Generalized billiards inside an infinite strip with periodic laws of reflection along the strip's boundaries

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AMS Classification (2000): *37D50, 37D45,37B10.*
Key words: *billiards, dynamical systems, spatial chaos, entropy.*

Abstract

A constructive description of generalized billiards is given, the billiards being inside an infinite strip with a periodic law of reflection off the strip's bottom and top boundaries. Each of the boundaries is equipped with the same periodic lattice, where the number of lattice's nodes between any two successive reflection points may be prescribed arbitrarily. For such billiards, a full description of the structure of the set of billiard trajectories is provided, the existence of spatial chaos is found, and the exact value of the spatial entropy in the class of monotonic billiard trajectories is found.

1 Introduction

Trajectories of a billiard ball inside a two-dimensional (plane) bounded table (a region) with a complicated boundary have, as a rule, a fairly complex structure, despite the simplicity of the ideal law of billiard reflection: *the angle of reflection equals the angle of incidence*. Currently, the behavior of billiard trajectories inside the following bounded plane regions are investigated in detail: Birkhoff's billiards (the table's boundary is a smooth closed *convex* curve); Sinai's billiards (bounded by a finite number of piecewise smooth curves whose smooth components are strictly convex inwards and intersect transversally); Bunimovich's billiards (bounded by several arcs of circles and straight line segments); polygonal billiards (the region is a connected polygon); and many others (see the books [7], [8], and [9]).

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Currently, the behavior of a billiard ball within infinite media having non-trivial *periodic* structures on their boundaries is being studied intensively. We just want to draw attention to a recent model of the behavior of a billiard ball within an infinite periodic horizontal channel, the bottom and the top boundaries of which are the union of isosceles triangles whose vertices lie inside the channel (see [4]). The "table" of this billiards is an infinite horizontal strip with "roughness" on its bottom and top straight line boundaries; the "roughness" is, in this case, the union of congruent triangles glued to the strip's boundaries inwardly and repeated periodically along the boundaries. This model is a variation of the famous standard model "Lorentz channel", in which both boundaries are the union of congruent semicircles situated periodically along the parallel sides of the strip (see [5] and [4]). Again, here we deal with the billiards inside a horizontal strip, where the small semicircles are glued to the strip's boundaries inwardly and periodically. Another recent paper [6] proposes a model of an elastic "bouncing billiard ball": the ball is subject to a constant vertical force and bounces on a one-dimensional periodically corrugated floor. The boundary of the "table", the corrugated floor, consists of the union of periodically repeated identical arcs of a semicircle. As before, we can also think of this situation as of the "roughness" superposed onto the horizontal line. When the radius of each arc goes to zero, the ideal billiard ball reflection from the "roughness" can be thought of as a non-ideal reflection from the ideal straight line (i.e., the line without any "roughness") obeying a special law of reflection. We call billiards with such a law of reflection "generalized billiards".

The authors of the above mentioned papers made numerical computations for the behavior of the billiard ball trajectories and presented numerical studies of the dynamics of ensembles of billiard balls within the billiard channels. Based on those studies, they made interesting observations concerning, among others, diffusive behavior, the velocity autocorrelation function and its spectral analysis (without mathematical proofs). Note that if the size of the "roughness" tends to zero, we get, in the limit, a "non-ideal", or "generalized", billiards within the strip. Namely, two special periodic functions are fixed on the strip's sides, and each of the functions describes a non-ideal law of reflection.

In the present paper, we suggest a model for the behavior of a billiard ball within an infinite horizontal strip, in which the law of reflection from its straight line boundaries is described by two functions satisfying some natural conditions that generalize the ideal law of reflection. The standard model of billiards (the "ideal reflection on the boundary") as well as the two previous models (channel of polygons, Lorentz channel, and the "bouncing ball") are just particular cases of our general model. In addition, the investigation of our model shows that there are some new types of trajectories which are impossible for the standard billiards, the so-called "branching" trajectories, "chaotic" trajectories, and bounded trajectories with infinitely many segments ("links").

Let us consider a billiards inside an infinite horizontal strip with two parallel boundaries (sides), in which the ideal law of reflection is substituted by a more complicated one; the law of reflection can be, generally speaking, different at different points of the strip's boundary or even be random. In the present paper, we investigate the complex dynamics of the trajectories in such billiards. The discrete dynamical system, arising from the generalized billiards inside the strip, determined on the set of reflection points, can also be useful in various applications.

In this paper, we first investigate generalized billiards inside an infinite strip, where the reflection off the bottom side of the boundary is spatially periodic with the period ℓ, and is ideal on its top side. This means that the spatially-periodic lattice on the bottom side of the strip divides it into an infinite sequence of intervals of length ℓ (each of which is closed on the left side and open on the right side), and when the billiard ball reflects off the bottom side of the boundary at some point of an interval, then the angle of reflection is determined uniquely by the angle of incidence and the position of the reflection point (or the local coordinate of the reflection point). A billiard trajectory (or orbit, for brevity) in such generalized billiards is the union of lateral sides of isosceles triangles, the bases of which are the segments between the successive reflection points on the bottom side of the boundary, while their vertices are the reflection points of ideal billiards. The reflection points on the bottom side of the strip code the sequence of intervals of length ℓ at which the billiard ball is reflected. Each interval on the bottom side at which the trajectory bounces off is said to be a *marked interval*. We call the sequence of marked intervals, or equivalently, the sequence of the numbers that enumerate those intervals, the *full skeleton of the billiard trajectory*.

The appearance of the word "skeleton" in this context can be explained as follows. Let η be a broken line, the kinks of which (i.e., the ends of the segments, or links, of the broken line) on the bottom boundary of the strip are the midpoints of the marked intervals, while those on the top boundary are the vertices of the isosceles triangles whose bases are the segments between the successive midpoints of the marked intervals. The broken line η, generally speaking, is not a billiard trajectory inside the strip, since the law of reflection at the η's vertices on the bottom boundary of the strip does not hold. Nevertheless, η holds the global information about the real billiard trajectory γ inside the same strip with the same reflection points at the same marked intervals: firstly, η can be constructed uniquely by the midpoints of the marked intervals, and secondly, η differs from γ by less than $\ell/2$ (e.g., all the distances between the corresponding reflection points of η and γ are less than $\ell/2$). Thus, the broken line η can be thought of as the skeleton of the billiard trajectory γ. Forgetting the broken line η, we just keep the name "skeleton" for the marked intervals and the numbers M_k that number the marked intervals.

We call the billiards with the ℓ-periodic law of reflection from the bottom side of the strip *the billiards with the property of universality*, if for an *arbitrary* sequence of marked intervals there *exists a unique* billiard trajectory for which the sequence of those marked intervals is its skeleton.

One of the main results of the present paper is the constructive description of the set of ℓ-periodic, along the bottom strip's side, laws of reflection for which the billiards has the property of universality. For such billiards, some properties of their trajectories are studied; in particular, the exact value of the spatial entropy is found for the class of *monotonic* billiard trajectories.

2 Special dynamical systems with reflections

In this section, we introduce main notions and definitions that are essential for describing the main results on generalized billiards in an infinite strip with the law of reflection periodic on its boundaries. We start with important examples; the examples motivate the notions introduced.

2.1 Gunnery with ricochet

Let a flash-spotting above the earth surface be done from a point x_0 along the positive x-axis (see Fig. 1). Let us assume also that the missile is just a material mass point; we ignore the windage and suppose that the gravity orthogonal to the x-axis is constant. Then the trajectory of the missile is a parabola. The range of the missile depends on the initial missile's energy and on the initial angle of the gunnery, φ_0, which satisfies $x_1 - x_0 = d\sin(2\varphi_0)$. Here, d stands for the maximal flight range of the missile corresponding to the angle $\varphi_0 = \pi/4$, and x_1 stands for the coordinate of the missile's landing.

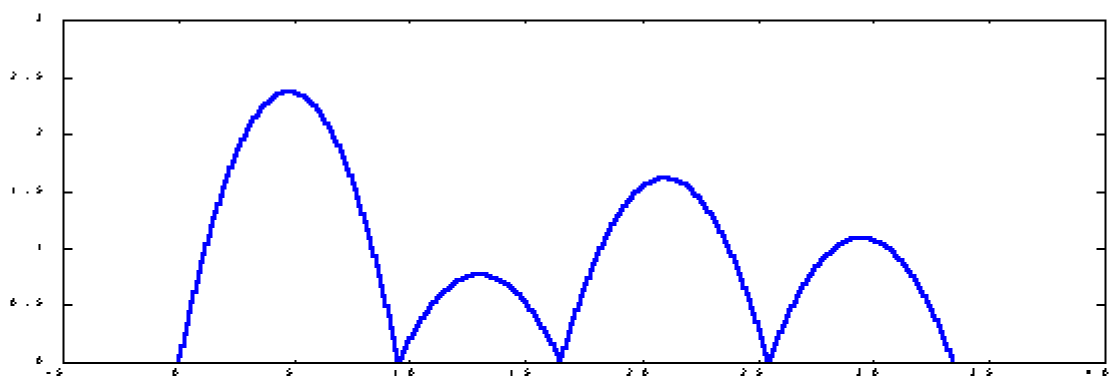

Figure 1: *Gunnery with ricochet*

Suppose that at the landing points the elastic ricochet occurs, so that the total kinetic energy of the missile is preserved. Then the missile will fly from point x_1 to point x_2. The range of the missile depends on the angle φ_1, under which the missile will ricochet, according to the same law: $x_2 - x_1 = d\sin(2\varphi_1)$. A new ricochet can occur at point x_2, then at point x_3, and so on.

In order to describe such repeated ricochet, it is important to know the law of reflection from the earth (in our particular case, the law of ricochet of the missile at each landing point). The law may be different at different landing points. It can be expressed in the form of an equation that ties up the consecutive angles of ricochet, φ_{k-1} φ_k, with the coordinate x_k of the landing point. Let us restrict ourselves with the angles smaller than $\pi/4$. Since there is a one-to-one correspondence between the missile departure angles, φ_k, launched from the points x_k, and the distances $x_{k+1} - x_k$, the law of ricochet can be represented in the form

$$f(x_{k+1} - x_k,\ x_k - x_{k-1},\ x_k) = 0 \quad . \tag{1}$$

Thus, the function f of three variables determines the general law of reflection. If the first argument of the function f is determined uniquely via the other two (i.e., it is a function of the remaining two arguments), then the law (1) determines the whole missile trajectory uniquely. Such a situation could indeed happen: the gunnery in the opposite direction could give the same trajectory. This will happen if the law (1) remains the same after changing the positive direction of the x-axis to the opposite one, i.e., if $f(x_k - x_{k-1},\ x_{k+1} - x_k,\ \ - x_k) = 0$. For such kind of laws, one can expand the initial trajectory in the opposite direction with the ricochet points $x_{-1},\ x_{-2},\ \dots$. As a result, a sequence $\{x_k|\ k \in \mathbf{Z}\}$ of landing points infinite in both directions arises. This sequence satisfies to the equation (1) and determines a discrete dynamical system.

2.2 The standard billiards inside an infinite strip

Let us consider the standard billiards inside an infinite strip of the width h: the bottom side of the strip coincides with the x-axis and the upper parallel side is distance h apart (Fig. 2).

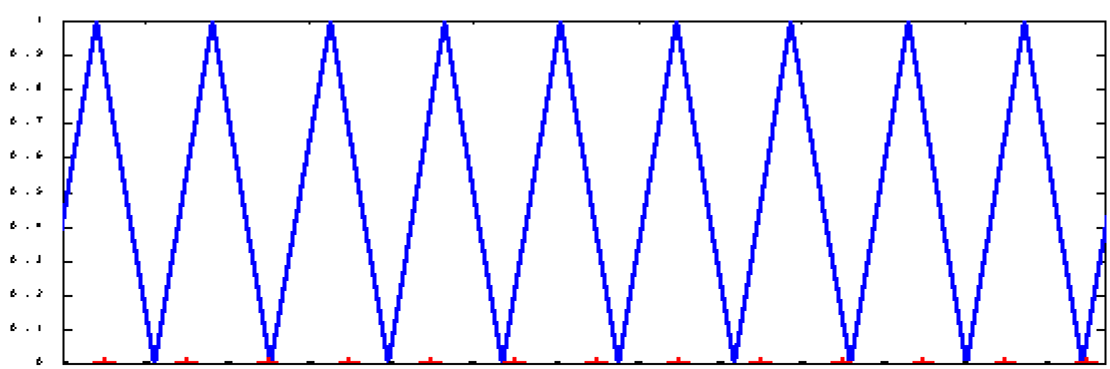

Figure 2: *The ideal billiards with the marked board*

The law of reflection in the standard billiards is ideal: *the angle of reflection equals the angle of incidence.* Cast aside the trivial case when the billiard trajectory is perpendicular to the strip's sides, we get that each remaining billiard trajectory is periodic along the x-axis with some period L. In addition, for each L there exists a billiard trajectory with period L which is determined uniquely up to an arbitrary shift along the x=axis. We can say nothing more substantial about trajectories of such billiards.

Interesting properties of trajectories of the ideal billiards arises when the boundary of the strip is marked periodically. We give respecting definitions that will also play an important role when studying the generalized billiards.

Definition 1. We say that a side of the strip is equipped with an *ℓ-periodic marking*, if the lattice of integers $n\ell$ ($n \in \mathbf{Z}$) is given, where $n\ell$'s are the midpoints of the semi-open intervals $I_n = [n\ell - \frac{\ell}{2},\ n\ell + \frac{\ell}{2})$. We call all the semi-open intervals I_n the *marked intervals.*

Definition 2. Let γ be a billiard trajectory inside the strip with the ℓ-periodic marking on the bottom side of the strip. We call the marked intervals containing points of reflection of the trajectory γ *the marked intervals.* We call the set of the marked intervals the *full skeleton* of the trajectory γ with respect to the strip's side equipped with the ℓ-periodic marking. At the same time, if x_k is the coordinate of the reflection point on the interval I_{M_k}, then

$$x_k = M_k\ell + \xi_k, \qquad -\ell/2 \le \xi_k < \ell/2,\ M_k \in \mathbf{Z}\ . \tag{2}$$

We call ξ_k *the local coordinate* of the reflection point x_k, and the set of integers M_k its *full skeleton*: $\{M_k\} = S(\gamma)$.

The notions introduced allow us to prove the following statements.

Proposition 1. *Consider the ideal billiard in an infinite strip with the ℓ-periodic marking of the bottom strip's side. The L-periodic trajectory in such a billiard ($L > 0$) is determined uniquely by its full skeleton if and only if the ratio L/ℓ is irrational.*

Proof. Without loss of generality we can set $\ell = 1$: this can be done by a dilation of the plane that changes the scales on the x- and y-axes uniformly and does not change the ideal billiard law of reflection.

Let us show first that the full skeleton $S(\gamma) = \{M_k\}$ of an arbitrary trajectory γ determines its period L uniquely. Indeed, after reflecting the billiard strip with respect to its sides N times, we straighten (unfold) the billiard trajectory, i.e., convert it into a straight line. The slope of the unfolded trajectory γ is defined by

$$\tan\alpha = \frac{Nh}{x_{k+N} - x_k}\ ,$$

where h is the width of the strip.

Plugging into this formula the expressions for x_k and x_{k+N} from the relation (2) and going to the limit as $N \to \infty$, we get

$$\tan\alpha = \lim_{N\to\infty} \frac{Nh}{(M_{k+N} - M_k)\ell}\ .$$

On the other hand, the period of the trajectory, L, is related to the angle α between the unfolded trajectory and the x-axis and the width h of the strip by the simple relation $\tan\alpha = \dfrac{h}{L/2}$ (see Fig. 2).

Consequently, the period of the billiard trajectory is indeed determined uniquely via its full skeleton. Since billiard trajectories with the same period are the same up to a parallel shift, we get the following important corollary: *different billiard trajectories with the same full skeleton can be obtained one from the other by a parallel shift along the x-axis.*

Let us denote by $Q(\gamma) \in [-1/2,\ 1/2)$ the set of local coordinates ξ_k of the reflection points of the billiard trajectory γ.

Suppose that the period of the trajectory γ is a *rational* number: $L = p/q$, where p and q are relatively prime integers. We prove that in this case there are non-trivial shifts of the trajectory γ that do not change the full skeleton. Indeed, in the case in question, the set $Q(\gamma)$ of local coordinates consists of q points. Let us shift the initial trajectory γ to the right at the distance $\varepsilon > 0$, which is strictly less than the distance from the set $Q(\gamma)$ to the right end of the interval $[-1/2,\ 1/2)$. Then the shifted trajectory has the same full skeleton as the initial trajectory. Therefore, if L is rational, the billiard trajectory is not determined uniquely by its full skeleton.

But if the period L of the trajectory γ is *irrational*, then the set $Q(\gamma)$ of local coordinates of the reflection points fills out the interval $[-1/2,\ 1/2)$ everywhere densely. The latter follows from the Jacobi theorem on the rotation of a circle with the circumference 1 through an irrational angle [10]. Let us prove that in this case, any non-trivial shift of the trajectory changes its full skeleton. Indeed, if γ is shifted to the right at a distance $\varepsilon > 0$, then at least one reflection point x_k will leave the marked interval numbered M_k, in which this point has been situated from the beginning. This will change the full skeleton. One can pick x_k as the point with the local coordinate $\xi_k \in (1/2 - \varepsilon, 1/2)$. Likewise, when shifting the trajectory to the left at a distance $\varepsilon > 0$, one can choose the point with the local coordinate ξ_k from the interval $(-1/2,\ -1/2 + \varepsilon)$ as x_k. So, if L is irrational, the full skeleton determines the billiard trajectory uniquely. ∎

Proposition 1 can be generalized for the ideal billiard with *different* marking on the sides of the strip. As before, we eliminate from our considerations trivial trajectories which are segments orthogonal to the boundary of the strip.

Proposition 2. *Let ℓ_k-periodic markings are fixed on both of the parallel sides of an infinite strip ($k = 1,\ 2$). If the ratio ℓ_1/ℓ_2 is irrational, then an arbitrary non-trivial billiard trajectory inside the strip is determined uniquely by its full skeleton, i.e. by the numbers of the marked intervals on both sides of the strip.*

Proof. Let us denote by L the period of a non-trivial billiard trajectory. Because ℓ_1/ℓ_2 is an irrational number, at least one of the numbers L/ℓ_1 or L/ℓ_2 is irrational. Applying the proposition 1 completes the proof. Q.E.D.

2.3 Generalized billiards inside an infinite strip

Consider billiards inside an infinite strip and suppose that the law of reflection on the upper side of the strip is ideal (i.e., the angle of reflection is equal to the angle of incidence on that side), and that the angle of reflection on the bottom side is determined uniquely by the angle of incidence and the coordinate of the reflection point. In this case, denoting by x_k the coordinates of the consecutive reflection points on the bottom side of the strip, we see that the trajectory γ is given by the union of the lateral sides of the isosceles triangles with the bases $(x_k,\ x_{k+1})$ and the third vertices on the upper side of the strip (Fig. 3).

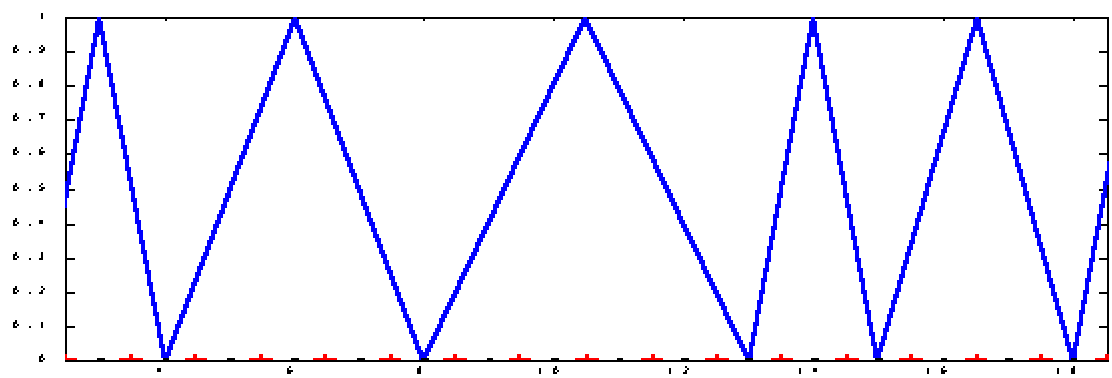

Figure 3: *A generalized billiards inside an infinite strip*

The trajectory γ just obtained differs from the trajectory of the missile introduced in the Subsection 2.1 in that the trajectory γ consists of isosceles triangles with the bases $(x_k,\ x_{k+1})$, while the trajectory of the missile consists of the union of isosceles parabolas with the same bases.

In the case of the generalized billiard inside the infinite strip, there is a one-to-one correspondence between the angles of reflection and the coordinates of the consecutive reflection points situated on the bottom side of the strip. That is why we can represent the law of reflection on the bottom side of the strip in the form of equation (1).

Let us restrict ourselves to the case where the law of reflection (1) is given by a function f periodic, with the period ℓ, in the third coordinate : $f(\bullet,\ \bullet,\ x_k + \ell) = f(\bullet,\ \bullet,\ x_k)$. In this case, the law of reflection (1) takes the form

$$f(x_{k+1} - x_k,\ x_k - x_{k-1},\ \xi_k) = 0 \ , \tag{3}$$

where ξ_k is the local coordinate of the reflection point x_k determined by (2).

2.4 Stationary solutions of non-linear diffusion chains

In the articles [1,2], in order to construct solutions of non-linear diffusion chains, a dynamical system of the same type as it was introduced above in the Subsections 2.1 – 2.3, had been introduced. The only difference is that instead of the parabolas from subsection 2.1, in [1,2] curves in the form of hyperbolic cosines between the consecutive points x_k and x_{k+1} are involved, i.e., curves given by:

$$y(x) = 1 - \frac{\cosh(x - 1/2 \cdot (x_{k+1} + x_k))}{\cosh(1/2 \cdot (x_{k+1} - x_k))} \quad . \tag{4}$$

Exactly the same trajectories arise for the motion of the missile from the Subsection 2.1, if one considers the gravity field on the upper half plane to be decreasing linearly with respect to the height.

In papers [1, 2], the law of reflection has the following form:

$$\tanh(x_{k+1} - x_k) - \tanh(x_k - x_{k-1}) = C \cdot \xi_k \ , \tag{5}$$

where C is a constant not depending on the trajectory. Fig. 4 shows an example of the graph of such a cosh-trajectory.

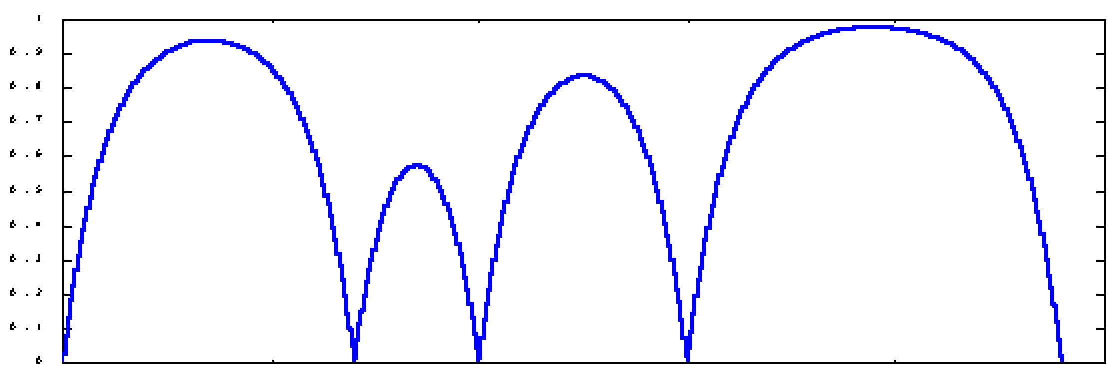

Figure 4: *COSH-trajectory*

2.5 On zeros of bounded solutions of some non-linear differential equations

Let $y(x)$ be a bounded solution of a differential equation. Let us reflect, for all $y < 0$, the graph of $y(x)$ with respect to the x-axis. We will get the graph of a new function, $g(x)$, which is positive everywhere except for its zeros (coinciding with the zeros of $y(x)$). This graph can be considered as a trajectory of a dynamical system similar to those introduced in the subsections 2.1–2.4, where the role of x_k is played by the zeros of the solution $y(x)$, and the arcs of the graph of $g(x)$ between every two consecutive zeros are described by a differential equation which is either the same as the initial one or can be expressed via the initial one if we keep in mind that $g(x) = -y(x)$ whenever $y(x) < 0$.

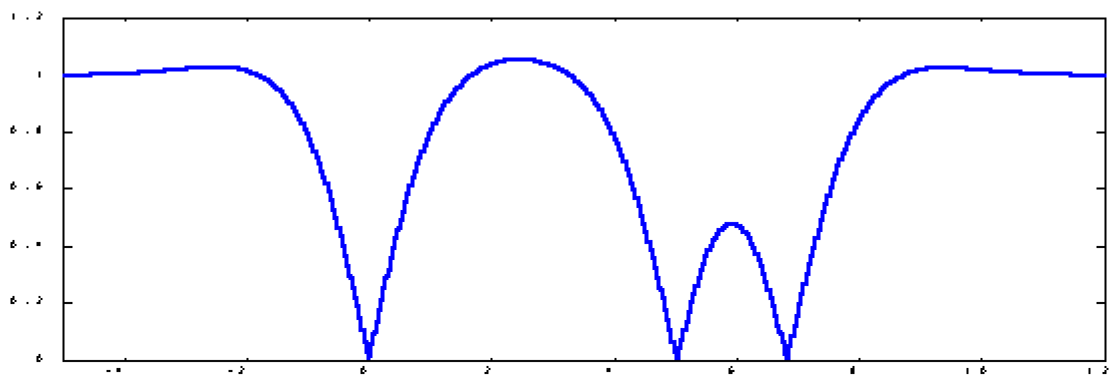

Figure 5: *The three-zero-solution trajectory*

As it was shown in the paper [3], the distance between the zeros of every bounded solution of the equation $-\gamma y^{(4)} + y'' - y + \text{sign} y = 0$ can be characterized by integers, i.e., by the skeleton of this solution, as well as it can be done for the trajectories of the generalized billiards. More precisely, if $y(x, \{x_k\})$ is a bounded solution of this equation that has a finite number of zeros $x_0 < x_1 < \ldots < x_m$, and the distances $a_k = x_k - x_{k-1}$ between the successive zeros satisfy $a_k \geq \pi/\beta$, where $\beta = \left(\frac{1}{2\sqrt{\gamma}} - \frac{1}{4\gamma}\right)^{1/2}$, then the solution $y(x)$ is determined uniquely, up to the sign and a shift along the x-axis, through its skeleton $s(y) = \{n_k\}_{k=1}^m$. Here n_k are positive integers, that are determined uniquely via the distances a_k between the successive zeros from the condition

$$\beta a_k = -\frac{\pi}{2} + \delta + n_k\pi + \varepsilon_k, \qquad |\varepsilon_k| < \frac{1}{4} \qquad \delta = \arctan(4\gamma - 1)^{-1/2}.$$

Moreover, any finite sequence $\{n_k\}_{k=1}^m$ of integers satisfying $n_k \geq 2$ is a skeleton of some bounded solution.

The graph of the trajectory corresponding to the solution y with three zeroes and the skeleton $s(y) = \{2, 1\}$ is given in Fig.5.

3 Billiards inside an infinite strip with the law of periodic reflection along one of the strip's sides

Let us consider first a billiards in an infinite strip with the law of reflection which is periodic along one of the sides of the strip, say, along the bottom side, and is ideal from its top side. As it was said in the Section 2.3., each trajectory γ in such a billiards can be represented by the union of the lateral sides of isosceles triangles with the bases $(x_k,\ x_{k+1})$ and vertices on the bottom side of the strip. The reflection points x_k are described by the discrete dynamical system (3) with the values of ξ_k satisfying condition (2). We remark that the numbering of the reflection points (i.e., the indices k) characterize the reflections from the bottom side seen consecutively in time. If a billiard trajectory does not have cuspidal points, then one always has $x_{k+1} > x_k$ for all k's; in this case, the billiard trajectory is said to be *monotonic*.

The main questions about the generalized billiards are the following:

1. Do there exist trajectories in the generalized billiards without cuspidal points, i.e., trajectories such that the reflection points $\{x_k\}$ satisfy the conditions $x_k < x_{k+1}$, $x_k \to \pm\infty$ as $k \to \pm\infty$?
2. Is the trajectory in the generalized billiards uniquely determined by its full skeleton? (See definition 2.)
3. What infinite sequences of integers can serve as full skeletons of trajectories in the generalized billiards?
4. How to find the billiard trajectory effectively by its full skeleton?

3.1 Generalized billiards with the property of universality

Definition 3. We say that the generalized billiards with the ℓ-periodic marking of the non-ideal side of the strip has the property of *universality*, if every double-sided sequence of integers $\{M_k,\ k \in \mathbf{Z}\}$ is the full skeleton of a uniquely determined billiard trajectory.

The positive answers to the questions posed above are implicitly contained in Definition 3. The main question arises:

Do there exist generalized billiards with the property of universality?

The answer to this question is given by the following

Theorem 1. *If $|C| > max(4,\ 4/\ell)$, then the generalized billiards with the ℓ-periodic law of reflection (5) on the bottom side of the strip and the ideal law of reflection on the top side has the property of universality.*

Proof. Each billiard trajectory γ with the reflection points x_k and with the corresponding local coordinates ξ_k satisfying (5), also satisfies the equation

$$\xi_k = \frac{1}{C}\Big[\tanh\Big((M_{k+1}-M_k)\ell+(\xi_{k+1}-\xi_k)\Big)-\tanh\Big((M_k-M_{k-1})\ell+(\xi_k-\xi_{k-1})\Big)\Big] . \quad (6)$$

Conversely, all the solutions of the equation (6) with the given integers M_k and satisfying the condition $-\ell/2 \le \xi_k < \ell/2$ generate the dynamical system $\{x_k\}$ by formula (2), and therefore, they also generate the trajectory γ that satisfies the law of reflection (5).

Thus, in order to prove the theorem, one needs to prove that for any sequence of integers $\{M_k\}$ there is a unique solution of equations (6) satisfying $|\xi_k| \le \ell/2$.

We will consider the equation (6) in the space $\ell_\infty(\mathbf{Z})$ of infinite sequences with the norm $\| \{\xi\} \| = \sup_{\mathbf{k}}\{|\xi_{\mathbf{k}}|\}$, where $\xi = \{\xi_k|\ k \in \mathbf{Z}\}$. Let us express (6) in the operator form:

$$\xi = \mathcal{A}(\xi) \quad . \quad (7)$$

From the explicit form for the operator $\mathcal{A}$, following from the equation (6) and the inequality $|\tanh\alpha - \tanh\beta| \le |\alpha-\beta|$, we obtain

$$\|\mathcal{A}(\xi)-\mathcal{A}(\eta)\|_{\ell_\infty} \le \frac{4}{|C|}\|\xi-\eta\|_{\ell_\infty} \quad .$$

Since $|C| > 4$ by the hypothesis of the theorem, the operator $\mathcal{A}$ is a contraction in the space $\ell_\infty(\mathbf{Z})$. Consequently, there exists a unique solution $\xi \in \ell_\infty(\mathbf{Z})$ of the equation (7).

It remains to show that the solution ξ does not exceed $\ell/2$ in the norm of $\ell_\infty(\mathbf{Z})$. Indeed, by virtue of (6) and by the hypothesis in the theorem that $|C| > 4/\ell$, we have $\|\xi\|_{\ell_\infty} \le 2/|C| < \ell/2$.

Constructively, the solution ξ can be obtained by the method of consecutive approximations, starting with $\xi = 0$. This method is very effective: the approximations converge to the solution with the rate of geometric progression. ■

Applying the method used in the proof of theorem 1, one can obtain sufficient conditions for general ℓ-periodic laws of reflection (3) that provide the property of universality for the generalized billiards.

Theorem 2. *Let the law of reflection (3) be presented in the form*

$$\xi_k = g(x_{k+1}-x_k,\ x_k-x_{k-1}), \quad (8)$$

where the absolute value of the function $g(x,y)$ is less than $\ell/2$ and $g(x,y)$ is a Lipschitz function with a constant $q < 1/4$, i.e. the following inequality holds:

$$|g(x_2,\ y_2)-g(x_1,\ y_1)| \le q(|x_2-x_1|+|y_2-y_1|), \quad q < 1/4 \ . \quad (9)$$

Then the generalized billiards with the law of reflection (8) on the bottom side of the strip has the property of universality.

Proof. It is enough to prove that for any positive integers $\{m_k\}$, where $m_k = M_{k+1} - M_k$, there exists a unique solution of the equations

$$\xi_k = g\Big(m_k\ell + (\xi_{k+1} - \xi_k),\ m_{k-1}\ell + (\xi_k - \xi_{k-1})\Big) \tag{10}$$

in the space $\ell_\infty(\mathbf{Z})$, and $\|\xi\| < \ell/2$. This equation has the form (7) with a contracting operator $\mathcal{A}$. Hence, the solution of (10) exists and unique, and, moreover, because of $|g(x,\ y)| < \ell/2$,the absolute values of all the ξ_k's are strictly less than $\ell/2$. ■

The conditions for the law of reflection (8) mentioned in the formulation of the Theorem 2 are sufficient for the billiards to have the property of universality. Those conditions can be weakened in a number of cases.

Let us consider as an example the law of reflection (8) in the following special form:

$$\xi_k = \frac{1}{2}\cdot\frac{\ell\cdot(x_{k+1} - x_k)}{2h + |x_{k+1} - x_k|}\ , \tag{11}$$

where ℓ is the length of the marked interval, and h is the width of the strip. Note that in the law (11), the angle of incidence does not participate at all, while the angle of reflection is determined by the local coordinate of the reflection point.

For the law (11), the function g taken from the equation (8) is Lipschitz with the constant $q = \ell/4h$. So, in order that the hypotheses of the Theorem 2 hold, one needs in principle to restrict the value ℓ/h. Fortunately, for the law (11) it turns out that the billiard has the property of universality **without any restrictions**.

Theorem 3. *A billiards with the ℓ-periodic law of reflection (11) on the bottom side of the strip and the ideal law of reflection on the top side has the property of universality.*

Proof. The law (11) has a simple geometric interpretation. Let γ be a billiard trajectory that reflects at a point x_k of the interval $I_{M_k} = [M_k\ell - \ell/2,\ M_k\ell + \ell/2)$ on its bottom side, and let the local coordinate ξ_k of this point be known. In order to construct the link of the trajectory γ with the end at the point $A = (x_k, 0)$, let us consider an auxiliary point Y on the plane having the coordinates $Y = (M_k\ell,\ -(\ell/2 - |\xi_k|))$. The point Y is located on the vertical line beneath the midpoint of the interval I_{M_k} at a distance from this midpoint equal to the distance from the point ξ_k to the closest end of the interval I_{M_k}. Let us draw a ray that connects the point Y with the reflection point $A = (x_k, 0) = (M_k\ell + \xi_k, 0)$ of the trajectory γ from the bottom side of the strip and find the point B at which this ray meets the top side of the strip. Then, as one can easily check, the segment AB will be the sought link of the billiard trajectory γ, emanated from point A and obeying the law of reflection (11). The ideal reflection of the link AB off the top strip's side will determine uniquely the next point of reflection, x_{k+1}, of the trajectory γ.

Let $x^{(1)}_{k+1}$ and $x^{(2)}_{k+1}$ be two constructed in that way bottom reflection points for the trajectory γ that correspond to the two initial points $x^{(1)}_k$ and $x^{(2)}_k$. Simple geometric considerations show that if the points $x^{(1)}_{k+1}$ $x^{(2)}_{k+1}$ are located at a distance d from each other, then the distance between their pre-images, the points $x_k^{(1)}$ $x_k^{(2)}$, does not exceed $\dfrac{\ell d}{4h+\ell}$.

We show now that an arbitrary sequence of integers M_k, $k \in \mathbf{Z}$, is a full skeleton for a billiard trajectory uniquely determined by this skeleton.

Indeed, construct on each of the marked intervals I_{M_k}, $k \in \mathbf{Z}$, a system of nested intervals $I^i_{M_k}$, $i = 0, 1, 2, \ldots$, the lengths of which tend to zero. We set $I^0_{M_k} = I_{M_k}$. If the intervals $I^i_{M_k}$ are already defined for all M_k's and a fixed i, then the interval $I^{i+1}_{M_k}$ can be obtained as the pre-image of the interval $I^i_{M_{k+1}}$ under the action of the billiard mapping (11). As we noted above, the lengths of the intervals-preimages are the lengths of the initial intervals divided by the factor $(4h+\ell)/\ell > 1$. Hence, the lengths of the nested intervals $I^i_{M_k}$ tend to zero as $i \to \infty$. Therefore, on each interval I_{M_k} the unique limiting point $x_k \in I_{M_k}$ is defined; is can be found as the intersection of the nested intervals $I^i_{M_k}$, $i = 0, 1, 2, \ldots$. Considering now x_k as successive reflection points on the bottom side of the strip, we get *a unique* billiard trajectory for which the set of the numbers M_k, $k \in \mathbf{Z}$ is the full skeleton. Thereby, the property of universality is proven. ■

It is seen from the proof of Theorem 3 that the fixed reflection point x_n determined by the law (11) is actually determined by just a part of the full skeleton $\{M_k\}_{k=n}^{\infty}$. Therefore two distinct skeletons, $\{M_k,\ k \in \mathbf{Z}\}$ and $\{\widetilde{M}_k,\ k \in \mathbf{Z}\}$, coinciding from some place: $M_k = \widetilde{M}_k$ for all $k \geq n$, will determine two *distinct* billiard trajectories, that coincide with each other only starting from the reflection points x_k, $k \geq n$. In other words, for billiards with the law of reflection (11), trajectories can stick to each other.

If we reverse "time" in the law (11), i.e., consider the law of reflection

$$\xi_k = -\frac{1}{2} \cdot \frac{\ell \cdot (x_k - x_{k-1})}{2h + |x_k - x_{k-1}|} ,$$

in which only the *angle of incidence* appears (whereas the angle of reflection does not appear), then we obtain a billiards which can have trajectories that branch apart after each common point of reflection. At the same time, the property of universality holds.

3.2 Monotonic billiard trajectories and symbolic dynamics

Let the sequence of the reflection points x_k of the trajectory γ on the bottom side be monotonic: $x_{k+1} > x_k$, $k \in \mathbf{Z}$. If the numbers M_k of the intervals I_k, from which a billiard trajectory reflects in that order, form a strictly monotonic sequence, then we say that this trajectory is *strictly monotonic*. In this case, each marked interval I_k can contain no more than one reflection point. We code the successive intervals I_k by the numbers 1, respectively 0, depending on whether the marked interval does, respectively does not, contain a reflected point. The obtained, infinite in both directions, sequence of zeros and ones will be called *the code of the billiard trajectory* γ and will be denoted by $C(\gamma)$: $C(\gamma) = \{..., 0, 1, 1, 0, 0, 0, 1, ...\}$.

If a billiards has the property of universality, then the code $C(\gamma)$ of a strictly monotonic trajectory γ determines the trajectory γ uniquely. Moreover, each sequence of zeros and ones can be the code of some monotonic billiard trajectory.

Therefore, the codes can be considered as a natural characteristic of strictly monotonic trajectories. At the same time, the shift of a billiard trajectory γ by the lattice period ℓ is equivalent to the Bernoulli shift of its code $C(\gamma)$.

Together with the code $C(\gamma)$ of a monotonic trajectory γ and the γ's full skeleton $S(\gamma) = \{M_k,\ k \in \mathbf{Z}\}$ we will consider the sequence of integers $m_k = M_{k+1} - M_k$, which we call the *skeleton* of the trajectory γ: $s(\gamma) = \{m_k,\ \ k \in \mathbf{Z}\}$. While the trajectory γ of a billiards with the property of universality can be determined uniquely by its full skeleton $S(\gamma)$, the same trajectory γ can be determined by its skeleton $s(\gamma)$ only up to a shift on an integer multiple the period ℓ of the lattice on the bottom side of the strip.

3.3 Generalized billiard trajectories with restrictions

If we set $C = 0$ in the law (5), then the law of reflection becomes ideal. On the other hand, it is supposed in the Theorem 1, that the constant C is big enough. It is interesting to study the billiard law of reflection with a fairly small constant C, for which the law of reflection is close to the ideal one. In this case, some restrictions for the skeleton of the trajectory appears.

For particular laws of reflection, like (6) or (8), the conditions for the smallness of the Lipschitz constant can hold not for all skeletons, but only for those with specific conditions. The following definition reflects such a peculiarity.

Definition 4. We say that the billiards from the Subsection 2.3 has an (a, b)-restriction, if each sequence of integers $\{m_k\}$ satisfying $a \le m_k \le b$ is the skeleton of some billiard trajectory, where this trajectory is determined uniquely (up to shifts by multiples of the lattice period) via its skeleton.

The following two theorems on reflection laws with the (a, b)-restriction take place.

Theorem 4. *The reflection law (5) with the lattice period $\ell = 1$ is a law with an $(a,\ +\infty)$-restriction, where $a = 1 + \ln 4 - \frac{1}{2}\ln|C| \ge 1$. In other words, an arbitrary sequence of integers $\{m_k\}$ $m_k \ge a$ is a skeleton of a unique (up to a shift) billiard trajectory.*

Proof. In the proof of Theorem 1, let us use the inequality $\Big|\tanh\alpha - \tanh\beta\Big| \le \dfrac{|\alpha-\beta|}{\cosh^2(d)}$ where simultaneously $\alpha,\ \beta \ge d > 0$. Then, if $m_k = M_{k+1} - M_k \ge a$,

$$\parallel A\xi - A\eta \parallel_{\ell_\infty} \le \frac{4}{|C|} \cdot \frac{\parallel \xi - \eta \parallel_{\ell_\infty}}{\cosh^2(a-1)} \quad .$$

The inequality $q = \dfrac{4}{|C|} \cdot \dfrac{1}{\cosh^2(a-1)} < \dfrac{16}{|C|e^{2(a-1)}} = 1$ yields that $\mathcal{A}$ is a contraction operator.

Inequality $|\xi| < 1/2$ also holds. Indeed, we have $m_k \geq a \geq 1$, so the arguments of the functions tanh in the equality (6) are not less than $a-1$. Hence, (6) implies: $|\xi_k| \leq \dfrac{1}{|C|}\Big(1 - \tanh(a-1)\Big) < \dfrac{2}{|C|} \cdot e^{-2(a-1)} = \dfrac{1}{8} < \dfrac{1}{2}$. ■

Remark. If the condition $\ell = 1$ in the formulation of Theorem 4 is not stipulated, then one needs to set a as follows:

$$a = \max\Big(1 + \frac{1}{2\ell} \cdot \ln\frac{16}{|C|},\ 1 + \frac{1}{2\ell} \cdot \ln\frac{4}{\ell|C|}\Big)\ .$$

The value $1 + \dfrac{1}{2\ell} \cdot \ln\dfrac{16}{|C|}$ in the written formula for a guarantees the contraction property for of the operator A, and the value $1+\dfrac{1}{2\ell}\cdot\ln\dfrac{4}{\ell|C|}$ guarantees that inequality $|\xi_k| < \ell/2$ holds.

Theorem 5. *Suppose that the Lipschitz condition (9) in the law (8) (see the formulation of Theorem 2) holds only when* $(a-1)\ell \leq x,\ y \leq (b+1)\ell$. *Then (8) is the law with an* (a,b)*-restriction.*

Proof. It suffices practically to repeat the proof of Theorem 2, applied only to sequences of integers m_k that satisfy $a \leq m_k \leq b$. ■

3.4 Generalized billiards with periodic law of reflection on both sides of the strip

Until now, we studied generalized billiards within an infinite strip with a periodic law of reflection on bottom side of the strip of width h and the ideal law on the upper side. Here we consider billiards with periodic laws of reflection on both sides of the strip. In the simplest case the laws of reflection are the same. It turns out that in this case, the problem can be reduced to a billiards with one ideal side. Indeed, reflecting the upper half of the strip together with the billiard trajectory with respect to the line of the symmetry of the strip (i.e., the middle line of the strip parallel to its boundary), yields a billiards inside a strip of the width $h/2$, in which the middle line plays the role of the ideal side.

Consider the general case in which an ℓ_1-periodic marking is given on the bottom side, while an ℓ_2-periodic marking is given on the upper side of the strip. We denote by $x_k^{(1)}$ and $x_k^{(2)}$ the consecutive coordinates of the reflection points on the bottom and on the top side, respectively. A billiard trajectory consists of infinitely many straight line segments (the links of the trajectory) successively connecting points

$$...;\ (x_{k-1}^{(1)}, 0);\ (x_k^{(2)}, h);\ (x_k^{(1)}, 0);\ (x_{k+1}^{(2)}, h);\ (x_{k+1}^{(1)}, 0); ...\ .$$

Denote by $\xi_k^{(1)}$, $\xi_k^{(2)}$ the local coordinates, and by $M_k^{(1)}$, $M_k^{(2)}$ the numbers of the marked intervals where the reflections occurs. Then, analogously to formula (2),

$$\left\{\begin{array}{lll} x_k^{(1)} & = & M_k^{(1)}\ell_1 + \xi_k^{(1)}, \qquad -\ell_1/2 \le \xi_k^{(1)} < \ell_1/2, \\ x_k^{(2)} & = & \sigma + M_k^{(2)}\ell_2 + \xi_k^{(2)}, \quad -\ell_2/2 \le \xi_k^{(2)} < \ell_2/2, \end{array}\right. \tag{12}$$

where $M_k^{(1)}$ $M_k^{(2)}$ are integers that constitute the *full skeleton* of the trajectory, $S(\gamma) = \{M_k^{(j)};\ k \in \mathbf{Z}, j = 1, 2\}$, and the number σ describes the shift of the ℓ_2-periodic marking on the top side of the strip.

The *property of universality* for billiards with periodic marking on both sides of the strip is introduced in the same manner as it was done for the billiards with marking on the non-ideal side only. Namely, each sequence of integers $\{M_k^{(j)};\ k \in \mathbf{Z}, j = 1, 2\}$ serves as a full skeleton of a uniquely determined billiard trajectory.

We suppose that, as in the formulation of the Theorem 2 (see formula (8)), the laws of reflection on both sides of the strip are periodic with the periods ℓ_1 and ℓ_2, respectively, and are given as follows:

$$\left\{\begin{array}{lll} \xi_k^{(1)} & = & g_1\left(x_{k+1}^{(2)} - x_k^{(1)},\ x_k^{(1)} - x_k^{(2)}\right), \\ \xi_k^{(2)} & = & g_2\left(x_k^{(1)} - x_k^{(2)},\ x_k^{(2)} - x_{k-1}^{(1)}\right), \end{array}\right. \tag{13}$$

with $g_1,\ g_2$ some continuous functions.

Theorem 6. *Suppose that the laws of reflection on both sides of the strip are periodic with the periods ℓ_1 and ℓ_2 and are given in the form (13). Suppose also that the functions g_j $(j = 1,\ 2)$, are Lipschitz with the constant $q < 1/4$, i.e., the following inequality holds:*

$$|g_j(x_2,\ y_2) - g_j(x_1,\ y_1)| \le q(|x_2 - x_1| + |y_2 - y_1|), \quad q < 1/4. \tag{14}$$

Suppose, in addition, that $|g_j(x,\ y)| < \ell_j/2$ for all values of the arguments. Then the generalized billiards with the law of reflection (13) has the property of universality. This means that each billiard trajectory is defined by its full skeleton $\{M_k^{(j)};\ k \in \mathbf{Z}, j = 1, 2\}$, and any two sequences of integers $\{M_k^{(j)}\}$, $j = 1, 2$, are a full skeleton for a particular billiard trajectory.

Proof.

Consider, for the billiard in question, an arbitrary trajectory γ, where $S(\gamma) = \{M_k^{(j)};\ k \in \mathbf{Z}, j = 1, 2\}$ is its full skeleton. Plugging expressions (12) into the formulae (13) yields for the skeleton and for the local coordinates of the reflection points $\xi_k^{(1)}$, $\xi_k^{(1)}$, of the trajectory γ the following equalities:

$$\left\{\begin{array}{lll} \xi_k^{(1)} & = & g_1\left(\sigma + M_{k+1}^{(2)}\ell_2 - M_k^{(1)}\ell_1 + \xi_{k+1}^{(2)} - \xi_k^{(1)},\ -\sigma - M_k^{(2)}\ell_2 + M_k^{(1)}\ell_1 + \xi_k^{(1)} - \xi_k^{(2)}\right), \\ \xi_k^{(2)} & = & g_2\left(-\sigma - M_k^{(2)}\ell_2 + M_k^{(1)}\ell_1 + \xi_k^{(1)} - \xi_k^{(2)},\ \sigma + M_k^{(2)}\ell_2 - M_{k-1}^{(1)}\ell_1 + \xi_k^{(2)} - \xi_{k-1}^{(1)}\right). \end{array}\right. \tag{15}$$

Conversely, if some sequence of integers $\{M_k^{(j)};\ k \in \mathbf{Z}, j = 1, 2\}$ and a sequence of real numbers $\xi_k^{(1)}$, $\xi_k^{(2)}$ satisfying $|\xi_k^{(1)}| < \ell_1/2$, $|\xi_k^{(2)}| < \ell_2/2$ satisfies also the equalities (15), then these sequences are, respectively, the full skeleton and the local coordinates of the reflection points of some billiard trajectory.

Therefore, in order to prove the theorem, it is enough to prove that for any sequence of integers $\{M_k^{(j)};\ k \in \mathbf{Z}, j = 1, 2\}$ the system of equations (15) with respect to the variables $\xi_k^{(1)}$, $\xi_k^{(2)}$ has a unique solution satisfying the inequalities $|\xi_k^{(1)}| < \ell_1/2$, $|\xi_k^{(2)}| < \ell_2/2$.

Let us consider the system (15) as an operator equation of the form (7) in the space $\ell_\infty^2 = \ell_\infty \times \ell_\infty$ with the norm $\| \xi \|_{\ell_\infty^2} = \max\Big(\| \xi^{(1)} \|_{\ell_\infty} , \| \xi^{(2)} \|_{\ell_\infty} \Big)$, where $\xi = (\xi^{(1)}, \xi^{(2)}) \in \ell_\infty^2$, $\xi^{(j)} = \{\xi_k^{(j)} |\ k \in \mathbf{Z}\} \in \ell_\infty$, $j = 1,\ 2$. From the explicit form of the operator $\mathcal{A}$, defined by the right hand sides of the system (15) and the hypotheses of the theorem it follows that the operator $\mathcal{A}$ is a contraction in the space ℓ_∞^2. Consequently, for each sequence of integers $\{M_k^{(j)};\ k \in \mathbf{Z}, j = 1, 2\}$ there exists a unique solution of the equation (15). The estimates $|\xi_k^{(1)}| < \ell_1/2$, $|\xi_k^{(2)}| < \ell_2/2$ follow directly from the following hypothesis of the theorem: $|g_j(x,\ y)| < \ell_j/2$. ■

3.5 Numerical methods for constructing the trajectory from its skeleton

Each billiard trajectory inside a strip is the union of successive segments (links) with the ends on the bottom and the top sides of the strip; the ends of the links are the reflection points of the billiard trajectory. So, in order to construct the billiard trajectory, it is sufficient to know the coordinates of successive reflection points on the strip's boundary. In turn, the abscises of the reflection points inside a strip with a periodic marking of the strip's boundary unambiguously tie up with the local coordinates of the reflection points and the full skeleton of the trajectory, – the numbers of the marked intervals to which they belong to. (See the formulae (2) for the billiards with the ideal law of reflection on the top side, and the formulae (11) in the general case.)

Thus, in order to construct the billiard trajectory γ from its full skeleton $S(\gamma)$, one needs to find first the local coordinates of the reflection points. The coordinates satisfy equations (6), (8), or (15) depending on the form of the reflection law on the strip's boundary. Since these equations are determined by the skeleton $S(\gamma)$ and the laws of reflection on the boundary, then, under the conditions shown in the Theorems **1, 2, 3, 4,** and **6**, the method of successive approximations converges as a geometric progression with a common ratio $q < 1$. This is the reason for this method to be so effective for constructing the billiard trajectory from its skeleton.

The numerically constructed trajectories with the ideal law of reflection on the upper side of the strip and the law (6) on the bottom side are represented on the Figures 6 – 14. We set, for the sake of simplicity, $\ell = 1$.

In Fig. 6, the billiard trajectory has the skeleton $s_6 = (..., \ 1, \ 1, \ 1, \ 4, \ 4, \ 4, \ ...)$, when the period jumps from 1 to 4. The trajectory in Fig. 7 is represented by its full skeleton $S_7 = (..., \ 2, \ 3, \ 4, \ 4, \ 4, \ 5, \ 6, \ ...)$, or its skeleton $s_7 = (..., \ 1, \ 1, \ 0, \ 0, \ 0, \ 1, \ 1, \ ...)$. This trajectory reflects three times from the same marked interval.

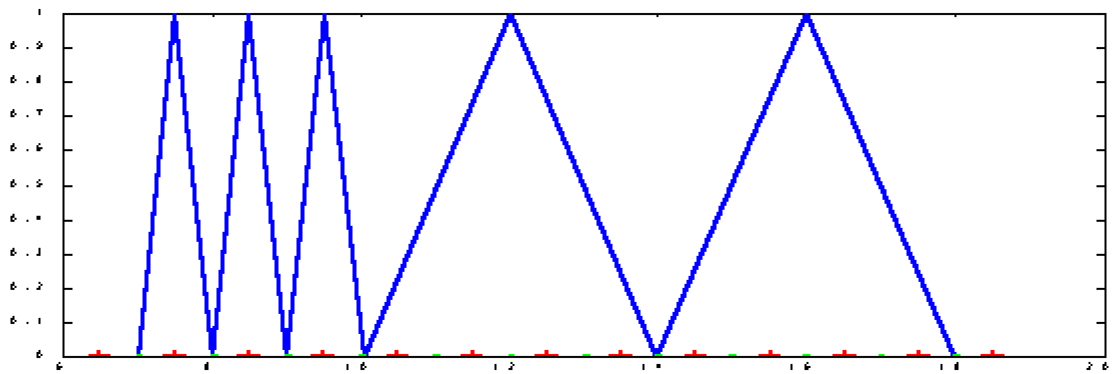

Figure 6: *A billiard trajectory with a change of its spatial period*

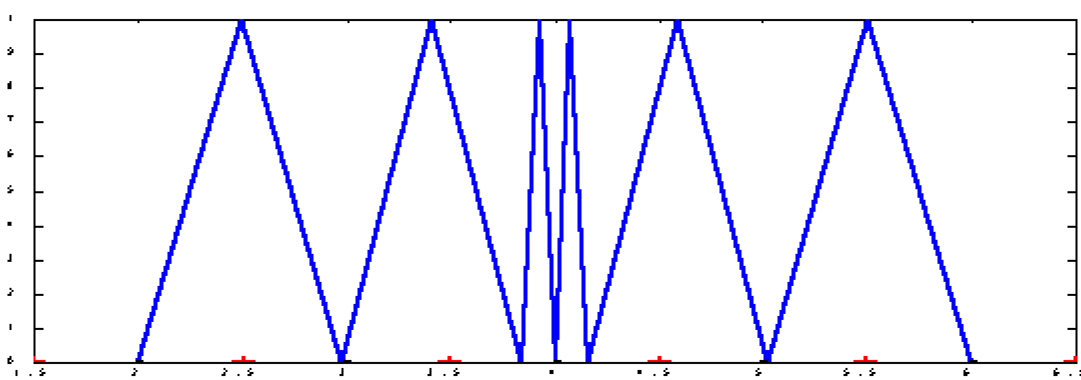

Figure 7: *A spatial periodic trajectory with local repetitions*

In Fig. 8 we can see a trajectory with local returns and with the skeleton $s_8 = (..., \ 2, \ -1, \ 2, \ -1, \ 2, \ -1, \ 2, \ ...)$. The skeleton of the trajectory in Fig. 9 is $s_9 = (..., \ 1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 2, \ 3, \ 4, \ 5, \ 5, \ 4, \ 3, \ 2, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1, \ ...)$; the period of this trajectory changes “smoothly”.

Fig. 10 represents a cyclic billiard trajectory with the full skeleton $S_{10} = (0, \ 5, \ 10, \ 0)$, while Fig. 11 represents a cyclic trajectory with local returns and the skeleton $s_{11} = (4, \ -1, \ 4, \ -2, \ 4, \ -1, \ 4, \ -2, \ 4, \ -1, \ 4, \ -17)$.

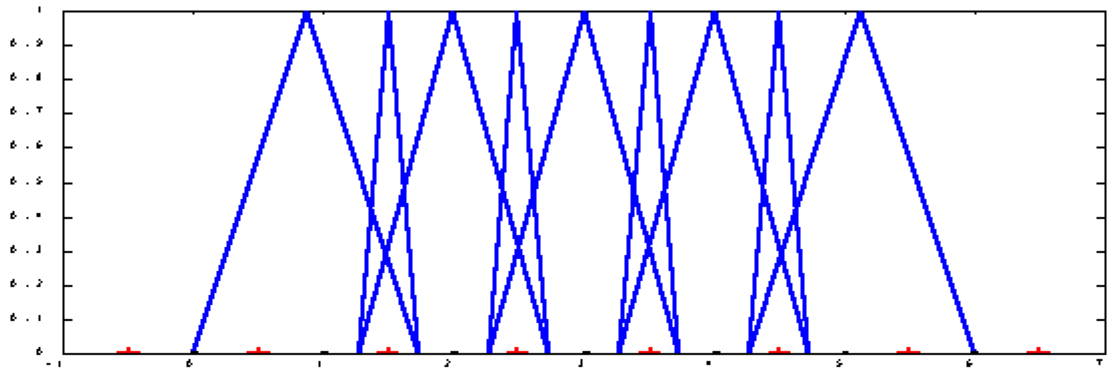

Figure 8: *A spatial periodic trajectory with local returns*

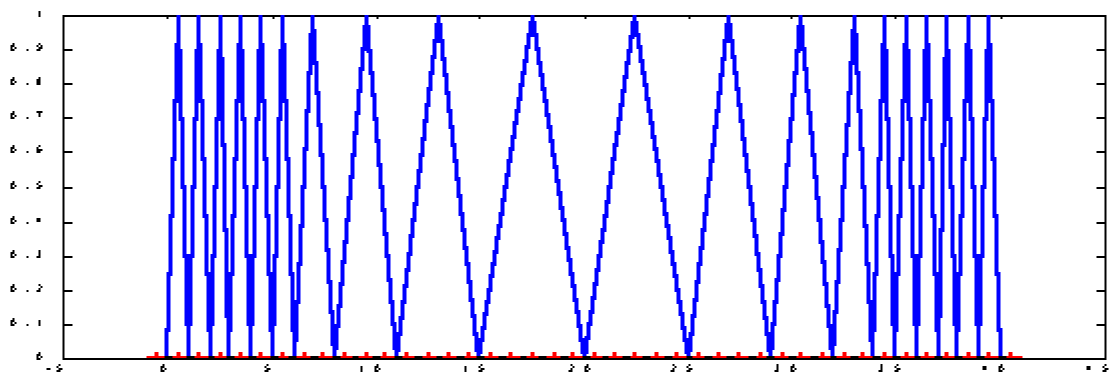

Figure 9: *A billiard trajectory with local change of its period*

The billiard trajectories of Figures 12, 13, 14 have, respectively, the skeletons $s_{12} = (..., 1, 1, 1, 10, 1, 1, 1, ...)$, $s_{13} = (..., 4, 1, 4, 1, 4, 1, 4, ...)$, and $s_{14} = (..., 1, 1, 0, 1, 1, 0, 1, 1, 0, ...)$.

4 Spatial chaos and spatial entropy for the generalized billiards with the property of universality

4.1 The structure of billiard trajectories

Consider inside an infinite strip of width h a broken line, the union of the links, that are straight line segments with the ends situated on the sides of the strip. All the ends of the links will be called *vertices* of the broken line. *A chain* will be understood as a broken line consisting of a finite ordered set of links, where every two consecutive

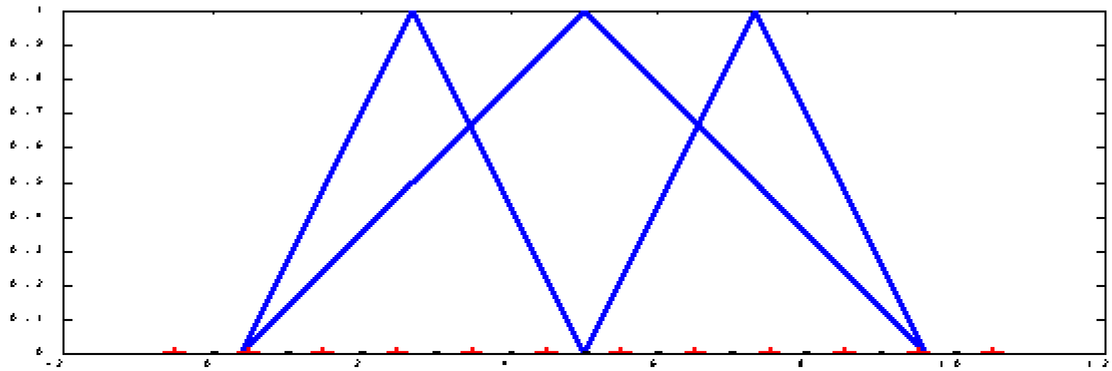

Figure 10: *A cyclic billiard trajectory*

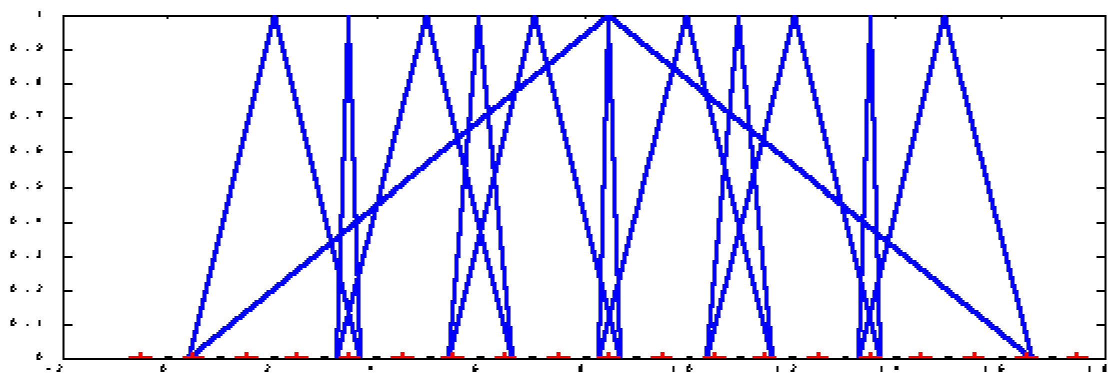

Figure 11: *A cyclic billiard trajectory with local returns*

links have a common vertex. A broken line with finite or infinite number of links, will be said to be *connected*, if any two of its links are the beginning and the end of some chain. A connected broken line inside a strip with no more than a countable set of links will be called *accessible* broken line.

We call an accessible broken line homeomorphic to a straight line an *infinite trajectory*. We call a homeomorphic line along with a chosen direction on the line *time*, where the homeomorphism assigns the *motion* of a point along this trajectory. Thereby, we have a way to say that a motion of a point along the trajectory is forward, respectively backward. Another class of accessible broken lines is the set of *cycles*, which are broken lines homeomorphic to a circle.

We will say that a broken line has an *ordered numbering of its vertices* with numbers **Z** if the vertices with consecutive numbers are the ends of the same link, and the movement along this numeration in the positive and negative directions cover each link at least once. As a matter of fact, the same vertex can be numbered several (might be countably many) times. There is a natural ordered numeration of the

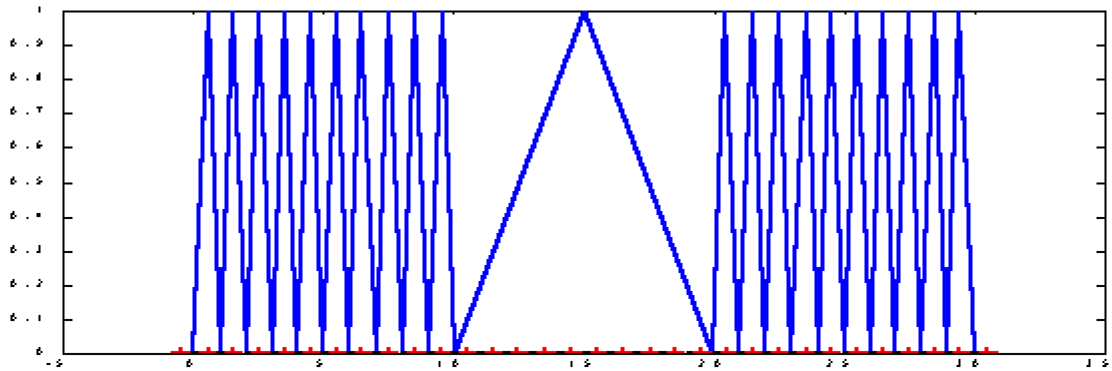

Figure 12: *A spatial periodic billiard trajectory with one perturbation*

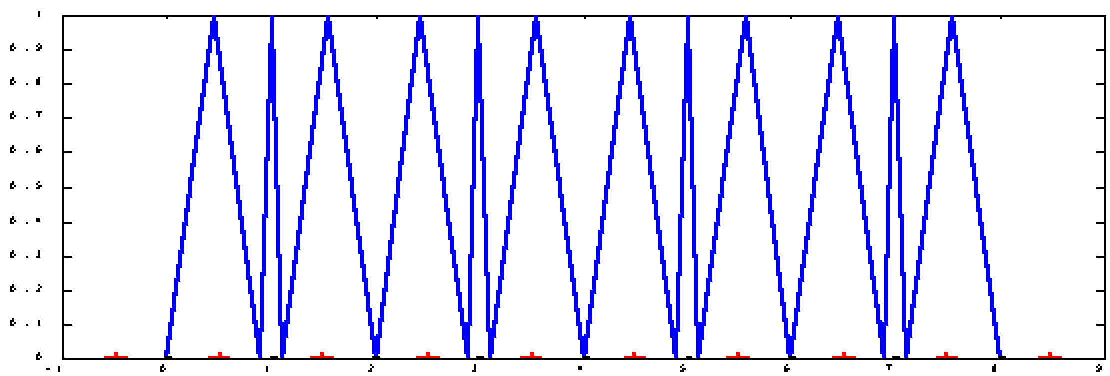

Figure 13: *A spatial periodic billiard trajectory with local repetitions*

vertices for infinite trajectories and cycles: in the first case, the order is given by time, and for a cycle, by the direction of the movement along the cycle. A little bit less evident is an ordered numeration of the vertices for a broken line consisting of a countable set of links, emanated from the same point, A, on the bottom side of the strip, to the integer even points on the top side: we number all the vertices of the broken line on the top boundary with even integers (the abscises of these points), and prescribe all the odd numbers to the vertex A. An ordered movement along this broken line leads to a double covering each of the links.

Lemma. *Each accessible broken line access an ordered numbering of its vertices.*

Proof. Let the links of the broken line be numbered by numbers in $\mathbf{Z}$ (which perhaps form a proper subset of $\mathbf{Z}$). Such a numeration is always possible because, by definition, an accessible broken line contains no more than a countable set of links. Without loss of generality, one can find a link numbered with a zero ("the zero link").

Consider a link with the smallest positive number, n_1. Since the accessible broken

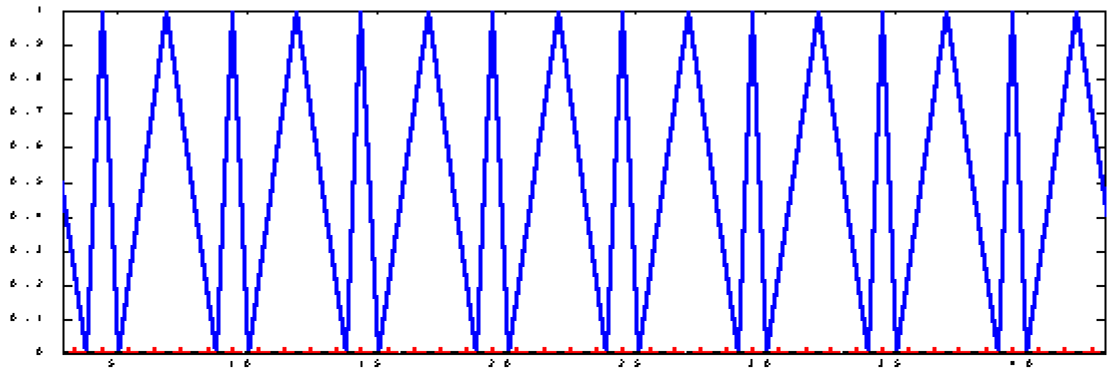

Figure 14: *A strictly monotonic spatial periodic billiard trajectory*

line is connected, there is a chain whose beginning link is the zero link and the ending link is that with number n_1; we denote this chain by $C(0,\ n_1)$. Number all the vertices of the chain with the consecutive integers from 0 to n_1+1. Further, consider the link with the smallest positive number, n_2, which is not involved into the chain $C(0,\ n_1)$. Consider the chain $C(n_1,\ n_2)$ with the initial link n_1 and the final link n_2. Continue numbering the vertices of the chain $C(0,\ n_1)$ for the naturally ordered set of vertices of the chain $C(n_1,\ n_2)$. This process is either infinite or finite. In the first case, we get the numeration of the vertices by all positive integers. In the second case, there is a final link numbered by some number n_m. The further numbering of the vertices with positive integers is as follows: we number in succession the ends of the link n_m. In both cases we will get a numbering of the vertices by all positive numbers.

In the same way, we number the vertices of an accessible broken line by all negative integers. It is clear that the described numeration by numbers in $\mathbf{Z}$ is an ordered numeration. ■

Definition 5. Let γ_1 and γ_2 be two accessible broken lines. We say that these broken lines are spaced from each other by less than ε in the Hausdorff metric, if in the ε-neighborhood of every point of each of the lines there is at least one point of some other line. In other words, each broken line lies inside the ε-neighborhood of the other line. We write this fact in the form $\rho(\gamma_1,\gamma_2) < \varepsilon$, where

$$\rho(\gamma_1,\gamma_2) := \max\{\sup_{x\in\gamma_1}\inf_{y\in\gamma_2}\ d(x,y),\ \ \sup_{y\in\gamma_2}\inf_{x\in\gamma_1}\ d(x,y)\}$$

is the Hausdorff distance between the broken lines γ_1 and γ_2, and $d(x,y)$ is the standard Euclidean distance between points x and y.

The appearance of the word "universality" in the name of a generalized billiards with the property of universality can be explained by the following result.

Theorem 7. *For each $\varepsilon > 0$, each billiards B_ε with the property of universality and with a ε-periodic law of reflection on both sides of the strip has the following prop-*

erty. If γ is an arbitrary accessible broken line, then there exists a billiard trajectory γ_ε in the billiard B_ε such that

$$\rho(\gamma, \gamma_\varepsilon) < \varepsilon \ . \tag{16}$$

In other words, an arbitrary accessible broken line can be approximated with an arbitrary precision in the Hausdorff metric by a billiard trajectory of the billiards B_ε.

Proof. Given $\varepsilon > 0$, let us consider a billiards B_ε with an ε-periodic law of reflection on both sides of the strip that has the property of universality, i.e., each set of integers $\{M_k^{(j)}\}$ $(j = 1, 2)$ is a full skeleton of a unique billiard trajectory in this billiards B_ε.

Let γ be an accessible broken line. According to the above lemma in this section, the broken line γ has an ordered numbering of its vertices. This numbering orders automatically the numbers of the marked intervals on the strip's sides on which the vertices of γ are situated. This gives a set of integers $\{M_k^{(j)}\}$, $j = 1, 2$. Consider this set as a full skeleton of the billiard trajectory γ_ε in the billiards B_ε. Each vertex of the broken line γ and the corresponding (i.e., with the same number) vertex of the billiard trajectory γ_ε lies, by construction, on the same marked interval of length ε. The correspondence between the vertices of the broken line and the vertices of the billiard trajectory generates a natural correspondence between their links. Both ends of the corresponding links belong to the same marked intervals of length ε on both sides of the strip. Therefore, the distance between such links is going to be less than ε. This yields (16). ∎

Next theorem shows that the generalized billiards with the property of universality possesses the total spatial chaos.

Theorem 8. *A generalized billiards with the property of universality has billiard trajectories that are everywhere dense inside the strip.*

Proof. Consider a billiards inside the strip of width h with the ℓ_1, ℓ_2-periodic marking of its boundary. The case of the billiards with an ideal top side and with a periodic marking of the bottom non-ideal side is covered by this more general case (see the beginning of the Subsection 3.4).

Let us introduce auxiliary values that are going to be used for the rest part of the proof. Let φ_k, $k \in \mathbf{Z}$, be acute angles satisfying $\sin\varphi_k = \varepsilon_k/2\ell$, where $\ell = \max(\ell_1, \ \ell_2)$, $\varepsilon_0 = \dfrac{\ell h}{2\sqrt{\ell^2 + h^2}}$, and $\varepsilon_k = \varepsilon_0/2^{|k|}$. We remark that $\varepsilon_0 < h/2$.

Draw auxiliary straight lines that make acute angles φ_k with the positive x-axis. These lines pass either through points X_k, if these points are at a distance less than ε_k from the strip's boundary, or through points $\widetilde{X_k}$ close to X_k, where each $\widetilde{X_k}$ is obtained from X_k by the vertical shift at a distance $\varepsilon_k/2$ towards the middle (horizontal) axis of the strip. Each of the lines drawn meets the bottom $(j = 1)$ and the top $(j = 2)$ sides of the strip on the marked intervals numbered by $M_k^{(j)}$, $j = 1, 2$. Then the segment of a billiard trajectory with the reflection points situated on the intervals $M_k^{(j)}$, $j = 1, 2$, intersects the circle $U_{\varepsilon_k}(X_k)$ centered at the point X_k and having radius ε_k. Indeed, the circle $B_{\varepsilon_k/2}$ centered at one of the points through which we drew the auxiliary straight lines lies both entirely inside the circle $U_{\varepsilon_k}(X_k)$ and inside the billiard strip, because $\varepsilon_k < h/2$. It is seen from the construction of the auxiliary straight lines that the segment of the billiard trajectory meets the circle $B_{\varepsilon_k/2}$, and, consequently, the circle $U_{\varepsilon_k}(X_k)$.

Let us consider the thus constructed sequence of integers $\{M_k^{(j)},\ k \in \mathbf{Z},\ j = 1, 2\}$ as the full skeleton of a billiard trajectory γ. The trajectory γ exists and is determined uniquely by this full skeleton because, by the hypothesis of the theorem, the given billiards has the property of universality.

We show now that the billiard trajectory γ is everywhere dense in the entire billiard strip. Indeed, let us consider an arbitrary point X inside the strip together with its ε-neighborhood, $U_\varepsilon(X)$, and show that γ meets $U_\varepsilon(X)$. Let us choose a point X_n from the countable everywhere dense set of points $\{X_k\}$ which is located at a distance less than $\varepsilon/2$ from the point X, where, at the same time, the inequality $\varepsilon_n < \varepsilon/2$ should hold. Then the point X_n together with its ε_n-neighborhood $U_{\varepsilon_n}(X_n)$ lies entirely inside the neighborhood $U_\varepsilon(X)$. Since γ intersects $U_{\varepsilon_n}(X_n)$, γ meets $U_\varepsilon(X)$, too. ■

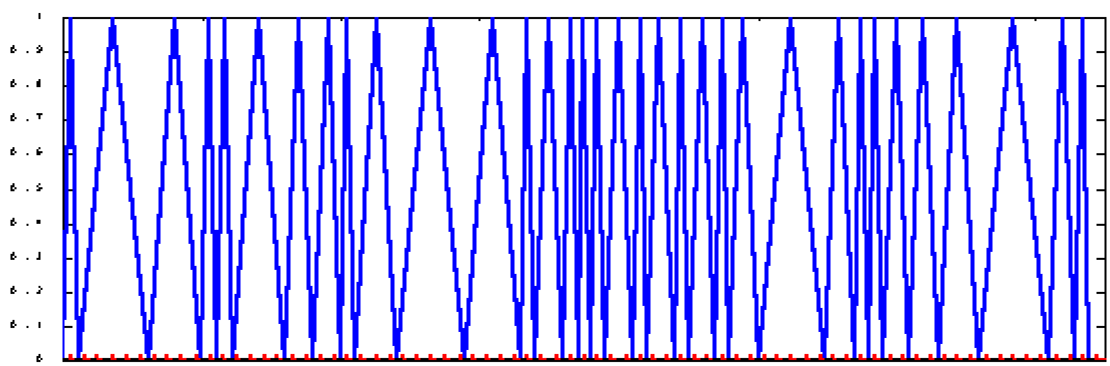

Figure 15: *Spatial chaotic billiard trajectory*

4.2 Spatial chaos

Since the skeletons of billiard trajectories in billiards with the property of universality and the ℓ-periodic law of reflection can be arbitrary sequences of integers m_k, in particular they can be randomly chosen, this leads to the existence of spatially “chaotic” trajectories in such billiards. In Fig. 15, a billiard trajectory γ inside a strip with the ideal law of reflection on the top side and with the 1-periodic law (5) on the bottom side is presented. A random sequence of integers $n \geq 1$ has been chosen with the probability $p_n = 2^{-n}$ as the full skeleton:

$$s(\gamma) = [1,5,4,1,1,4,2,2,1,3,5,4,1,2,1,1,1,2,1,2,1,2,1,2,5,2,1,1,2,2,3,5,2,1] ,$$

This sequence leads to a spatial-chaotic billiard trajectory.

Following the papers [2] [3], we introduce the notion of spatial chaos in billiards inside a strip.

Definition 6. We say that a generalized billiards possesses *spatial chaos*, if for each $\varepsilon > 0$ there is a $d(\varepsilon) > 0$ such that for each monotonic sequence of points $X_k = (x_k,\ y_k)$ from the billiard strip satisfying $0 \leq y_k \leq h$ and $x_{k+1} - x_k > d(\varepsilon)$, there is a strictly monotonic billiard trajectory γ such that the distances from the points X_k to the trajectory γ is strictly less than ε:

$$d(X_k, \gamma) := \inf_{Y \in \gamma} d(X_k, Y) < \varepsilon \ .$$

Theorem 9. *A generalized billiards inside a strip with the property of universality possesses spatial chaos.*

Proof. Consider a billiards inside the strip with the $\ell_1,\ \ell_2$-periodic marking of its boundary. Let ε be an arbitrary small positive number, $\varepsilon < \dfrac{\ell h}{2\sqrt{\ell^2 + h^2}}$. Set $d(\varepsilon) = 2\ell\left(\dfrac{h}{\varepsilon} + 1\right)$, where h is the width of the billiard strip, and $\ell = \max(\ell_1,\ \ell_2)$. Let an arbitrary sequence of points $X_k = (x_k,\ y_k)$ in the strip satisfy $x_{k+1} - x_k > d(\varepsilon)$. Let us draw through every neighborhood $U_\varepsilon(X_k)$ of the point X_k, as we did in the proof of Theorem 8, the line making an angle φ with the positive x-axis such that $\sin\varphi = \varepsilon/2\ell$. The intersection points of these lines with the marked intervals situated on the boundary of the strip determine the full skeleton $S(\gamma) = \{M_k^{(j)},\ k \in \mathbf{Z},\ j = 1, 2\}$ of the sought trajectory γ, where the trajectory is strictly monotonic and passes through the ε-neighborhoods of all the points $X_k = (x_k,\ y_k)$. The strict monotonicity of the trajectory γ follows from the fact that the projections to the bottom side of the strip of all the γ skeleton's intervals, located on the top side of the strip, alternate, with no intersection, with the marked γ skeleton's intervals on the bottom side. ■

4.3 Spatial entropy

Consider all strictly monotonic billiard trajectories (see definition in the Subsection 3.2) that are periodic along the x-axis with the period T. Let us call them T-*periodic* trajectories and denote by $S(T)$ the number of all the T-periodic trajectories.

Definition 7. *The spatial entropy* of a billiards with respect to strictly monotonic periodic trajectories is by definition the number

$$\eta = \overline{\lim}_{T\to\infty} \ \frac{1}{T} \ln S(T) \ . \tag{17}$$

Theorem 10. *The spatial entropy of a billiards with the property of universality inside a strip with the ℓ-periodic law of reflection along the bottom side of the strip and the ideal law of reflection on the top side is equal to*

$$\eta = \frac{1}{\ell} \ln 2 \ . \tag{18}$$

Proof. Consider periods T that are multiple of ℓ, i.e., set $T = N\ell$, where N is an integer. There is a one-to-one correspondence between the strictly monotonic sequences and their codes. Therefore, $S(N\ell)$ equals the number of non-trivial N-periodic codes. The number of such codes equals $2^N - 1$. Hence

$$\eta = \lim_{N\to\infty} \frac{1}{N\ell} \ln(2^N - 1) = \frac{1}{\ell} \ln 2 \ .$$

■

5 Acknowledgements

I.L. and L.N. express their gratitude to DFG through a financial support of the project 436 UKR 113/61 and thanks the Institute of Applied Mathematics at the University of Bonn for the hospitality. G.G. would like to express his gratitude to the A.v. Humboldt Foundation for the financial support, and to the Department of Physics at Bielefeld University and the Bielefeld-Bonn-Stochastic Research Center BiBoS for their hospitality.

References

[1] I.L. Nizhnik: *Time domain analysis of nonlinear lattices solutions*, Institute of Mathematics of the Ukraine Academy of Sciences, Kiev (1998), Preprint 98.1

[2] L.P. Nizhnik, M. Hasler, I.L. Nizhnik: *Stable stationary solutions in reaction-diffusion systems consisting of a 1-d array of stable cells*, International Journal of Bifurcation and Chaos, 2 (2002), 261-279

[3] S. Albeverio, I.L. Nizhnik: *Spatial chaos in a fourth-order nonlinear parabolic equation*, Physics Letters A 288 (2001), pp. 299-304

[4] D. Alonso, A. Ruiz, I. de Vega: *Transport in polygonal billiards*, Physica D, 187 (2004), pp. 184-199

[5] D. Alonso, R. Artuzo, G. Casati, I. Guarneri: *Heat conductivity and dynamical instability*, Phys. Rev. Lett., 82 (1999), p. 1859

[6] L. Matyas, R. Klages: *Irregular diffusion in the bouncing ball billiard*, Physica D, 187 (2004), pp. 165-183

[7] S. Tabachnikov: *Billiards*, Panoramas et Syntheses, No. 1, 1995

[8] G. Galperin, A. Zemliakov: *Mathematics of Billiards*, Moscow, "Nauka", 1986

[9] G. Galperin, N. Chernov: *Mathematical Billiards*, Moscow, "Znanie", 1990

[10] Ya. G. Sinai: *Ergodic Theory*, Princeton University Press, 1980

[11] I. Kornfeld, Ya. G. Sinai: *Ergodic Theory*, Moscow, Nauka, 1980

[12] N. I. Chernov, R. Markarian: *Introduction to the Ergodic Theory of Chaotic Billiards*, 2000.

[13] L. A. Bunimovich: *On billiards close to dispersing ones*, Sbornik: Mathematics, 1974, Vol. 94, pp. 49-73

[14] N. I. Chernov: *A new proof of Sinai's formula for calculating of the entropy of hyperbolic billiards. Application to the Lorentz gas and the Bunimovich stadium*, Functional Analysis and Its Applications, Vol 25, 1991, pp. 50-69