) $f \in \mathcal{F} \Rightarrow P(Uf) = \curlyvee_{\mathcal{E}(\mathcal{U}')} \left\{ \sum_{i=1}^{n} P_{B^{G_i}} s_i \mid \sum_{i=1}^{n} s_i \leq Uf, s_i \in \mathcal{S}_0, G_i \in \mathcal{T}_0 \right\}$
Particularly we have

$$P_{B^G} s \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s \quad \forall\, G \in \mathcal{T}_0,\ s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}.$$

Proof. If we replace U by the kernel $f \to U(f_0 \cdot f)$ we may assume that U is bounded. Using Lemma 2.5 and Proposition 2.3 we deduce that the map $P_0 : \mathcal{U}(\mathcal{F}) \to b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$ given by

$$P_0(Uf) = \curlyvee_{\mathcal{E}(\mathcal{U}')} \left\{ \sum_{i=1}^{n} P_{B^{G_i}} s_i \mid \sum_{i=1}^{n} s_i \leq Uf, s_i \in \mathcal{S}_0, G_i \in \mathcal{T}_0 \right\}$$

verifies the following properties:
1) $P_0(Uf_1 + Uf_2) = P_0(Uf_1) + P_0(Uf_2)$ for all $f_1, f_2 \in \mathcal{F}$.
2) $P_{B^G}(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_0(Uf) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Uf$ for all $f_1, f_2 \in \mathcal{F}$.
3) $f_1, f_2 \in \mathcal{F}, s \in b\mathcal{E}(\mathcal{U}), Uf_1 \leq Uf_2 + s \Rightarrow P_0(Uf_1) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} P_0(Uf_2) + s$.
Using Lemma 2.6 it follows that there exists a kernel P on $(E, \mathcal{B})$ uniquely determined such that
1) $P(Uf) = P_0(Uf) \quad \forall f \in \mathcal{F}$
2) $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B} \Rightarrow Ps \in b\mathcal{E}(\mathcal{U}') \cap p\mathcal{B}$
3) $s, t \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B},\ s \leq t \Rightarrow Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Pt \preccurlyeq_{\mathcal{E}(\mathcal{U}')} t$.
From the definition of P it follows $P_{B^G} Uf \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Uf$ for all $f \in \mathcal{F}$ and $G \in \mathcal{T}_0$ and so

$$P_{B^G} s \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s \quad \forall s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}.$$

□

3 Shih's Theorem

We recall (see [2], [4, ch. I]) that a *Ray cone* on $(E, \mathcal{B})$ with respect to $\mathcal{U}$ is a convex cone $\mathcal{R}$ of bounded $\mathcal{B}$-measurable $\mathcal{U}$-excessive functions such that there exists a bounded sub Markovian resolvent $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ on $(E, \mathcal{B})$ such that $(p\mathcal{B}) \cap \mathcal{E}(\mathcal{U}) = (p\mathcal{B}) \cap \mathcal{E}(\mathcal{V})$ and such that

1) $1 \in \mathcal{R}$; 2) $\mathcal{R}$ is min-stable; 3) $V_0(p(\mathcal{R} - \mathcal{R})) \subset \mathcal{R}$; 4) $V_\alpha(\mathcal{R}) \subset \mathcal{R}\ \forall\ \alpha > 0$, 5) $\mathcal{R}$ is separable with respect to the uniform norm; 6) The σ-algebra on E generated by $\mathcal{R}$ coincides with $\mathcal{B}$.

For any countable subset $\mathcal{S}$ of $bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$, there exists a Ray cone $\mathcal{R}$ with $\mathcal{R} \supset \mathcal{S}$.

If $\mathcal{R}$ is a Ray cone with respect to $\mathcal{U}$ then the topology $\mathcal{T}_\mathcal{R}$ on E generated by $\mathcal{R}$, called *Ray topology*, is such that the topological space $(E, \mathcal{T}_\mathcal{R})$ is a Lusin topological space such that the Borel sets in $(E, \mathcal{T}_\mathcal{R})$ are exactly the Borel sets in $(E, \mathcal{T})$. Moreover from ([2], Proposition 1.2) it follows that for any compact subset K in $(E, \mathcal{T}_\mathcal{R})$, any $s \in \mathcal{R}$ and any finite measure μ on $(E, \mathcal{B})$ we have

$$\mu(R^K s) = \inf\{\mu(R^G s) \mid G \in \mathcal{T}_\mathcal{R}, K \subset G\}.$$

Let further $\mathcal{R}'$ be a Ray cone with respect to $\mathcal{U}'$ and $\mathcal{T}_{\mathcal{R}'}$ the Ray topology on E generated by $\mathcal{R}'$. Using the above considerations the topological space $(E, \mathcal{T}_{\mathcal{R}'})$ is a Lusin topological space for which the Borel sets are exactly the Borel sets from $(E, \mathcal{T})$. Also for any compact subset K of $(E, \mathcal{T}_{\mathcal{R}'})$, any $t \in \mathcal{R}'$ and any finite measure μ on $(E, \mathcal{B})$ we have

$$\mu('R^K t) = \{\inf \mu('R^G s) \mid G \in \mathcal{T}_{\mathcal{R}'}, K \subset G\}.$$

In the sequel since $(p\mathcal{B}) \cap \mathcal{E}(\mathcal{U}) \subset (p\mathcal{B}) \cap \mathcal{E}(\mathcal{U}')$ we can choose $\mathcal{R}$ and $\mathcal{R}'$ such that $\mathcal{R} \subset \mathcal{R}'$ and moreover there exists $f_0 \in p\mathcal{B}$, $0 < f_0 \leq 1$, such that Vf_0 is bounded and $Uf_0 \in \mathcal{R}$. For simplicity we write $\mathcal{T}' = \mathcal{T}_{\mathcal{R}'}$. Consequently for any compact subset K in $(E, \mathcal{T}')$ and any finite measure μ on $(E, \mathcal{B})$ we get

$$\mu(R^K Uf_0) = \inf \mu(R^G Uf_0) \mid G \in \mathcal{T}', G \supset K\},\ \mu('R^K Uf_0) = \inf \mu('R^G Uf_0) \mid G \in \mathcal{T}', G \supset K\}./$$

Remark. If $\mathcal{U}$ (resp. $\mathcal{U}'$) is the sub-Markovian resolvent associated with the right process X (resp. X') with $(E, \mathcal{T})$ as state space, then the preceding relations hold replacing $\mathcal{T}$ instead of $\mathcal{T}'$ if the processes X and X' are assumed to be Hunt processes.

Proposition 3.1. *Let K be a $\mathcal{T}'$-compact subset of E, μ be a finite measure on $(E, \mathcal{B})$ and s be a bounded regular $\mathcal{U}$-excessive function, $\mathcal{B}$-measurable. Then we have*

$$\mu(R^K s) = \inf\{\mu(R^G s) \mid G \in \mathcal{T}', G \supset K\}$$

$$\mu('R^K s) = \inf\{\mu('R^G s) \mid G \in \mathcal{T}', G \supset K\}.$$

Proof. Firstly let $f \in bp\mathcal{B}$ such that $f \leq f_0$. We have

$$\mu(R^K Uf) \leq \inf\{\mu(R^G(Uf)) \mid G \in \mathcal{T}', G \supset K\}$$

$$\mu(R^K U(f_0 - f)) \leq \inf\{\mu(R^G U(f_0 - f)) \mid G \in \mathcal{T}', G \supset K\}$$

$$\inf_{\substack{G \supset K \\ G \in \mathcal{T}}} \mu(R^G(Uf)) + \inf_{\substack{G \supset K \\ G \in \mathcal{T}'}} \mu(R^G(U(f_0 - f)) =$$

$$= \inf_{\substack{G \supset K \\ G \in \mathcal{T}'}} \mu(R^G(Uf_0)) = \mu(R^K(Uf_0)) = \mu(R^K(Uf)) + \mu(R^K(U(f_0 - f)))$$

and therefore

$$\mu(R^K Uf) = \inf_{\substack{G \supset K \\ G \in \mathcal{T}'}} \mu(R^G Uf).$$

Analogously we get $\mu('R^K Uf) = \inf_{\substack{G \supset K \\ G \in \mathcal{T}'}} \mu('RUf)$. Let further $(f_m)_m$ be a sequence in $p\mathcal{B}$ such that $f_m \leq mf_0$ for all $m \in \mathbb{N}$ and $Uf_m \nearrow s$. Since s in regular we get $\inf_m(s - Uf_m) = 0$. For all $m \in \mathbb{N}$ we have $Uf_m \leq s \leq Uf_m + \mathcal{R}(s - Uf_m)$ and so

$$R^G Uf_m \leq R^G s \leq R^G Uf_m + \mathcal{R}(s - Uf_m),\ R^G Uf_m \leq 'R^G s \leq 'R^G Uf_m + R(s - Uf_m)$$

for all $G \in \mathcal{T}'$. From the first part of the proof we get

$$\mu(R^K Uf_m) = \inf_{\substack{G \supset K \\ G \in \mathcal{T}'}} \mu(R^G Uf_m),\ \mu('R^K Uf_m) = \inf_{\substack{G \supset K \\ G \in \mathcal{T}'}} \mu(R^G Uf_m)\ \forall m \in \mathbb{N}$$

and so

$$\inf_{\substack{G\supset K\\ G\in\mathcal{T}'}} \mu(R^G s) \le \mu(R^K U f_m) + \mu(R(s-Uf_m)),$$
$$\inf_{\substack{G\supset K\\ G\in\mathcal{T}'}} \mu('R^G s) \le \mu('R^K U f_m) + \mu(R(s-Uf_m)).$$

Since $\inf_m R(s-Uf_m)=0$ we get

$$\inf_{\substack{G\supset K\\ G\in\mathcal{T}'}} \mu(R^G s) \le \mu(R^K s),\ \inf_{\substack{G\supset K\\ G\in\mathcal{T}'}} \mu('R^G s) \le \mu('R^K s).$$

□

In the sequel we denote by $\mathcal{F}$ a countable subset of $bp\mathcal{B}$ such that $f_0\in\mathcal{F}$ and

$$f\in\mathcal{F}\Rightarrow \exists\,\alpha>0 \text{ with } f\le \alpha f_0,\ \mathcal{F}+\mathcal{F}\subset\mathcal{F}; Q_+\mathcal{F}\subset\mathcal{F},$$

$$f_1,f_2\in\mathcal{F}\Rightarrow \sup(f_1,f_2),\inf(f_1,f_2)\in\mathcal{F},\ f_1,f_2\in\mathcal{F}, f_1\le f_2\Rightarrow f_2-f_1\in\mathcal{F}$$

and such that the monotone class in $bp\mathcal{B}$ generated by $\mathcal{F}$ is equal with $bp\mathcal{B}$. Also we denote by $\mathcal{S}_0$ a countable subset of $(bp\mathcal{B})\cap\mathcal{E}(\mathcal{U})$ such that $U(\mathcal{F})\subset\mathcal{S}_0$, $Q_+(\mathcal{S}_0)\subset\mathcal{S}_0$ and $s,t\in\mathcal{S}_0\Rightarrow s+t$, $s\curlyvee t$, $R(s-t)\in\mathcal{S}_0$ where $s\curlyvee t$ means supremum between s and t with respect to the specific order $\preccurlyeq_{\mathcal{E}(\mathcal{U})}$:

We consider a countable base $\mathcal{G}_0$ for the topology $\mathcal{T}'$ which is closed under finite union and finite intersection. Using Theorem 2.7 there exists a kernel P on $(E,\mathcal{B})$, uniquely determined such that

1) $P(bp\mathcal{B}\cap\mathcal{E}(\mathcal{U}))\subset bp\mathcal{B}\cap\mathcal{E}(\mathcal{U}')$

2) $s,t\in bp\mathcal{B}\cap\mathcal{E}(\mathcal{U})$, $s\le t\Rightarrow Ps\preccurlyeq_{\mathcal{E}(\mathcal{U}')} Pt\preccurlyeq_{\mathcal{E}(\mathcal{U}')} t$

3) $f\in\mathcal{F}\Rightarrow PUf=\curlyvee_{\mathcal{E}(\mathcal{U}')}\left\{\sum_{i=1}^n P_{B_i^G} s_i \mid \sum_{i=1}^n s_i\le Uf, s_i\in\mathcal{S}_0, G_i\in\mathcal{G}_0\right\}.$

Proposition 3.2. *For any σ-balayage B and $s\in bp\mathcal{B}\cap\mathcal{E}(\mathcal{U})$ we have*

$$P_B s\preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps.$$

Proof. Assume that s is regular. Firstly we show that the function $Ps-(R^Ks-{}'R^Ks)$ is $\mathcal{U}'$-strongly supermedian. Indeed, let μ, ν bee two finite measures on $(E,\mathcal{B})$ such that $\mu\le_{\mathcal{E}(\mathcal{U}')}\nu$. and let $(G_n)_n$ be a decreasing sequence in $\mathcal{G}_0$ such that

$$\inf_n(\mu+\nu)(R^{G_n}s)=(\mu+\nu)(R^Ks),\ \inf_n(\mu+\nu)('R^{G_n}s)=(\mu+\nu)('R^Ks),$$

where $\mathcal{G}_0$ in the countable base of $\mathcal{T}'$ closed with respect to finite union and finite intersection which appear in the definition of P. Since $P_{B^{G_n}}s\preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps$ and $B^{G_n}s-{}'B^{G_n}s+{}'B^{G_n}P_{B^{G_n}}=P_{B^{G_n}}s$ it follows that $B^{G_n}s-{}'B^{G_n}s\preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps$ and so

$$\mu(Ps-(B^{G_n}s-{}'B^{G_n}s))\le\nu(Ps-(B^{G_n}s-{}'B^{G_n}s)).$$

Passind $n\to\infty$ we deduce that

$$\mu(Ps-(R^Ks-{}'R^Ks))\le\nu(Ps-(R^Ks-{}'R^Ks)).$$

Since B is a σ-balayage then $b(B) \in \mathcal{B}$ and it is a $\mathcal{U}$-basic and $\mathcal{U}'$-basic set, and so there exists an increasing sequence $(K_n)_n$ of compact subsets of $b(B)$ with

$$\mu(Bs) = \sup_n \mu(R^{K_n} s),\ \mu('Bs) = \sup_n \mu('R^{K_n} s).$$

From the above considerations it follows

$$\mu(Ps - (Bs - 'Bs)) \le \nu(Ps - (Bs - 'Bs))$$

and so $Ps - (Bs - B's)$ is $\mathcal{U}'$-strongly supermedian and therefore it is $\mathcal{U}'$-excessive. Hence $Bs - 'Bs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps$ and so

$$P_B s := (Bs - 'Bs) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} \circ \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps.$$

If $s \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$ then there exists an increasing sequence $(s_n)_n$ of regular $\mathcal{U}$-excessive functions $s_n \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ with $s_n \nearrow s$. We have

$$P_B s_n \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps_n \quad \forall n \in \mathbb{N}$$

and so $P_B s \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps$. □

Theorem 3.3. *Let $s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ and $u \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U}')$ be such that $Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} u \le s$. Then $u \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$. Particularly for any $s \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$ we have $Ps \in (bp\mathcal{B}) \cap \mathcal{E}(\mathcal{U})$.*

Proof. For any σ-balayage, using Proposition 3.2, we have $P_B u \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} u$ and thus by Proposition 2.4 we deduce that $Bu \le u$. Consequently $u \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$. □

Theorem 3.4. *For any absorbent point $a \in E$ with respect to $\mathcal{U}'$ we have*

$$PUf_0 \le B^{E\setminus\{a\}} Uf_0.$$

Particularly we have

$$P(Uf_0)(a) < Uf_0(a).$$

Proof. In the proof of this theorem we shall write simply $\preccurlyeq$, $\curlywedge$, $\curlyvee$ instead of $\preccurlyeq_{\mathcal{E}(\mathcal{U}')}$, $\curlywedge_{\mathcal{E}(\mathcal{U}')}$, $\curlyvee_{\mathcal{E}(\mathcal{U}')}$ respectively. By Theorem 2.7 we have

$$P(Uf_0) = \curlyvee \left\{ \sum_{i=1}^{n} P_{B^{G_i}} s_i \mid \sum_{i=1}^{n} s_i \le Uf_0, s_i \in \mathcal{S}_0, G_i \in \mathcal{G}_0 \right\}$$

Since a is an absorbent point with respect to $\mathcal{U}'$ we have, for any $s \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$

$$B^G s \curlywedge 'B^G s \succcurlyeq 'B^{\{a\}} s \quad \forall G \in \mathcal{G}_0, G \ni a$$

and so

$$P_{B^G} s \preccurlyeq B^G s - 'B^{\{a\}} s, \quad \forall G \in \mathcal{G}_0, G \ni a,$$

$P_{B^G} s(a) = 0$ for all $G \in \mathcal{G}_0, G \ni a$. Analogously if $a \notin b(B^{G_i})$ then we have $'B^G s(a) = 0$ and so $P_{B^G} s(a) = B^G s(a)$ for any $s \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U})$.

Let $(s_i^{(k)})_{\substack{1\le i\le n\\ 1\le k\le m}}$ be a finite system in $\mathcal{S}_0$ with $\sum_{i=1}^{n} s_i^k \le Uf_0$ for all $1 \le k \le m$, $(G_i^k)_{\substack{1\le k\le m\\ 1\le i\le n}}$ a finite system in $\mathcal{G}_0$. We put

$$u^k = \sum_{\substack{i=1\\ a\in b(B^{G_i})}}^{n} P_{B^{G_i^k}} s_i^k, \quad v^k := \sum_{\substack{i=1\\ a\notin b(R^{G_i})}}^{n} P_{B^{G_i^k}} s_i^k.$$

We have

$$u^k(a) = 0, \quad v^k(a) = \sum_{\substack{i=1\\ a\notin b(R^{G_i})}}^{n} B^{G_i} s_i(a),$$

$$v^k = \sum_{\substack{i=1\\ a\notin b(B^{G_i^k})}}^{n} (B^{G_i^k} s_i^k - {}'B^{\{a\}}(B^{G_i^k} s_i)) + \sum_{\substack{i=1\\ a\notin b(B^{G_i^k})}}^{n} B^{G_i^k} s_i^k(a) \cdot {}'B^{\{a\}}1 \preccurlyeq$$

$$\sum_{\substack{i=1\\ a\notin b(B^{G_i^k})}}^{n} (B^{G_i^k} s_i - {}'B^{\{a\}}(B^{G_i^k} s_i)) + B^{E\setminus\{a\}} Uf_0(a) \cdot {}'B^{\{a\}}1$$

and

$$\curlyvee_{k=1}^{m} \left(\sum_{i=1}^{n} P_B G_i^k s_i^k \right) \preccurlyeq \sum_{k=1}^{n} u^k + \curlyvee_{k=1}^{m} v^k.$$

Since $B^{G_i^k} s_i^k - {}'B^{\{a\}} B^{G_i^k} s_i^k \in \mathcal{E}(\mathcal{U}')$ and $(B^{G_i^k} s_i^k - {}'B^{\{a\}} B^{G_i^k} s_i^k)(a) = 0$, it follows that

$$[\curlyvee_{k=1}^{m} (\sum_{i=1}^{m} P_{B^{G_i^k}} s_i)](a) \le B^{E\setminus\{a\}} Uf_0(a).$$

Hence $P(Uf_0)(a) \le B^{E\setminus\{a\}} Uf_0(a)$, $P(Uf_0) \le B^{E\setminus\{a\}} Uf_0$. □

Theorem 3.5. *The kernel P is an exact subordination operator with respect to $\mathcal{U}$ and the sub-Markovian resolvent $\mathcal{V} = (V_\alpha)_{\alpha\ge 0}$ on $(E, \mathcal{B})$ associated with P by*

$$Vf = Uf - PUf \;\forall\; f \in p\mathcal{B}, Uf > 0$$

is exact subordonate to $\mathcal{U}$ and $\mathcal{E}(\mathcal{V}) = \mathcal{E}(\mathcal{U}')$.

Proof. Since by Theorem 3.3 and Theorem 3.5 we have $P(\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}) \subset \mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ and

$$s, t \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U}), s \le t \Rightarrow Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Pt \preccurlyeq_{\mathcal{E}(\mathcal{U}')} s,$$

$$s \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U}), u \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U}'), Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} u \le s \Rightarrow u \in b\mathcal{E}(\mathcal{U}).$$

From Theorem 1.10 it follows that P is an exact subordination operator with respect to $\mathcal{U}$ such that any $M \in \mathcal{B}$, $M \subset E\setminus E_P$ is absorbent with respect $\mathcal{U}'$. Particularly any point $a \in E\setminus E_P$ is absorbent with respect to $\mathcal{U}'$ and so by Theorem 3.4 we have $a \in E_P$ contradiction. Hence $E = E_P$ and therefore from Theorem 1.10 we get $\mathcal{E}(\mathcal{V}) = \mathcal{E}(\mathcal{U}')$. □

Theorem 3.6. *Let Q be a kernel on $(E, \mathcal{B})$ such that $Q(b\mathcal{E}(\mathcal{U})) \subset b\mathcal{E}(\mathcal{U})$ and such that*

$$s, t \in bp\mathcal{B} \cap \mathcal{E}(\mathcal{U}), s \le t \Rightarrow Qs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qt \preccurlyeq_{\mathcal{E}(\mathcal{U}')} t.$$

Then the following assertions are equivalent:

1) Q is an exact subordination operator with respect to $\mathcal{U}$ such that the set $\{s-Qs/s \in \mathcal{E}(\mathcal{U})\cap p\mathcal{B}\}$ is solid in $\mathcal{E}(\mathcal{U}')$ with respect to the natural order.

2) For any $u \in \mathcal{E}(\mathcal{U}')\cap p\mathcal{B}$ such that there exists $s \in b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}$ with

$$Qs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} u \leq s$$

we have $u \in b\mathcal{E}(\mathcal{U})$.

3) For any σ-balayage B we have

$$P_B s \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qs \quad \forall s \in b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}$$

4) $Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qs \quad s \in b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}$.

Proof. From Theorem 1.10 it follows that 1) $\Leftrightarrow$ 2).

2) $\Rightarrow$ 3). Let B be a σ-balayage. We show that for any $s \in b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}$ we have $Bs - {}'Bs = Q(Bs) - {}'BQ(Bs)$. Indeed, since ${}'B(Bs - Q(Bs)) \leq Bs - QBs$ and ${}'B(Bs - QBs) \in b\mathcal{E}(\mathcal{U}')\cap p\mathcal{B}$, it follows that the function

$$u := {}'B(Bs - QBs) + Q(Bs)$$

belongs to $b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}$ and $u \leq Bs$.

On the other hand we have

$$u \geq {}'BBs = {}'Bs \text{ on } b(B)$$

and so $u \geq Bs$. Therefore $u = Bs$ and so $Bs - {}'Bs = Q(Bs) - {}'BQ(Bs)$. From the relation $Bs - {}'Bs = Q(Bs) - {}'B(Q(Bs))$ we get $P_B s = (Bs - {}'Bs) \curlyvee_{\mathcal{E}(\mathcal{U}')} 0 \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Q(Bs) \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qs$.

3) $\Rightarrow$ 4). For any $f \in \mathcal{F}$ we have

$$\begin{aligned} PUf &= \curlyvee_{\mathcal{E}(\mathcal{U}')} \left\{ \sum_{i=1}^{n} P_{B^{G_i}} s_i \mid \sum_{i=1}^{n} s_i \leq Uf, s_i \in \mathcal{S}_0, G_i \in \mathcal{G}_0 \right\} \\ &\preccurlyeq_{\mathcal{E}(\mathcal{U}')} \left\{ \sum_{i=1}^{n} Qs_i \mid \sum_{i=1}^{n} s_i \leq Uf, s_i \in \mathcal{S}_0 \right\} \preccurlyeq_{\mathcal{E}(\mathcal{U}')} QUf. \end{aligned}$$

Since $bp\mathcal{B}$ is the monotone class in $bp\mathcal{B}$ generated by $\mathcal{F}$ we deduce

$$Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qs \quad \forall s \in b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}.$$

4) $\Rightarrow$ 2). Let $s \in b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}$ and let B be a σ-balayage. Let $u \in b\mathcal{E}(\mathcal{U}')\cap p\mathcal{B}$ be such that $Qs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} u \leq s$. We have $Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} u \leq s$ and therefore, by Proposition 2.4, $u \in b\mathcal{E}(\mathcal{U})$. □

Proposition 3.7. *Let Q be an exact subordination operator with respect to $\mathcal{U}$ such that*

$$s, t \in bp\mathcal{B}\cap \mathcal{E}(\mathcal{U}),\ s \leq t \Rightarrow Qs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qt \preccurlyeq_{\mathcal{E}(\mathcal{U}')} t,$$

$$Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qs \quad \forall s \in b\mathcal{E}(\mathcal{U})\cap p\mathcal{B}.$$

Let further $\mathcal{W}$ be the Markovian resolvent of kernel on $(E, \mathcal{B})$ having as initial kernel

$$Wf := Uf - QUf \quad \forall f \in p\mathcal{B}, Uf < \infty.$$

Then there exists $g \in p\mathcal{B}^{(u)}$, $0 \leq g \leq 1$ such that

$$QUf = P(U(gf)) + U((1-g)f) \quad \forall f \in p\mathcal{B}.$$

If moreover $E_Q = E$ then we have

$$\mathcal{E}(\mathcal{W}) = \mathcal{E}(\mathcal{U}').$$

Proof. Let $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ be the sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ having as initial kernel V where $Vf = Uf - PUf, \quad f \in p\mathcal{B}, Uf < \infty$. From

$$Ps \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Qs \quad \forall s \in b\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$$

we deduce

$$Wf \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Vf = Uf - PUf \quad \forall f \in p\mathcal{B}, f \leq f_0,$$

where $f_0 \in p\mathcal{B}$, $0 < f_0 \leq 1$, $Uf_0 < \infty$. From $Wf = Uf - QUf \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Uf - PUf = Vf$ for all $f \in p\mathcal{B}$, $f \leq f_0$ it follows that there exists $g \in pb\mathcal{B}^{(u)}$, $0 \leq g \leq 1$ such that

$$Wf = V(gf) \quad \forall f \in p\mathcal{B}, f \leq f_0.$$

i.e.

$$Uf - QUf = U(gf) - PU(gf) \quad \forall f \in p\mathcal{B}, f \leq f_0$$

$$QUf = PU(gf) + U((1-g)f) \quad \forall f \in p\mathcal{B}, f \leq f_0$$

and so $Q(Uf) = P(U(gf)) + U((1-g)f) \quad f \in p\mathcal{B}$.

Assume that $E_Q = E$. Then $Wf_0 > 0$ on E and so for any $t \in \mathcal{E}(\mathcal{U}')$ we have $\inf(t, nWf_0) \in \mathcal{E}(\mathcal{W})$ for all n. Consequently $t \in \mathcal{E}(\mathcal{W})$ and so since $\mathcal{E}(\mathcal{W}) \subset \mathcal{E}(\mathcal{U}')$ we get $\mathcal{E}(\mathcal{W}) = \mathcal{E}(\mathcal{U}')$. □

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