Infinite interacting diffusion particles I: Equilibrium process and its scaling limit

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Abstract

A stochastic dynamics $(\mathbf{X}(t))_{t\geq 0}$ of a classical continuous system is a stochastic process which takes values in the space Γ of all locally finite subsets (configurations) in $\mathbb{R}^d$ and which has a Gibbs measure μ as an invariant measure. We assume that μ corresponds to a symmetric pair potential $\phi(x-y)$. An important class of stochastic dynamics of a classical continuous system is formed by diffusions. Till now, only one type of such dynamics—the so-called gradient stochastic dynamics, or interacting Brownian particles—has been investigated. By using the theory of Dirichlet forms from [27], we construct and investigate a new type of stochastic dynamics, which we call infinite interacting diffusion particles. We introduce a Dirichlet form $\mathcal{E}_\mu^\Gamma$ on $L^2(\Gamma;\mu)$, and under general conditions on the potential ϕ, prove its closability. For a potential ϕ having a "weak" singularity at zero, we also write down an explicit form of the generator of $\mathcal{E}_\mu^\Gamma$ on the set of smooth cylinder functions. We then show that, for any Dirichlet form $\mathcal{E}_\mu^\Gamma$, there exists a diffusion process that is properly associated with it. Finally, in a way parallel to [17], we study a scaling limit of interacting diffusions in terms of convergence of the corresponding Dirichlet forms, and we also show that these scaled processes are tight in $C([0,\infty),\mathcal{D}')$, where $\mathcal{D}'$ is the dual space of $\mathcal{D}{:=}C_0^\infty(\mathbb{R}^d)$.

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1 Introduction

A stochastic dynamics $(\mathbf{X}_t)_{t\geq 0}$ of a classical continuous system is a stochastic process which takes values in the space Γ of all locally finite subsets (configurations) in $\mathbb{R}^d$ and which has a Gibbs measure μ as an invariant measure. We assume that μ corresponds to a symmetric, translation invariant pair potential $\phi(x-y)$ and activity $z>0$.

An important class of stochastic dynamics of a classical continuous system is formed by diffusions. Till now, only one type of such dynamics—the so-called gradient stochastic dynamics, or interacting Brownian particles—has been investigated. This diffusion process informally solves the following system of stochastic differential equations:

$$
\begin{aligned}
&dx(t) = -\sum_{y(t)\in\mathbf{X}(t),\, y(t)\neq x(t)} \nabla\phi(x(t)-y(t))\,dt + \sqrt{2}\,dB^x(t), \qquad x(t)\in\mathbf{X}(t), \qquad (1.1)\\
&\mathbf{X}(0)=\gamma\in\Gamma,
\end{aligned}
$$

where $(B^x)_{x\in\gamma}$ is a sequence of independent Brownian motions. The study of such diffusions has been initiated by R. Lang [24] (see also [42, 13]), who considered the case $\phi\in C_0^3(\mathbb{R}^d)$ using finite-dimensional approximations of stochastic differential equations. More singular ϕ, which are of particular interest from the point of view of statistical mechanics, have been treated by H. Osada [32] and M. Yoshida [46]. These authors were the first to use the Dirichlet form approach from [27] for the construction of such processes. However, they could not write down the corresponding generator explicitly, hence could not prove that their processes actually solve (1.1) weakly. This, however, was proved in [5] (see also the survey paper [36]) by showing an integration by parts formula for the respective Gibbs measures.

But the gradient stochastic dynamics is, of course, not the unique diffusion process which has μ as an invariant measure. Indeed, let us consider the following system of stochastic differential equations:

$$dx(t) = \sqrt{2}\, \exp\left[\frac{1}{2} \sum_{y(t)\in \mathbf{X}(t),\, y(t)\neq x(t)} \phi(x(t)-y(t))\right] dB^x(t), \qquad x(t)\in \mathbf{X}(t), \tag{1.2}$$
$$\mathbf{X}(0) = \gamma \in \Gamma.$$

At least informally, one sees that this dynamics does also leave μ invariant. Note that, in system (1.1), the information about the interaction between particles is concentrated in the drift term, while in system (1.2) the interaction is in the diffusion coefficient and the drift term is absent. This is why we shall call a process that (weakly) satisfies (1.2) infinite interacting diffusion particles, or just interacting diffusions. Note that, if there is no interaction ($\phi = 0$), both processes solving (1.1) and (1.2) coincide.

In this paper, using the Dirichlet form approach (see [27]), under very wide conditions on ϕ (more pricisely, under (A1), (A2) or (A1), (A3), see below), we construct an equilibrium process that weakly solves (1.2). The problem of constructing a solution for (1.2) which starts from a given configuration, or a given distribution, is still open. Actually, this problem may be studied by using the ideas developed for the Hamiltonian and gradient stochastic dynamics, see [21], that is, by obtaining equations for the time evolution of correlation functions, or corresponding generating (Bogoliubov) functionals. This will be the subject of our forthcoming research.

There also exists another type of stochastic dynamics of a classical continuous system—the so-called Glauber-type dynamics, which is a spatial birth-and-death process, see [22].

Let us briefly describe the contents of the paper. After some preliminary information about Gibbs measures in Section 2, we construct a bilinear form $\mathcal{E}_\mu^\Gamma$ on $L^2(\Gamma;\mu)$ in Section 3. This form is defined on the set of smooth cylinder functions as follows:

$$\mathcal{E}_\mu^\Gamma(F,G) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} zm(dx)\, \langle \nabla_x F(\gamma\cup\{x\}), \nabla_x G(\gamma\cup\{x\})\rangle, \tag{1.3}$$

where m denotes Lebesgue measure on $\mathbb{R}^d$. By using the Georgii–Nguyen–Zessin identity, one gets an equivalent representation of $\mathcal{E}_\mu^\Gamma$:

$$\mathcal{E}_\mu^\Gamma(F,G) = \int_\Gamma \mu(d\gamma) \sum_{x\in\gamma} \exp\left[\sum_{y\in\gamma\setminus\{x\}} \phi(x-y)\right] \langle \nabla_x F(\gamma), \nabla_x G(\gamma)\rangle, \tag{1.4}$$

where $\nabla_x F(\gamma)$ is defined as in (3.10). We show that $\mathcal{E}_\mu^\Gamma$ is a pre-Dirichlet form. To compare our situation with the gradient dynamics, let us recall that, in the latter case, the corresponding Dirichlet form $\widetilde{\mathcal{E}}_\mu^\Gamma$ for F, G as in (1.3) looks as follows (see [5]):

$$\begin{aligned}\widetilde{\mathcal{E}}_\mu^\Gamma(F,G) &= \int_\Gamma \mu(d\gamma) \sum_{x\in\gamma} \langle \nabla_x F(\gamma), \nabla_x G(\gamma)\rangle \\ &= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} zm(dx)\, \exp\left[-\sum_{y\in\gamma\setminus\{x\}} \phi(x-y)\right] \langle \nabla_x F(\gamma\cup\{x\}), \nabla_x G(\gamma\cup\{x\})\rangle.\end{aligned}$$

In Section 4, we prove the closability of $\mathcal{E}_\mu^\Gamma$ under fairly general conditions on the potential ϕ, and in Section 5, assuming additionally that the function $e^{\phi(x)}$ is integrable in a neighborhood of zero (which still admits some "weak" singularity of ϕ), we write down the generator of $\mathcal{E}_\mu^\Gamma$ on the set of smooth cylinder functions. This generator looks as follows:

$$H_\mu^\Gamma F(\gamma) = -\sum_{x\in\gamma} \exp\left[\sum_{y\in\gamma\setminus\{x\}} \phi(x-y)\right] \Delta_x F(\gamma). \tag{1.5}$$

In Section 6, we prove the existence of a conservative diffusion process which is properly associated with the (closed) Dirichlet form $\mathcal{E}_\mu^\Gamma$. This process lives, in general, in the bigger space $\ddot{\Gamma}$ of all locally finite multiple configurations, but we prove, analogously to [38], that the process indeed lives in Γ in case $d \geq 2$. (If $d = 1$, one cannot, of course, exclude collisions of particles.) According to (1.5), the constructed diffusion process informally solves (1.2).

Section 7 is devoted to the study of a scaling limit of the constructed process $(\mathbf{X}(t))_{t\geq 0}$. The scaling we study is the same as the one considered by many authors for the gradient stochastic dynamics. The scaled process $(\mathbf{X}_\epsilon(t))_{t\geq 0}$ is defined by

$$\mathbf{X}_\epsilon(t) := S_{\mathrm{out},\,\epsilon}(S_{\mathrm{in},\,\epsilon}(\mathbf{X}(\epsilon^{-2}t))), \qquad t \geq 0,\ \epsilon > 0, \tag{1.6}$$

and we are interested in the scaling limit as $\epsilon \to 0$. The first scaling in (1.6), $S_{\mathrm{in},\,\epsilon}$, scales the positions of particles inside the configuration space as follows:

$$\Gamma \ni \gamma \mapsto S_{\mathrm{in},\,\epsilon}(\gamma) := \{\epsilon x \mid x \in \gamma\} \subset \Gamma.$$

The second scaling, $S_{\mathrm{out},\,\epsilon}$, leads out of the configuration space and is given by

$$\Gamma \ni \gamma \mapsto S_{\mathrm{out},\,\epsilon}(\gamma) := \epsilon^{d/2}\gamma - \epsilon^{-d/2}\rho\, dx \in \mathcal{D}',$$

where we identify the configuration with the corresponding sum of Dirac measures, ρ is the first correlation function of μ, and $\mathcal{D}'$ is the dual space of $\mathcal{D} := C_0^\infty(\mathbb{R}^d)$.

T. Brox showed in [9] that, in the low activity-high temperature regime, the Gibbs measure μ converges under the scaling $S_{\mathrm{out},\,\epsilon} S_{\mathrm{in},\,\epsilon}$ to a corresponding white noise measure ν_c with covariance operator $c\,\mathrm{Id}$, where the constant $c > 0$ is explicitly given through the first and second moments of μ. However, T. Brox believed that there is no limiting Markov process for the scaling limit of the gradient stochastic dynamics. Then, H. Rost gave some heuristic arguments for the existence of a limiting generalized Ornstein–Uhlenbeck process [39]. In the celebrated paper [44], H. Spohn described a proof of convergence of the scaled processes in the case where the underlying potential ϕ is smooth, compactly supported, and positive, and $d \le 3$. In [18], M. Z. Guo and G. Papanicolaou tried to prove convergence of the corresponding resolvents, however their considerations were on a more heuristic level. Finally, in the recent paper [17], in the case of a general potential ϕ, the authors proved convergence of the processes on the level of convergence of the associated Dirichlet forms. Furthermore, the tightness of the processes in $C([0,\infty),\mathcal{D}')$ was proven, and the convergence of the processes in law was shown under the assumption that the Boltzmann–Gibbs principle holds.

In this paper, we follow the approach of [17]. So, we show that, on the level of convergence of the associated Dirichlet forms, the scaled processes $(\mathbf{X}_\epsilon(t))_{t\ge0}$ converge to the generalized Ornstein–Uhlenbeck process $(\mathbf{N}(t))_{t\ge0}$ in $\mathcal{D}'$ that informally satisfies the following stochastic differential equation:

$$d\mathbf{N}(t,x) = \frac{1}{c}\,\Delta\mathbf{N}(t,x)\,dt + \sqrt{2}\,d\mathbf{W}(t,x), \tag{1.7}$$

where $(\mathbf{W}(t))_{t\ge0}$ is a Brownian motion on $\mathcal{D}'$ with covariance operator $-\Delta$. We recall that the limiting process $(\widetilde{\mathbf{N}}(t))_{t\ge0}$ of the gradient stochastic dynamics satisfies:

$$d\widetilde{\mathbf{N}}(t,x) = \frac{\rho}{c}\,\Delta\widetilde{\mathbf{N}}(t,x)\,dt + \sqrt{2\rho}\,d\mathbf{W}(t,x). \tag{1.8}$$

Thus, comparing (1.7) and (1.8), we see that the gradient stochastic dynamics and the interacting diffusions have great similarity on the macroscopic level, though they have different bulk diffusion coefficients: ρ/c for the former stochastic dynamics, and $1/c$ for the latter.

We finish this paper by proving the tightness of the scaled processes $(\mathbf{X}_\epsilon(t))_{t\ge0}$ in $C([0,\infty),\mathcal{D}')$. To complete the proof of the convergence in law of the scaled processes, one still needs to prove the Boltzmann–Gibbs principle in our situation, which remains an open problem.

It is also possible to study an invariance principle (scaling limit) of a tagged particle of interacting diffusions (cf. [33, 34]). This will be the subject of future research.

Finally, we would like to mention that, though some proofs of the results of this paper use the ideas and techniques developed for the gradient dynamics, for convenience of the reader we have tried to make this paper self-contained as possible.

2 Gibbs measures on configuration spaces

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all subsets of $\mathbb{R}^d$ which are locally finite:

$$\Gamma_{\mathbb{R}^d} := \big\{\, \gamma \subset \mathbb{R}^d \mid |\gamma_\Lambda| < \infty \text{ for each compact } \Lambda \subset \mathbb{R}^d \,\big\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure

$$\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d),$$

where ε_x is the Dirac measure with mass at x, and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. The space Γ can be endowed with the relative topology as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology on Γ with respect to which all maps

$$\Gamma \ni \gamma \mapsto \langle f, \gamma\rangle := \int_{\mathbb{R}^d} f(x)\, \gamma(dx) = \sum_{x\in\gamma} f(x), \qquad f \in \mathcal{D},$$

are continuous. Here, $\mathcal{D} := C_0^\infty(\mathbb{R}^d)$ is the space of all infinitely differentiable real-valued functions on $\mathbb{R}^d$ with compact support. We shall denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra on Γ.

Let π_z, $z > 0$, denote the Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity measure $zm(dx)$. This measure can be characterized by its Laplace transform

$$\int_\Gamma \exp[\langle f, \gamma\rangle]\, \pi_z(d\gamma) = \exp\left(\int_{\mathbb{R}^d} (e^{f(x)} - 1)\, zm(dx)\right), \qquad f \in \mathcal{D}.$$

We refer e.g. to [45, 4] for a detailed discussion of the construction of the Poisson measure on the configuration space.

Now, we proceed to consider Gibbs measures. A pair potential is a Borel measurable function $\phi\colon \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ such that $\phi(-x) = \phi(x)$ for all $x \in \mathbb{R}^d$. We shall also suppose that $\phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. Let $\mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ denote the set of all open, relatively compact sets in $\mathbb{R}^d$. Then, for $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$, the conditional energy $E^\phi_\Lambda\colon \Gamma \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E^\phi_\Lambda(\gamma) := \begin{cases} \sum\limits_{\{x,y\}\subset\gamma,\, \{x,y\}\cap\Lambda\neq\varnothing} \phi(x-y), & \text{if } \sum\limits_{\{x,y\}\subset\gamma,\, \{x,y\}\cap\Lambda\neq\varnothing} |\phi(x-y)| < \infty, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

Given $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$, define for $\gamma \in \Gamma$ and $\Delta \in \mathcal{B}(\Gamma)$

$$\begin{aligned} \Pi^{z,\phi}_\Lambda(\gamma, \Delta) :=\, & \mathbf{1}_{\{Z^{z,\phi}_\Lambda < \infty\}}(\gamma)\, [Z^{z,\phi}_\Lambda(\gamma)]^{-1} \\ & \times \int_\Gamma \mathbf{1}_\Delta(\gamma_{\Lambda^{\mathrm{c}}} + \gamma'_\Lambda) \exp\big[-E^\phi_\Lambda(\gamma_{\Lambda^{\mathrm{c}}} + \gamma'_\Lambda)\big]\, \pi_z(d\gamma'), \end{aligned} \tag{2.2}$$

where $\Lambda^{\mathrm{c}} := \mathbb{R}^d \setminus \Lambda$ and

$$Z^{z,\phi}_\Lambda(\gamma) := \int_\Gamma \exp\big[-E^\phi_\Lambda(\gamma_{\Lambda^{\mathrm{c}}} + \gamma'_\Lambda)\big]\, \pi_z(d\gamma'). \tag{2.3}$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a grand canonical Gibbs measure with interaction potential ϕ if it satisfies the Dobrushin–Lanford–Ruelle equation

$$\mu\Pi^{z,\phi}_\Lambda = \mu \qquad \text{for all } \Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d). \tag{2.4}$$

Let $\mathcal{G}(z, \phi)$ denote the set of all such probability measures μ.

We rewrite the conditional energy E^{ϕ}_{Λ} in the following form

$$E^{\phi}_{\Lambda}(\gamma) = E^{\phi}_{\Lambda}(\gamma_{\Lambda}) + W(\gamma_{\Lambda} \mid \gamma_{\Lambda^c}),$$

where the term

$$W(\gamma_{\Lambda} \mid \gamma_{\Lambda^c}) := \begin{cases} \sum\limits_{x\in\gamma_{\Lambda},\, y\in\gamma_{\Lambda^c}} \phi(x-y), & \text{if } \sum\limits_{x\in\gamma_{\Lambda},\, y\in\gamma_{\Lambda^c}} |\phi(x-y)| < \infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

describes the interaction energy between γ_{Λ} and γ_{Λ^c}. Analogously, we define $W(\gamma' \mid \gamma'')$ when $\gamma' \cap \gamma'' = \varnothing$.

Any $\mu \in \mathcal{G}(z,\phi)$ satisfies the Georgii–Nguyen–Zessin identity

$$\int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)\, F(\gamma, x) = \int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} zm(dx)\, \exp\big[-W(\{x\} \mid \gamma)\big] F(\gamma + \varepsilon_x, x), \quad (2.5)$$

where $F : \Gamma \times \mathbb{R}^d \to [0, +\infty]$ is a measurable function ([31, Theorem 2], see also [23, Theorem 2.2.4]). In fact, this identity uniquely characterizes the Gibbs measures in the sense that any probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ belongs to $\mathcal{G}(z,\phi)$ if and only if μ satisfies (2.5), cf. [31, Theorem. 2].

Let us now describe two classes of Gibbs measures which appear in classical statistical mechanics of continuous systems [40, 41]. For every $r = (r^1, \ldots, r^d) \in \mathbb{Z}^d$, we define the cube

$$Q_r := \left\{ x \in \mathbb{R}^d \mid r^i - \frac{1}{2} \le x^i < r^i + \frac{1}{2} \right\}.$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$, we set $\gamma_r := \gamma_{Q_r}$, $r \in \mathbb{Z}^d$. For $N \in \mathbb{N}$ let Λ_N be the cube with side length $2N-1$ centered at the origin in $\mathbb{R}^d$, Λ_N is then a union of $(2N-1)^d$ unit cubes of the form Q_r.

For $\Lambda \subset \mathbb{R}^d$, by Γ_{Λ} we denote the subset of Γ consisting of all configurations $\gamma \in \Gamma$ such that $\gamma = \gamma_{\Lambda}$.

Now, we formulate conditions on the interaction.

(S) (*Stability*) There exists $B \ge 0$ such that, for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and for all $\gamma \in \Gamma_{\Lambda}$,

$$E^{\phi}_{\Lambda}(\gamma) \ge -B|\gamma|.$$

Notice that the stability condition automatically implies that the potential ϕ is semi-bounded from below.

(SS) (*Superstability*) There exist $A > 0$, $B \ge 0$ such that, if $\gamma \in \Gamma_{\Lambda_N}$ for some N, then

$$E^{\phi}_{\Lambda_N}(\gamma) \ge \sum_{r\in\mathbb{Z}^d} \big(A|\gamma_r|^2 - B|\gamma_r|\big).$$

This condition is evidently stronger than (S).

(LR) (*Lower regularity*) There exists a decreasing positive function $a\colon \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r\in\mathbb{Z}^d} a(\|r\|) < \infty$$

and for any Λ', Λ'' which are finite unions of cubes Q_r and disjoint, with $\gamma' \in \Gamma_{\Lambda'}$, $\gamma'' \in \Gamma_{\Lambda''}$,

$$W(\gamma' \mid \gamma'') \geq - \sum_{r',r''\in\mathbb{Z}^d} a(\|r' - r''\|)|\gamma'_{r'}|\,|\gamma''_{r''}|.$$

Here, $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^d$.

(I) (*Integrability*) We have

$$\int_{\mathbb{R}^d} |1 - e^{-\phi(x)}|\, m(dx) < +\infty.$$

We also need

(UI) (*Uniform integrability*) We have

$$\int_{\mathbb{R}^d} |1 - e^{-\phi(x)}|\, m(dx) < z^{-1} \exp(-1 - 2B),$$

where B is as in (S).

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called tempered if μ is supported by

$$S_\infty := \bigcup_{n=1}^{\infty} S_n,$$

where

$$S_n := \left\{ \gamma \in \Gamma \mid \forall N \in \mathbb{N} \sum_{r\in\Lambda_N\cap\mathbb{Z}^d} |\gamma_r|^2 \leq n^2 |\Lambda_N \cap \mathbb{Z}^d| \right\}.$$

By $\mathcal{G}^t(z,\phi) \subset \mathcal{G}(z,\phi)$ we denote the set of all tempered grand canonical Gibbs measures (Ruelle measures for short). Due to [41] the set $\mathcal{G}^t(z,\phi)$ is non-empty for all $z > 0$ and any potential ϕ satisfying conditions (SS), (LR), and (I). Furthermore, the set $\mathcal{G}(z,\phi)$ is not empty for potentials satisfying (S) and (UI), or equivalently, for stable potentials in the low activity-high temperature regime, see e.g. [29, 30]. A measure $\mu \in \mathcal{G}(z,\phi)$ in the latter case is constructed as a limit of finite volume Gibbs measures corresponding to empty boundary conditions.

Let us now recall the so-called Ruelle bound (cf. [41]).

Proposition 2.1 *Suppose that either conditions* (SS), (LR), (I) *are satisfied and* $\mu \in \mathcal{G}^t(z,\phi)$, $z > 0$, *or conditions* (S), (UI) *are satisfied and* $\mu \in \mathcal{G}(z,\phi)$ *is the Gibbs measure constructed as a limit of finite volume Gibbs measures with empty boundary conditions.*

Then, for any $n \in \mathbb{N}$, *there exists a non-negative measurable symmetric function* $k_\mu^{(n)}$ *on* $(\mathbb{R}^d)^n$ *such that, for any measurable symmetric function* $f^{(n)} : (\mathbb{R}^d)^n \to [0, \infty]$,

$$\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1,\dots,x_n)\,\mu(d\gamma)$$
$$= \frac{1}{n!}\int_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\dots,x_n)k_\mu^{(n)}(x_1,\dots,x_n)\,m(dx_1)\cdots m(dx_n),$$

and

$$\forall (x_1,\dots,x_n)\in(\mathbb{R}^d)^n: \quad k_\mu^{(n)}(x_1,\dots,x_n)\le \xi^n, \tag{2.6}$$

where $\xi > 0$ *is independent of* n.

The functions $k_\mu^{(n)}$, $n \in \mathbb{N}$, are called correlation functions of the measure μ, while (2.6) is called the Ruelle bound.

The above proposition particularly implies that, for any $\varphi \in \mathcal{D}$, $\varphi \geq 0$, and $n \in \mathbb{N}$,

$$\int_\Gamma \langle\varphi,\gamma\rangle^n\,\mu(d\gamma) < \infty, \tag{2.7}$$

that is, any measure μ as in Proposition 2.1 has all local moments finite.

3 The bilinear form $\mathcal{E}_\mu^\Gamma$

In what follows, we fix a measure μ as in Proposition 2.1. In this section, we shall construct a pre-Dirichlet form $\mathcal{E}_\mu^\Gamma$ on the space $L^2(\Gamma, \mu)$.

We introduce the set $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma)$ of all functions of the form

$$\Gamma \ni \gamma \mapsto F(\gamma) = g_F(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle), \tag{3.1}$$

where $N \in \mathbb{N}$, $\varphi_1,\dots,\varphi_N \in \mathcal{D}$, and $g_F \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$.

We define

$$\mathcal{E}_\mu^\Gamma(F,G) := \int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} zm(dx)\,\langle\nabla_x F(\gamma+\varepsilon_x), \nabla_x G(\gamma+\varepsilon_x)\rangle, \tag{3.2}$$

where $F, G \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma)$. Here, ∇_x denotes the gradient in the x variable and $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^d$. For any F of the form (3.1), we have

$$\mathcal{E}_\mu^\Gamma(F) \le \int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} zm(dx)\left(\sum_{i=1}^N |\partial_i g_F(\langle\varphi_1,\gamma+\varepsilon_x\rangle,\dots,\langle\varphi_N,\gamma+\varepsilon_x\rangle)|\,|\nabla\varphi_i(x)|\right)^2$$
$$\le \int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} zm(dx)\ \mathrm{const}\sum_{i=1}^N |\nabla\varphi_i(x)|^2 < \infty,$$

where $\partial_j\, g_F$ means derivative with respect to the j-th coordinate and, as usual, we set $\mathcal{E}_\mu^\Gamma(F) := \mathcal{E}_\mu^\Gamma(F, F)$. Thus, the the right-hand side of (3.2) is well-defined.

In order to get an alternative representation of the form $\mathcal{E}_\mu^\Gamma$, we shall suppose the following:

(A1) There exists $r > 0$ such that

$$\sup_{x \in B(r)^c} \phi(x) < \infty,$$

where $B(r)$ denotes the closed ball in $\mathbb{R}^d$ of radius r centered at the origin.

Lemma 3.1 *In addition to the conditions of Proposition* 2.1, *let* ϕ *also satisfy* (A1). *Then, for* $\mu \otimes dx$*-a.e.* $(\gamma, x) \in \Gamma \times \mathbb{R}^d$

$$\sum_{y \in \gamma} |\phi(x - y)| < \infty \tag{3.3}$$

and for μ*-a.a.* $\gamma \in \Gamma$:

$$\sum_{y \in \gamma \setminus \{x\}} |\phi(x - y)| < \infty \qquad \text{for each } x \in \gamma. \tag{3.4}$$

Proof. To show (3.3), it suffices to prove that, for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$,

$$\sum_{y \in \gamma_{(\Lambda^r)^c}} |\phi(x - y)| < \infty \qquad \text{for } \mu \otimes m\text{-a.e. } (\gamma, x) \in \Gamma \times \Lambda, \tag{3.5}$$

where $\Lambda^r {:=} \{y \in \mathbb{R}^d : d(y, \Lambda) \le r\}$, $d(y, \Lambda)$ denoting the distance from y to Λ. By (I) and (A1), we have

$$\int_{B(r)^c} |\phi(x)| \, m(dx) < \infty. \tag{3.6}$$

Therefore, by Proposition 2.1

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_\Lambda m(dx) \sum_{y \in \gamma_{(\Lambda^r)^c}} |\phi(x - y)| \\
&\quad = \int_\Lambda m(dx) \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dy) \, |\phi(x - y)| \mathbf{1}_{(\Lambda^r)^c}(y) \\
&\quad = \int_\Lambda m(dx) \int_{\mathbb{R}^d} m(dy) k_\mu^{(1)}(y) |\phi(x - y)| \mathbf{1}_{(\Lambda^r)^c}(y) \\
&\quad \le \xi \int_\Lambda m(dx) \int_{(\Lambda^r)^c} m(dy) \, |\phi(x - y)| \\
&\quad \le \xi m(\Lambda) \int_{B(r)^c} |\phi(y)| \, m(dy) < \infty,
\end{aligned}$$

which implies (3.5).

Analogously, to show (3.4) it suffices to prove that, for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$,

$$\sum_{x \in \gamma_\Lambda} \sum_{y \in \gamma_{(\Lambda^r)^c}} |\phi(x - y)| < \infty \qquad \text{for } \mu\text{-a.e. } \gamma \in \Gamma. \tag{3.7}$$

By Proposition 2.1 and (3.6)

$$\begin{aligned}
&\int_\Gamma \sum_{x\in\gamma_\Lambda} \sum_{y\in\gamma_{(\Lambda^r)^c}} |\phi(x-y)|\,\mu(d\gamma)\\
&\qquad = 2\int_{(\mathbb{R}^d)^2} \mathbf{1}_\Lambda(x)\mathbf{1}_{(\Lambda^r)^c}(y)|\phi(x-y)|k_\mu^{(2)}(x,y)\,m(dx)\,m(dy)\\
&\qquad \le 2\xi^2 m(\Lambda)\int_{B(r)^c} |\phi(y)|\,m(dy) < \infty,
\end{aligned}$$

which implies (3.7) ■

By using (2.5), (3.2), and Lemma 3.1, we have for any $F,G\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$

$$\begin{aligned}
\mathcal{E}_\mu^\Gamma(F,G) &= \int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} zm(dx)\,\exp\left[-\sum_{y\in\gamma}\phi(x-y)\right]\\
&\qquad\times\exp\left[\sum_{y\in\gamma}\phi(x-y)\right]\langle\nabla_x F(\gamma+\varepsilon_x),\nabla_x G(\gamma+\varepsilon_x)\rangle\\
&= \int_\Gamma S^\Gamma(F,G)\,d\mu. \qquad (3.8)
\end{aligned}$$

Here,

$$\begin{aligned}
S^\Gamma(F,G)(\gamma) &:= \int_{\mathbb{R}^d} A(\gamma,x)\langle\nabla_x F(\gamma),\nabla_x G(\gamma)\rangle\,\gamma(dx),\\
A(\gamma,x) &:= \exp\left[\sum_{y\in\gamma\setminus\{x\}}\phi(x-y)\right], \qquad x\in\gamma,\ \mu\text{-a.e. } \gamma\in\Gamma, \qquad (3.9)
\end{aligned}$$

and for any $\gamma\in\Gamma$ and $x\in\mathbb{R}^{d^d}$

$$\nabla_x F(\gamma) := \nabla_y F(\gamma-\varepsilon_x+\varepsilon_y)\big|_{y=x}. \qquad (3.10)$$

Note that, since $F\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$, it naturally extends from Γ to all of $\mathcal{D}'$:=dual of $\mathcal{D}$. In particular, $F(\gamma)$ is defined if γ is a signed measure such that $|\gamma|$ is finite on compacts.

Lemma 3.2 *Let the conditions of Lemma* 3.1 *be satisfied. Let* $F_1,\dots,F_N,G_1,\dots,G_N\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$, $\phi,\psi\in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$. *Then, for* μ*-a.a.* $\gamma\in\Gamma$,

$$\begin{aligned}
&S^\Gamma(\phi(F_1,\dots,F_N),\psi(G_1,\dots,G_N))(\gamma)\\
&\qquad = \sum_{i,j=1}^N \partial_i\phi(F_1(\gamma),\dots,F_N(\gamma))\,\partial_j\psi(G_1(\gamma),\dots,G_N(\gamma))S^\Gamma(F_i,G_j)(\gamma).
\end{aligned}$$

Proof. Immediate by (3.9). ■

Lemma 3.3 *Let the conditions of Lemma* 3.1 *be satisfied. Then,* $S^\Gamma(F,G)=0$ μ*-a.e. for all* $F,G\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$ *such that* $F=0$ μ*-a.e.*

Proof. Let $F\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$, $F=0$ μ-a.e. Then, for any $r>0$,

$$\begin{aligned} 0 &= \int_\Gamma \left(\int_{B(r)} A(\gamma,x)\,\gamma(dx) \right) |F(\gamma)|\,\mu(d\gamma) \\ &= \int_\Gamma \mu(d\gamma) \int_{B(r)} zm(dx)|F(\gamma\cup\{x\})|. \end{aligned}$$

Hence, $F(\gamma\cup\{x\})=0$ for $\mu\otimes m$-a.e. $(\gamma,x)\in\Gamma\times\mathbb{R}^d$. For any fixed $\gamma\in\Gamma$, $\mathbb{R}^d\ni x\mapsto F(\gamma\cup\{x\})\in\mathbb{R}$ is a smooth function. Hence, for μ-a.e. $\gamma\in\Gamma$, $F(\gamma\cup\{x\})=0$ for all $x\in\mathbb{R}^d$, and so $S^\Gamma(F){:=}S^\Gamma(F,F)=0$ μ-a.e. on Γ. Using the Cauchy–Schwarz inequality, we obtain the assertion. ■

Proposition 3.1 *Let the conditions of Proposition* 2.1 *be fulfilled and let* ϕ *also satisfy* (A1). *Then,* $(\mathcal{E}_\mu^\Gamma,\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *is a pre-Dirichlet form on* $L^2(\Gamma;\mu)$ (*i.e., if* $(\mathcal{E}_\mu^\Gamma,\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *is closable, then its closure* $(\mathcal{E}_\mu^\Gamma,D(\mathcal{E}_\mu^\Gamma))$ *is a Dirichlet form*).

Proof. Since $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$ is dense in $L^2(\Gamma;\mu)$, the assertion follows by Lemmas 3.2, 3.3 directly from [27, Chap. I, Proposition 4.10] (see also [27, Chap. II, Exercise 2.7]). ■

4 Closability of the (pre-)Dirichlet form $\mathcal{E}_\mu^\Gamma$

In this section, we shall prove that, under some condition on the potential ϕ, the bilinear form $(\mathcal{E}_\mu^\Gamma,\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ is closable on $L^2(\Gamma;\mu)$. So, in what follows, we suppose the following:

(A2) Let

$$\Phi(x){:=}\phi(x)\vee 0,\qquad x\in\mathbb{R}^d,$$

and for $\mu\otimes m$-a.e. $(\gamma,x)\in\Gamma\times\mathbb{R}^d$ we set

$$\rho(\gamma,x){:=}\exp\left[-\sum_{y\in\gamma}\Phi(x-y)\right].$$

Then, for μ-a.e. $\gamma\in\Gamma$, $\rho(\gamma,\cdot)=0$ m-a.e. on

$$\mathbb{R}^d\setminus\Big\{\,x\in\mathbb{R}^d \mid \int_{\Lambda_x}\rho(\gamma,\cdot)^{-1}\,dm<\infty \text{ for some open neighborhood } \Lambda_x \text{ of } x\,\Big\}.$$

Remark 4.1 Let us suppose that $\phi\in C(\mathbb{R}^d\setminus\{0\})$. Then, condition (A2) is satisfied if there exists $R>0$ such that $\phi(x)\le 0$ for $|x|\ge R$. Indeed, in this case, for each $\gamma\in\Gamma$, $\rho(\gamma,\cdot)$ is a positive continuous function on $\mathbb{R}^d\setminus\gamma$, which evidently yields (A2). Alternatively, if μ is a Ruelle measure, for (A2) to hold it suffices that, for each $\gamma\in S_\infty$, the series $\sum_{y\in\gamma}\Phi(\cdot-y)$ converges locally uniformly on $\mathbb{R}^d\setminus\gamma$. For Gibbs measures in low activity-high temperature regime, in the latter condition the set S_∞ can be replaced by the set of all configurations $\gamma\in\Gamma$ satisfying, for all $N\in\mathbb{N}$,

$$|\gamma_{\Lambda_N}|\le C(\gamma)m(\Lambda_N),\qquad C(\gamma)>0,$$

for a fixed sequence $\{\Lambda_N\}_{N=1}^\infty\subset\mathcal{O}_c(\mathbb{R}^d)$ such that $\Lambda_N\subset\Lambda_{N+1}$, $m(\Lambda_{N+1}\setminus\Lambda_N)\ge N+1$, $N\in\mathbb{N}$, and $\bigcup_{N=1}^\infty\Lambda_N=\mathbb{R}^d$, which is also a set of full μ measure (cf. [23, Theorem 5.2.4], see also [43, Proposition 1]).

Theorem 4.1 *Let the conditions of Proposition* 2.1 *be fulfilled and let* ϕ *also satisfy* (A1) *and* (A2). *Then, the bilinear form* $(\mathcal{E}_\mu^\Gamma,\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *is closable on* $L^2(\Gamma;\mu)$.

Proof. Let

$$\Psi(x):=\phi(x)\wedge 0,\qquad x\in\mathbb{R}^d.$$

We now define an auxiliary bilinear form

$$\mathcal{E}_{\mu,\Psi}^\Gamma(F,G):=\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}\gamma(dx)\,\exp\left[\sum_{y\in\gamma\setminus\{x\}}\Psi(x-y)\right]\langle\nabla_xF(\gamma),\nabla_xG(\gamma)\rangle,\tag{4.1}$$

where $F,G\in D(\mathcal{E}_{\mu,\Psi}^\Gamma):=\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$. By (3.8), (3.9), (4.1), and Lemma 3.1

$$\mathcal{E}_\mu^\Gamma(F)\ge\mathcal{E}_{\mu,\Psi}^\Gamma(F),\qquad F\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma),\tag{4.2}$$

and particularly the bilinear form $\mathcal{E}_{\mu,\Psi}^\Gamma$ is well defined. Using (2.5), (4.1), and Lemma 3.1, we get

$$\begin{aligned}\mathcal{E}_{\mu,\Psi}^\Gamma(F)&=\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}zm(dx)\,\exp\left[-\sum_{y\in\gamma}\phi(x-y)+\sum_{y\in\gamma}\Psi(x-y)\right]|\nabla_xF(\gamma+\varepsilon_x)|^2\\&=\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}zm(dx)\,\exp\left[-\sum_{y\in\gamma}\Phi(x-y)\right]|\nabla_xF(\gamma+\varepsilon_x)|^2.\end{aligned}\tag{4.3}$$

Claim. The bilinear form $(\mathcal{E}_{\mu,\Psi}^\Gamma,\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ is closable on $L^2(\Gamma;\mu)$.

For μ-a.e. $\gamma\in\Gamma$, we define a measure $\sigma_\gamma(dx):=\rho(\gamma,x)m(dx)$ on $\mathbb{R}^d$, and we introduce the following biliear form on the space $L^2(\mathbb{R}^d;\sigma_\gamma)$:

$$\mathcal{E}_{\sigma_\gamma}(f,g):=\int_{\mathbb{R}^d}\langle\nabla f,\nabla g\rangle\,d\sigma_\gamma,\qquad f,g\in\mathcal{D}^{\sigma_\gamma},$$

where $\mathcal{D}^{\sigma_\gamma}$ denote the σ_γ-classes determined by $\mathcal{D}$. Then, by [6, Theorem 5.3] or [10, Theorem 6.2], it follows from (A2) that the form $(\mathcal{E}_{\sigma_\gamma},\mathcal{D}^{\sigma_\gamma})$ is closable for μ-a.e. $\gamma\in\Gamma$. Notice also that

$$\exp\left[\sum_{y\in\gamma\setminus\{x\}}\Psi(x-y)\right]\le 1\qquad\text{for }\mu\text{-a.e. }\gamma\in\Gamma.$$

Now, the proof of the claim is completely analogous to the proof of [10, Theorem 6.3] (see also the survey paper [36]).

Let $(F_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$ such that

$$\|F_n\|_{L^2(\mu)} \to 0 \quad \text{as } n\to\infty \tag{4.4}$$

and

$$\mathcal{E}_\mu^\Gamma(F_n - F_k) \to 0 \quad \text{as } n,k\to\infty. \tag{4.5}$$

To prove the closability of $\mathcal{E}_\mu^\Gamma$, it suffices to show that there exits a subsequence $(F_{n_k})_{k=1}^\infty$ such that

$$\mathcal{E}_\mu^\Gamma(F_{n_k}) \to 0 \quad \text{as } k\to\infty.$$

By (4.2) and (4.5),

$$\mathcal{E}_{\mu,\Psi}^\Gamma(F_n - F_k) \to 0 \quad \text{as } n,k\to\infty. \tag{4.6}$$

By the claim, the form $\mathcal{E}_{\mu,\Psi}^\Gamma$ is closable on $L^2(\Gamma;\mu)$, and therefore (4.4) and (4.6) imply that

$$\mathcal{E}_\Psi(F_n) \to 0 \quad \text{as } n\to\infty.$$

From here and (4.3)

$$\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} zm(dx)\, \exp\left[-\sum_{y\in\gamma}\Phi(x-y)\right] |\nabla_x F_n(\gamma+\varepsilon_x)|^2 \to 0 \quad \text{as } n\to\infty.$$

Therefore, there exists a subsequence $(F_{n_k})_{k=1}^\infty$ such that

$$\exp\left[-\sum_{y\in\gamma}\Phi(x-y)\right] |\nabla_x F_{n_k}(\gamma+\varepsilon_x)|^2 \to 0 \qquad \text{as } k\to\infty \text{ for } \mu\otimes m\text{-a.e. } (\gamma,x)\in\Gamma\times\mathbb{R}^d. \tag{4.7}$$

By Lemma 3.1,

$$\sum_{y\in\gamma}|\Phi(x-y)| \le \sum_{y\in\gamma}|\phi(x-y)| < \infty \qquad \text{for } \mu\otimes m\text{-a.e. } (\gamma,x)\in\Gamma\times\mathbb{R}^d. \tag{4.8}$$

Thus, by (4.7) and (4.8),

$$|\nabla_x F_{n_k}(\gamma+\varepsilon_x)| \to 0 \quad \text{as } k\to\infty \text{ for } \mu\otimes m\text{-a.e. } (\gamma,x)\in\Gamma\times\mathbb{R}^d.$$

Therefore, by Fatou's lemma,

$$\begin{aligned}
\mathcal{E}_\mu^\Gamma(F_{n_k}) &= \int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} zm(dx)|\nabla_x F_{n_k}(\gamma+\varepsilon_x)|^2\\
&= \int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} zm(dx)|\nabla_x F_{n_k}(\gamma+\varepsilon_x) - \lim_{l\to\infty}\nabla_x F_{n_l}(\gamma+\varepsilon_x)|^2\\
&\le \liminf_{l\to\infty}\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} zm(dx)|\nabla_x F_{n_k}(\gamma+\varepsilon_x) - \nabla_x F_{n_l}(\gamma+\varepsilon_x)|^2\\
&= \liminf_{l\to\infty}\mathcal{E}_\mu^\Gamma(F_{n_k} - F_{n_l}),
\end{aligned}$$

which by (4.5) can be made arbitrarily small for k large enough. ■

In what follows, we shall denote by $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ the closure of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$.

5 Another condition of closability of the form $\mathcal{E}_\mu^\Gamma$, the generator of $\mathcal{E}_\mu^\Gamma$

Though we have given in Section 4 a condition on ϕ ensuring the closability of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$, we have no information about the generator of $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$, except for the fact that it exists. In this section, under a stronger restriction on the growth of the potential ϕ at zero, we shall show that the domain of the generator contains $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$, and we shall give an explicit formula for the action of the generator on this set.

Let us introduce the following condition on the potential ϕ:

(A3) Let $r>0$ be as in (A1). We have

$$\int_{B(r)} e^{\phi(x)}\, m(dx) < \infty.$$

Notice that condition (A3) still admits that $\phi(x)\to+\infty$ as $x\to 0$.

Theorem 5.1 *Let the conditions of Proposition* 2.1 *be fulfilled and let* ϕ *also satisfy* (A1) *and* (A3). *Then, for any* $F,G\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$,

$$\mathcal{E}_\mu^\Gamma(F,G) = \int_\Gamma (H_\mu^\Gamma F)(\gamma) G(\gamma)\, \mu(d\gamma),$$

where

$$(H_\mu^\Gamma F)(\gamma){:=}-\sum_{x\in\gamma} \exp\left[\sum_{y\in\gamma\setminus\{x\}} \phi(x-y)\right] \Delta_x F(\gamma) \qquad \text{for } \mu\text{-a.e. } \gamma\in\Gamma,$$
$$\Delta_x F(\gamma){:=}\Delta_y F(\gamma-\varepsilon_x+\varepsilon_y)\big|_{y=x},$$

Δ *denoting the Laplacian on* $\mathbb{R}^d$, *and* H_μ^Γ *is an operator in* $L^2(\Gamma;\mu)$ *with domain* $D(H_\mu^\Gamma){:=}\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$.

Corollary 5.1 *Under the conditions of Theorem* 5.1, *the bilinear form* $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *is closable on* $L^2(\Gamma;\mu)$ *and the operator* $(H_\mu^\Gamma, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *has a Friedrichs extension, which we denote by* $(H_\mu^\Gamma, D(H_\mu^\Gamma))$.

Remark 5.1 Notice that, in the above theorem, we do not demand condition (A2) to hold.

Proof of Theorem 5.1. First, we note that, for any $F\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$ and $\gamma\in\Gamma$, the function $f(x){:=}F(\gamma+\varepsilon_x)-F(\gamma)$ belongs to $\mathcal{D}$ and $\nabla f(x)=\nabla_x F(\gamma+\varepsilon_x)$. Therefore, we have, for any $F,G\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$,

$$\mathcal{E}_\mu^\Gamma(F,G) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} zm(dx)\, \langle \nabla_x F(\gamma+\varepsilon_x), \nabla_x G(\gamma+\varepsilon_x)\rangle$$

$$= -\int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} zm(dx)\, \Delta_x F(\gamma+\varepsilon_x) G(\gamma+\varepsilon_x)$$
$$= \int_\Gamma (H_\mu^\Gamma F)(\gamma) G(\gamma)\, \mu(d\gamma),$$

and we have to show that $H_\mu^\Gamma F \in L^2(\Gamma;\mu)$.

As easily seen, it suffices to prove that, for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$,

$$\int_\Gamma \left(\sum_{x\in\gamma_\Lambda} \exp\left[\sum_{y\in\gamma\setminus\{x\}} \phi(x-y) \right] \right)^2 \mu(d\gamma) < \infty. \tag{5.1}$$

By (A1), (A3) and (3.6)

$$\int_{\mathbb{R}^d} |1-e^{\varphi(x)}|\, m(dx) < \infty. \tag{5.2}$$

Hence, using [20, Lemma 5.2] and Proposition 2.1, we get

$$\int_\Gamma \sum_{x\in\gamma_\Lambda} \exp\left[2 \sum_{y\in\gamma\setminus\{x\}} \phi(x-y) \right] \mu(d\gamma)$$
$$= \int_\Lambda zm(dx) \int_\Gamma \mu(d\gamma) \exp\left[\sum_{y\in\gamma} \phi(x-y) \right]$$
$$= \int_\Lambda zm(dx) \left(1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^n (e^{\phi(x-y_i)}-1) k_\mu^{(n)}(y_1,\dots,y_n)\, m(dy_1)\cdots m(dy_n) \right)$$
$$\le \int_\Lambda zm(dx) \left(1 + \sum_{n=1}^\infty \frac{\xi^n}{n!} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^n |1-e^{\phi(x-y_i)}|\, m(dy_1)\cdots m(dy_n) \right)$$
$$= zm(\Lambda) \sum_{n=0}^\infty \frac{\xi^n}{n!} \left(\int_{\mathbb{R}^d} |1-e^{\phi(y)}|\, m(dy) \right)^n < \infty. \tag{5.3}$$

Next, applying equality (2.5) twise, we get from (5.2):

$$\int_\Gamma \sum_{x_1\in\gamma_\Lambda} \sum_{x_2\in\gamma_\Lambda\setminus\{x_1\}} \exp\left[\sum_{y_1\in\gamma\setminus\{x_1\}} \phi(x_1-y_1) + \sum_{y_2\in\gamma\setminus\{x_2\}} \phi(x_2-y_2) \right] \mu(d\gamma)$$
$$= \int_\Gamma \mu(d\gamma) \int_\Lambda zm(dx_1) \int_\Lambda zm(dx_2) \exp\Bigg[-\sum_{y_1\in\gamma} \phi(x_1-y_1) - \sum_{y_2\in\gamma\cup\{x_1\}} \phi(x_2-y_2)$$
$$+ \sum_{y_1\in\gamma\cup\{x_2\}} \phi(x_1-y_1) + \sum_{y_2\in\gamma\cup\{x_1\}} \phi(x_2-y_2) \Bigg]$$
$$= \int_\Gamma \mu(d\gamma) \int_\Lambda zm(dx_1) \int_\Lambda zm(dx_2)\, e^{\phi(x_1-x_2)}$$

$$\leq z^2 m(\Lambda) \int_{\mathbb{R}^d} |1 - e^{\phi(x)}|\, m(dx) + z^2 m(\Lambda)^2 < \infty. \tag{5.4}$$

From (5.3) and (5.4) we conclude (5.1), and so the theorem is proved. ■

Corollary 5.2 *Let the conditions of Proposition* 2.1 *be fulfilled and let* ϕ *also satisfy either* (A1), (A2) *or* (A1), (A3). *Then,* $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ *is a Dirichlet form on* $L^2(\Gamma; \mu)$.

Proof. Immediate by Proposition 3.1, Theorem 4.1, and Corollary 5.1. ■

6 Quasi-regularity and diffusions

The diffusion process corresponding to the Dirichlet form $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ will, in general, live on the bigger state space $\ddot{\Gamma}$ consisting of all $\mathbb{Z}_+$-valued Radon measures on $\mathbb{R}^d$ (which is Polish, see e.g. [19]). Since $\Gamma \subset \ddot{\Gamma}$ and $\mathcal{B}(\ddot{\Gamma}) \cap \Gamma = \mathcal{B}(\Gamma)$, we can consider μ as a measure on $(\ddot{\Gamma}, \mathcal{B}(\ddot{\Gamma}))$ and correspondingly $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ as a Dirichlet form on $L^2(\ddot{\Gamma}; \mu)$.

The definition of quasi-regularity given in [27, Chap. IV, Def. 3.1] obviously simplifies now as follows: $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ on $L^2(\ddot{\Gamma}; \mu)$ is quasi-regular if and only if there exists an $\mathcal{E}_\mu^\Gamma$-nest $(K_n)_{n\in\mathbb{N}}$ consisting of compact sets in $\ddot{\Gamma}$.

We recall that a sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of $\ddot{\Gamma}$ is called an $\mathcal{E}_\mu^\Gamma$-nest if

$$\left\{ F \in D(\mathcal{E}_\mu^\Gamma) \mid F = 0 \text{ on } \ddot{\Gamma} \setminus A_n \text{ for some } n \in \mathbb{N} \right\}$$

is dense in $D(\mathcal{E}_\mu^\Gamma)$ with respect to the norm

$$\| \cdot \|_{\mathcal{E}_{\mu,1}^\Gamma} := \left(\mathcal{E}_\mu^\Gamma(\cdot, \cdot) + (\cdot, \cdot)_{L^2(\ddot{\Gamma}, \mu)} \right)^{1/2}.$$

Proposition 6.1 *Under the conditions of Corollary* 5.2, *the Dirichlet form* $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ *is quasi-regular.*

Proof. By [28, Proposition 4.1], it suffices to show that there exists a bounded, complete metric $\overline{\rho}$ on $\ddot{\Gamma}$ generating the vague topology such that, for all $\gamma \in \ddot{\Gamma}$, $\overline{\rho}(\cdot, \gamma) \in D(\mathcal{E}_\mu^\Gamma)$ and $S^\Gamma(\overline{\rho}(\cdot, \gamma)) \leq \eta$ μ-a.e. for some $\eta \in L^1(\ddot{\Gamma}; \mu)$ (independent of γ). The proof of this fact is quite analogous to the proof of [28, Proposition 4.8]. Let us outline the main changes needed.

We introduce the space $\mathcal{V}$ as the completion of $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \ddot{\Gamma})$ with respect to the norm

$$|F|_\Gamma := \left(\int S^\Gamma(F)\, d\mu \right)^{1/2} + \int |F|\, d\mu, \qquad F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \ddot{\Gamma}).$$

Here, $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \ddot{\Gamma})$ denotes the set of all functions on $\ddot{\Gamma}$ of the form (3.1). The formulation and the proof of [28, Lemma 4.2] now carries over to our case. In particular, $\mathcal{V}$

is continuously embedded into $L^1(\Gamma;\mu)$ and S^Γ extends uniquely to a bilinear map from $(\mathcal{V},|\cdot|_\Gamma)\times(\mathcal{V},|\cdot|_\Gamma)$ into $L^1(\ddot{\Gamma};\mu)$.

Lemma 4.3 in [28] now reads as follows: Let $f\in\mathcal{D}$. Then, $\langle f,\cdot\rangle\in\mathcal{V}$ and

$$S^\Gamma(\langle f,\cdot\rangle)(\gamma)=\int A(\gamma,x)S(f)(x)\,\gamma(dx)\qquad\text{for }\mu\text{-a.e. }\gamma\in\ddot{\Gamma}.$$

Here, $S(f){:=}S(f,f)$ and $S(f,g)(x){:=}\langle\nabla f(x),\nabla g(x)\rangle$, $f,g\in\mathcal{D}$, $x\in\mathbb{R}^d$. The proof is again the same.

Next, we consider the following norm on $\mathcal{D}$:

$$|f|_E{:=}\left(\int S(f)\,dm\right)^{1/2}+\int|f|k_\mu^{(1)}\,dm,$$

where $k_\mu^{(1)}$ is the first correlation function of μ. Recall that, by Proposition 2.1, $k_\mu^{(1)}\le\xi$ for some $\xi>0$. Furthermore, by (2.5) and Lemma 3.1,

$$k_\mu^{(1)}(x)=\int_\Gamma\exp\left[-\sum_{y\in\gamma}\phi(x-y)\right]\mu(d\gamma),\qquad x\in\mathbb{R}^d,$$

and consequently, applying once more Lemma 3.1, we see that $k_\mu^{(1)}(x)>0$ for m-a.e. $x\in\mathbb{R}^d$. Thus, the measures m and $k_\mu^{(1)}m$ are equivalent.

We evidently have

$$|f|_E\ge|\langle f,\cdot\rangle|_\Gamma\qquad\text{for all }f\in\mathcal{D}.\tag{6.1}$$

Let $\overline{\mathcal{D}}$ denote the completion of $\mathcal{D}$ with respect to $|\cdot|_E$. The counterpart of [28, Lemma 4.4] in our case reads as follows:

Lemma 6.1 *The inclusion map* $i:(\mathcal{D},|\cdot|_E)\subset(L^1(\mathbb{R}^d;k_\mu^{(1)}m),\|\cdot\|_{L^1(\mathbb{R}^d;k_\mu^{(1)}m)})$ *extends uniquely to a continuous inclusion* $\overline{i}:\overline{\mathcal{D}}\hookrightarrow L^1(\mathbb{R}^d;k_\mu^{(1)}m)$. *Furthermore,* S *extends uniquely to a bilinear continuous map from* $(\overline{\mathcal{D}},|\cdot|_E)\times(\overline{\mathcal{D}},|\cdot|_E)$ *to* $L^1(\mathbb{R}^d;m)$ *satisfying* (S1)–(S3) *in* [28] *with* $\mathcal{D}$ *replaced with* $\overline{\mathcal{D}}$.

Proof. Let $f_n\in\mathcal{D}$, $n\in\mathbb{N}$, be an $|\cdot|_E$-Cauchy sequence such that $f_n\to0$ in $L^1(\mathbb{R}^d;k_\mu^{(1)}m)$ as $n\to\infty$. Then, by (6.1), $(\langle f_n,\cdot\rangle)_{n\in\mathbb{N}}$ is a $|\cdot|_\Gamma$-Cauchy sequence in $\mathcal{V}$ such that $\langle f_n,\cdot\rangle\to0$ in $L^1(\Gamma;d\mu)$ as $n\to\infty$. Hence, by what has been proved above, $|\langle f_n,\cdot\rangle|_\Gamma\to0$ as $n\to\infty$. Therefore, $\int S^\Gamma(\langle f_n,\cdot\rangle)\,d\mu\to0$ as $n\to\infty$, and so $\int S(f_n)\,dm\to0$ as $n\to\infty$. Consequently, $|f_n|_E\to0$ as $n\to\infty$. The remaining parts of the assertion can then be easily shown. ■

We define

$$\begin{aligned}\mathcal{F}C_{\mathrm{b}}^\infty(\overline{\mathcal{D}},\ddot{\Gamma}){:=}\big\{\,&g(\langle f_1,\cdot\rangle,\dots,\langle f_N,\cdot\rangle)\mid N\in\mathbb{N},\\&f_1,\dots,f_N\ m\text{-versions of elements in }\overline{\mathcal{D}},\ g\in C_{\mathrm{b}}^\infty(\mathbb{R}^N)\,\big\}.\end{aligned}$$

The proof of the following assertion is the same as that of [28, Proposition 4.6] if one uses Lemma 6.1 instead of [28, Lemma 4.4]:

Lemma 6.2 *We have* $\mathcal{F}C_{\mathrm{b}}^\infty(\overline{\mathcal{D}},\ddot\Gamma)\subset D(\mathcal{E}_\mu^\Gamma)$ *and for* $F=g_F(\langle f_1,\cdot\rangle,\dots,\langle f_N,\cdot\rangle)$, $G=g_G(\langle g_1,\cdot\rangle,\dots,\langle g_M,\cdot\rangle)\in\mathcal{F}C_{\mathrm{b}}^\infty(\overline{\mathcal{D}},\ddot\Gamma)$

$$S^\Gamma(F,G)(\gamma)=\sum_{i=1}^N\sum_{j=1}^M \partial_i g_F(\langle f_1,\cdot\rangle,\dots,\langle f_N,\cdot\rangle)\\ \times\partial_j g_G(\langle g_1,\cdot\rangle,\dots,\langle g_M,\cdot\rangle)\int A(\gamma,x)S(f_i,g_j)(x)\,\gamma(dx)$$

for μ*-a.e.* $\gamma\in\ddot\Gamma$*. Furthermore, for all* $f\in\overline{\mathcal{D}}$, $\langle f,\cdot\rangle\in\mathcal{V}$ *and*

$$S^\Gamma(\langle f,\cdot\rangle)(\gamma)=\int A(\gamma,x)S(f)(x)\,\gamma(dx)\qquad\text{for } \mu\text{-a.e. } \gamma\in\ddot\Gamma.$$

Next, a counterpart of [28, Lemma 4.7] can be easily formulated and proved. In particular, as norm on $\overline{\mathcal{D}}\cap L^2(\mathbb{R}^d;k_\mu^{(1)}\,dm)$ we take

$$|f|_{E,2}:=\left(\int S(f)\,dm+\int f^2k_\mu^{(1)}\,dm\right)^{1/2}.$$

The formulation and the proof of [28, Lemma 4.10] now remain without essential changes, and therefore, using Lemma 6.2, we get analogously to the proof of [28, Lemma 4.11] that $F_k\in D(\mathcal{E}_\mu^\Gamma)$ and

$$S^\Gamma(F_k)(\gamma)\le\int A(\gamma,x)\widetilde\chi_{j_k}^2(x)\,\gamma(dx)\quad\text{for } \mu\text{-a.e. } \gamma\in\ddot\Gamma,\tag{6.2}$$

where F_k and $\widetilde\chi_{j_k}$ are as in [28] (the function $F_k(\cdot)=F_k(\cdot,\gamma_0)$ is defined for any fixed $\gamma_0\in\ddot\Gamma$, while the function $\widetilde\chi_{j_k}$ is independent of γ_0).

Finally, we set

$$c_k:=\left(1+\int\widetilde\chi_{j_k}^2\,dm\right)^{-1/2}2^{-k/2},\qquad k\in\mathbb{N},$$

(since each $\widetilde\chi_{j_k}$ is bounded and has compact support, $\int\widetilde\chi_{j_k}^2\,dm<\infty$). Evidently, $c_k\to0$ as $k\to\infty$. We define

$$\overline\rho(\gamma_1,\gamma_2):=\sup_{k\in\mathbb{N}}c_k\big(F_k(\gamma_1,\gamma_2)\big),\qquad\gamma_1,\gamma_2\in\ddot\Gamma.$$

By [28, Theorem 3.6], $\overline\rho$ is a bounded, complete metric on $\ddot\Gamma$ generating the vague topology.

Analogously to the proof of [28, Proposition 4.8], we conclude by (6.2) that, for any fixed $\gamma_0\in\ddot\Gamma$, $\overline\rho(\cdot,\gamma_0)\in D(\mathcal{E}_\mu^\Gamma)$ and

$$S^\Gamma(\overline\rho(\cdot,\gamma_0))\le\eta\qquad\mu\text{-a.e.},$$

where

$$\eta(\gamma):=\sup_{k\in\mathbb{N}}\left(2^{-k}\left(1+\int\widetilde\chi_{j_k}^2\,dm\right)^{-1}\int A(\gamma,x)\widetilde\chi_{j_k}^2(x)\,\gamma(dx)\right).$$

Evidently,

$$\int \eta\, d\mu \leq \sum_{k=1}^{\infty} 2^{-k} \left(1 + \int \widetilde{\chi}_{j_k}^2\, dm\right)^{-1} \int \widetilde{\chi}_{j_k}^2\, dm < \infty,$$

which concludes the proof of the proposition. ■

Proposition 6.2 *Under the conditions of Corollary* 5.2, $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ *has the local property* (*i.e.*, $\mathcal{E}_\mu^\Gamma(F,G) = 0$ *provided* $F, G \in D(\mathcal{E}_\mu^\Gamma)$ *with* $\operatorname{supp}(|F|\mu) \cap \operatorname{supp}(|G|\mu) = \varnothing$).

Proof. Identical to the proof of [28, Proposition 4.12]. ■

As a consequence of Propositions 6.1, 6.2 and [27, Chap. IV, Theorem 3.5, and Chap. V, Theorem 1.11], we obtain the main result of this section.

Theorem 6.1 *Let the conditions of Proposition* 2.1 *be fulfilled and let* ϕ, *in addition, satisfy either* (A1), (A2) *or* (A1), (A3). *Then, there exists a conservative diffusion process* (*i.e., a conservative strong Markov process with continuous sample paths*)

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}(t))_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}})$$

on $\ddot{\Gamma}$ (*cf.* [11]) *which is properly associated with* $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$, *i.e., for all* ($\mu$-*versions of*) $F \in L^2(\ddot{\Gamma};\mu)$ *and all* $t > 0$ *the function*

$$\ddot{\Gamma} \ni \gamma \mapsto p_t F(\gamma) := \int_{\mathbf{\Omega}} F(\mathbf{X}(t))\, d\mathbf{P}_\gamma \tag{6.3}$$

is an $\mathcal{E}_\mu^\Gamma$-*quasi-continuous version of* $\exp(-tH_\mu^\Gamma)F$, *where* H_μ^Γ *is the generator of* $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ (*cf.* [27, Chap. 1, Sect. 2]). $\mathbf{M}$ *is up to* μ-*equivalence unique* (*cf.* [27, Chap. IV, Sect. 6]). *In particular,* $\mathbf{M}$ *is* μ-*symmetric* (*i.e.,* $\int G\, p_t F\, d\mu = \int F\, p_t G\, d\mu$ *for all* $F, G : \ddot{\Gamma} \to \mathbb{R}_+$, $\mathcal{B}(\ddot{\Gamma})$-*measurable*) *and has* μ *as an invariant measure.*

In the above theorem, $\mathbf{M}$ can be taken to be canonical, i.e., $\mathbf{\Omega} = C([0,\infty) \to \ddot{\Gamma})$, $\mathbf{X}(t)(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \mathbf{\Omega}$, $(\mathbf{F}_t)_{t\geq 0}$ together with $\mathbf{F}$ is the corresponding minimum completed admissible family (cf. [14, Section 4.1]) and $\mathbf{\Theta}_t$, $t \geq 0$, are the corresponding natural time shifts.

We recall that by $(H_\mu^\Gamma, D(H_\mu^\Gamma))$ we denote the generator of the closed form $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$.

Theorem 6.2 $\mathbf{M}$ *from Theorem* 6.1 *is up to* μ-*equivalence* (*cf.* [27, Definition 6.3]) *unique between all diffusion processes* $\mathbf{M}' = (\mathbf{\Omega}', \mathbf{F}', (\mathbf{F}'_t)_{t\geq 0}, (\mathbf{\Theta}'_t)_{t\geq 0}, (\mathbf{X}'(t))_{t\geq 0}, (\mathbf{P}'_\gamma)_{\gamma\in\ddot{\Gamma}})$ *on* $\ddot{\Gamma}$ *having* μ *as an invariant measure and solving the martingale problem for* $(-H_\mu^\Gamma, D(H_\mu^\Gamma))$, *i.e., there exists a set* $\Gamma_0 \in \mathcal{B}(\ddot{\Gamma})$ *such that* $\ddot{\Gamma} \setminus \Gamma_0$ *is* $\mathcal{E}_\mu^\Gamma$-*exceptional* (*so, in particular,* $\mu(\Gamma_0) = 1$) *and such that for all* $G \in D(H_\mu^\Gamma)$

$$\widetilde{G}(\mathbf{X}'(t)) - \widetilde{G}(\mathbf{X}'(0)) + \int_0^t (H_\mu^\Gamma G)(\mathbf{X}'(s))\, ds, \qquad t \geq 0,$$

is an $(\mathbf{F}'_t)$-*martingale under* $\mathbf{P}'_\gamma$ *for all* $\gamma \in \Gamma_0$. (*Here,* $\widetilde{G}$ *denotes a quasi-continuous version of* G, *cf.* [27, Ch. IV, Proposition 3.3].)

Proof. The statement of the theorem follows directly from (the proof of) [7, Theorem 3.5]. ■

Our next aim is to show that the diffusion process $\mathbf{M}$ properly associated with $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ lives, in fact, on the space $\Gamma = \Gamma_{\mathbb{R}^d}$ provided $d \geq 2$.

Theorem 6.3 *Let the conditions of Theorem* 6.1 *be satisfied and let* $d \geq 2$. *Then the set* $\ddot{\Gamma} \setminus \Gamma$ *is* $\mathcal{E}_\mu^\Gamma$*-exceptional.*

Proof. We modify the proof of [38, Proposition 1 and Corollary 1] according to our situation. For the convenience of the reader we shall present the proof completely.

It suffices to prove the result locally, that is to show that, for every fixed $a \in \mathbb{N}$, the set

$$N := \left\{ \gamma \in \ddot{\Gamma} : \sup(\gamma(\{x\}) : x \in [-a, a]^d) \geq 2 \right\}$$

is $\mathcal{E}_\mu^\Gamma$-exceptional. By [38, Lemma 1], we need to prove that there exists a sequence $(u_n)_{n=1}^\infty \subset D(\mathcal{E}_\mu^\Gamma)$ such that each u_n, $n \in \mathbb{N}$, is a continuous function on $\ddot{\Gamma}$, $u_n \to \mathbf{1}_N$ pointwise as $n \to \infty$, and $\sup_{n\in\mathbb{N}} \mathcal{E}_\mu^\Gamma(u_n) < \infty$.

Let $f \in C_0^\infty(\mathbb{R})$ be such that $\mathbf{1}_{[0,1]} \leq f \leq \mathbf{1}_{[-1/2,3/2)}$ and $|f'| \leq 3 \times \mathbf{1}_{[-1/2,3/2)}$. For any $n \in \mathbb{N}$ and $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$, define a function $f_i^{(n)} \in \mathcal{D}$ by

$$f_i^{(n)}(x) := \prod_{k=1}^d f(nx_k - i_k), \qquad x \in \mathbb{R}^d.$$

Let also $I_i^{(n)}(x) := \prod_{k=1}^d \mathbf{1}_{[-1/2,3/2)}(nx_k - i_k)$, $x \in \mathbb{R}^d$, and note that $f_i^{(n)} \leq I_i^{(n)}$. Since

$$\partial_j f_i^{(n)}(x) = nf'(nx_j - i_j) \prod_{k \neq j} f(nx_k - i_k),$$

we get

$$|\nabla f_i^{(n)}(x)|^2 \leq 9n^2 d I_i^{(n)}(x). \tag{6.4}$$

Let $\psi \in C_{\mathrm{b}}^\infty(\mathbb{R})$ be such that $\mathbf{1}_{[2,\infty)} \leq \psi \leq \mathbf{1}_{[1,\infty)}$ and $|\psi'| \leq 2 \times \mathbf{1}_{(1,\infty)}$. Set $\mathcal{A} := \mathbb{Z}^d \cap [-na, na]^d$ and define continuous functions

$$\ddot{\Gamma} \ni \gamma \mapsto u_n(\gamma) := \psi\left(\sup_{i\in\mathcal{A}} \langle f_i^{(n)}, \gamma \rangle\right), \qquad n \in \mathbb{N}.$$

Evidently, $u_n \to \mathbf{1}_N$ pointwise as $n \to \infty$. Furthermore, by an appropriate approximation of the function

$$\mathbb{R}^{|\mathcal{A}|} \ni (y_1, \dots, y_{|\mathcal{A}|}) \mapsto \sup_{i\in\mathcal{A}} y_i$$

by $C_{\mathrm{b}}^\infty(\mathbb{R}^{|\mathcal{A}|})$ functions (compare with [37, Lemma 3.2] and [28, Lemma 4.7]), we conclude that, for each $n \in \mathbb{N}$, $u_n \in D(\mathcal{E}_\mu^\Gamma)$ and

$$S^\Gamma(u_n)(\gamma) \leq \left(\psi'\left(\sup_{i\in\mathcal{A}} \langle f_i^{(n)}, \gamma \rangle\right)\right)^2 \sum_{x\in\gamma} A(\gamma, x) \sup_{i\in\mathcal{A}} |\nabla f_i^{(n)}(x)|^2 \qquad \text{for } \mu\text{-a.e. } \gamma \in \ddot{\Gamma}. \tag{6.5}$$

Next, we have for each $\gamma \in \ddot{\Gamma}$

$$\left(\psi'\left(\sup_{i\in\mathcal{A}}\langle f_i^{(n)},\gamma\rangle\right)\right)^2 \le 4\times \mathbf{1}_{\{\sup_{i\in\mathcal{A}}\langle f_i^{(n)},\gamma\rangle>1\}} \le 4\times \mathbf{1}_{\{\sup_{i\in\mathcal{A}}\langle I_i^{(n)},\gamma\rangle\ge 2\}}, \tag{6.6}$$

where we used the fact that $\langle I_i^{(n)},\gamma\rangle$ is an integer. Thus, by (6.4)–(6.6)

$$\begin{aligned} S^\Gamma(u_n)(\gamma) &\le 4\times \mathbf{1}_{\{\sup_{i\in\mathcal{A}}\langle I_i^{(n)},\gamma\rangle\ge 2\}} \sum_{x\in\gamma} A(\gamma,x)\sup_{i\in\mathcal{A}}\left(9n^2 d I_i^{(n)}(x)\right) \\ &\le 36n^2 d\sum_{i\in\mathcal{A}} \mathbf{1}_{\{\langle I_i^{(n)},\gamma\rangle\ge 2\}}\sum_{x\in\gamma} A(\gamma,x)\mathbf{1}_{[-a-1,a+1]^d}(x) \qquad \text{for } \mu\text{-a.e. } \gamma\in\ddot{\Gamma}\ . \end{aligned}$$

Consequently,

$$\begin{aligned} \int S^\Gamma(u_n)\,d\mu &\le 36n^2 d\sum_{i\in\mathcal{A}}\int_{\ddot{\Gamma}}\mu(d\gamma)\int_{[-a-1,a+1]^d} zm(dx)\,\mathbf{1}_{\{\langle I_i^{(n)},\gamma+\varepsilon_x\rangle\ge 2\}}(\gamma,x) \\ &= 36n^2 d\sum_{i\in\mathcal{A}}\int_{\ddot{\Gamma}}\mu(d\gamma)\bigg(\int_{\{I_i^{(n)}=1\}} zm(dx)\,\mathbf{1}_{\{\langle I_i^{(n)},\gamma\rangle\ge 1\}}(\gamma) \\ &\qquad + \int_{[-a-1,a+1]^d\setminus\{I_i^{(n)}=1\}} zm(dx)\,\mathbf{1}_{\{\langle I_i^{(n)},\gamma\rangle\ge 2\}}(\gamma)\bigg) \\ &\le 36n^2 dz\sum_{i\in\mathcal{A}}\left(m(\{I_i^{(n)}=1\})\mu(\{\langle I_i^{(n)},\cdot\rangle\ge 1\})\right. \\ &\qquad \left.+ (2a+2)^d\mu(\{\langle I_i^{(n)},\cdot\rangle\ge 2\})\right). \end{aligned} \tag{6.7}$$

By using [41, Theorem 5.5], we easily conclude that there exist constants $c_1, c_2 \in (0,\infty)$, independent of i and n, such that for all $i\in\mathcal{A}$ and $n\in\mathbb{N}$

$$\begin{aligned} \mu(\{\langle I_i^{(n)},\cdot\rangle\ge 1\}) &\le c_1 m(\{I_i^{(n)}=1\}), \\ \mu(\{\langle I_i^{(n)},\cdot\rangle\ge 2\}) &\le c_2 m(\{I_i^{(n)}=1\})^2. \end{aligned} \tag{6.8}$$

Thus, by (6.7) and (6.8), there exists $c_3\in(0,\infty)$, independent of n, such that for all $n\in\mathbb{N}$

$$\int S^\Gamma(u_n)\,d\mu \le c_3 n^2\sum_{i\in\mathcal{A}} m(\{I_i^{(n)}=1\})^2. \tag{6.9}$$

Since $|\mathcal{A}| = (2na+1)^d$ and $m(\{I_i^{(n)}=1\}) = (2/n)^d$, we finally get from (6.9):

$$\begin{aligned} \mathcal{E}_\mu^\Gamma(u_n) &\le c_3 n^2(2na+1)^d\left(\frac{2}{n}\right)^{2d} \\ &\le \text{const}_d \qquad \text{for all } n\in\mathbb{N} \end{aligned}$$

for some $\text{const}_d\in(0,\infty)$, provided $d\ge 2$. ■

As a direct consequence of Theorem 6.3, we get

Corollary 6.1 *Let the conditions of Theorem* 6.1 *be satisfied and let* $d\ge 2$*. Then, the assertions of Theorems* 6.1 *and* 6.2 *hold with* $\ddot{\Gamma}$ *replaced by* Γ.

7 Scaling limit of the stochastic dynamics

Throughout this section, we shall suppose that ϕ satisfies (S) and (UI) with $z = 1$ and $\mu \in \mathcal{G}(1, \phi)$ is the measure corresponding to the construction with empty boundary conditions. We shall now discuss a scaling limit of the diffusion process constructed in Theorem 6.1, the scaling being absolutely analogous to the one considered in [9, 39, 44, 18, 17].

7.1 Scaling of the process

First, let us briefly recall a result of Brox [9] on a scaling limit of Gibbs measures.

First scaling. We scale the position of the particles inside the configuration space Γ as follows:

$$\Gamma \ni \gamma \mapsto S_{\mathrm{in},\,\epsilon}(\gamma) := \{\epsilon x \mid x \in \gamma\} \in \Gamma, \qquad \epsilon > 0.$$

Let us define the image measure

$$\widetilde{\mu}_\epsilon := S^*_{\mathrm{in},\,\epsilon}\mu. \tag{7.1}$$

As easily seen, $\widetilde{\mu}_\epsilon$ is an element of $\mathcal{G}(\epsilon^{-d}, \phi_\epsilon)$ with $\phi_\epsilon := \phi(\epsilon^{-1}\cdot)$. Furthermore, $\widetilde{\mu}_\epsilon$ satisfies (UI) and corresponds to the construction with empty boundary conditions.

Second scaling. This scaling leads us out of the configuration space and is given by

$$\Gamma \ni \gamma \mapsto S_{\mathrm{out},\,\epsilon}(\gamma) := \epsilon^{d/2}(\gamma - k^{(1)}_{\widetilde{\mu}_\epsilon} m) \in \Gamma_\epsilon, \qquad \epsilon > 0,$$

where $\Gamma_\epsilon := S_{\mathrm{out},\,\epsilon}(\Gamma) \subset \mathcal{D}'$, and as before $\mathcal{D}'$ is the topological dual of $\mathcal{D}$ (where both $\mathcal{D}$ and $\mathcal{D}'$ are equipped with their respective usual locally convex topologies). We consider Γ_ϵ as a topological subspace of $\mathcal{D}'$, thus Γ_ϵ is equipped with the corresponding Borel σ-algebra. Obviously, $S_{\mathrm{out},\,\epsilon} : \Gamma \to \Gamma_\epsilon$ is continuous, hence Borel-measurable. Since it is also one-to-one and since both Γ and $\mathcal{D}'$ are standard measurable spaces, it follows by [35, Chap. V, Theorem 2.4] that Γ_ϵ is a Borel subset of $\mathcal{D}'$ and that $S^{-1}_{\mathrm{out},\,\epsilon} : \Gamma_\epsilon \to \Gamma$ is also Borel-measurable. The function $k^{(1)}_{\widetilde{\mu}_\epsilon}$ is the first correlation function of $\widetilde{\mu}_\epsilon$. It easily follows from (7.1) that

$$k^{(1)}_{\widetilde{\mu}_\epsilon} = \epsilon^{-d}\rho,$$

where $\rho := k^{(1)}_\mu$ is the first correlation function of the measure μ (which is a constant because of the translation invariance of the measure μ). Thus,

$$S_{\mathrm{out},\,\epsilon}(\gamma) = \epsilon^{d/2}\gamma - \epsilon^{-d/2}\rho m =: \gamma_\epsilon.$$

We now set

$$\mu_\epsilon := S^*_{\mathrm{out},\,\epsilon}\widetilde{\mu}_\epsilon = S^*_{\mathrm{out},\,\epsilon}S^*_{\mathrm{in},\,\epsilon}\mu. \tag{7.2}$$

Let

$$u^{(2)}_\mu(x_1, x_2) := k^{(2)}_\mu(x_1, x_2) - k^{(1)}_\mu(x_1)k^{(2)}_\mu(x_2) = k^{(2)}_\mu(x_1, x_2) - \rho^2$$

denote the second Ursell function of the measure μ. By [9, Theorem 4.5] or [40, Chapter 4], we have

$$\int_{\mathbb{R}^d} |u^{(2)}_\mu(x, 0)|\, m(dx) < \infty,$$

and let

$$c{:=}\rho+\int_{\mathbb{R}^d} u_\mu^{(2)}(x,0)\,m(dx)$$

(which is the compressibility of the Gibbs state μ). We define a Gaussian measure ν_c on $(\mathcal{D}',\mathcal{B}(\mathcal{D}'))$ by its Fourier transform

$$\int_{\mathcal{D}'}\exp\big(i\langle\varphi,\omega\rangle\big)\,\nu_c(d\omega)=\left(-\frac{c}{2}\int_{\mathbb{R}^d}\varphi(x)^2\,m(dx)\right),\qquad \varphi\in\mathcal{D}.$$

We have the following result (cf. [9, Theorem 6.5]):

Proposition 7.1 *Let us assume that the potential ϕ satisfies* (S) *and* (UI) *with $z=1$, and let $\mu\in\mathcal{G}(1,\phi)$ be the Gibbs measure corresponding to the construction with empty boundary conditions. For each $\epsilon>0$, consider μ_ϵ, defined by* (7.2)*, as a probability measure on $(\mathcal{D}',\mathcal{B}(\mathcal{D}'))$. Then, the family of measures $(\mu_\epsilon)_{\epsilon>0}$ converges weakly on $\mathcal{D}'$ to the Gaussian measure ν_c.*

For simplicity of notation, in what follows we shall exclude the case $d=1$. However, all our further considerations do also work in that case.

The scaled process of our interest is defined by

$$\mathbf{X}_\epsilon(t){:=}S_{\mathrm{out},\,\epsilon}(S_{\mathrm{in},\,\epsilon}\mathbf{X}(\epsilon^{-2}t)),\qquad t\geq 0,\ \epsilon>0,$$

where $(\mathbf{X}(t))_{t\geq0}$ is the process constructed in Corollary 6.1. Next, for each $\epsilon>0$, we construct a Dirichlet form $\mathcal{E}_\epsilon$ such that $(\mathbf{X}_\epsilon(t))_{t\geq0}$ is the unique process which is properly associated to $\mathcal{E}_\epsilon$.

Since the transformation $S_{\mathrm{out},\,\epsilon}$ is invertible, we can define a unitary operator $\mathcal{S}_{\mathrm{out},\,\epsilon}: L^2(\Gamma_\epsilon;\mu_\epsilon)\to L^2(\Gamma;\widetilde{\mu}_\epsilon)$ by setting $\mathcal{S}_{\mathrm{out},\,\epsilon}F$ to be the $\widetilde{\mu}_\epsilon$-class represented by $\widetilde{F}\circ S_{\mathrm{out},\,\epsilon}$ for any μ_ϵ-version $\widetilde{F}$ of $F\in L^2(\Gamma_\epsilon;\mu_\epsilon)$. Using this operator, we define a bilinear form $(\mathcal{E}_\epsilon,D(\mathcal{E}_\epsilon))$ on $L^2(\Gamma_\epsilon,\mu_\epsilon)$ as the image of the bilinear form $(\mathcal{E}^\Gamma_{\widetilde{\mu}_\epsilon},D(\mathcal{E}^\Gamma_{\widetilde{\mu}_\epsilon}))$ under $\mathcal{S}^{-1}_{\mathrm{out},\,\epsilon}$:

$$\mathcal{E}_\epsilon(F,G){:=}\mathcal{E}^\Gamma_{\widetilde{\mu}_\epsilon}(\mathcal{S}_{\mathrm{out},\,\epsilon}F,\mathcal{S}_{\mathrm{out},\,\epsilon}G),\qquad F,G\in D(\mathcal{E}_\epsilon),\tag{7.3}$$

where $D(\mathcal{E}_\epsilon){:=}\mathcal{S}^{-1}_{\mathrm{out},\,\epsilon}\big(D(\mathcal{E}^\Gamma_{\widetilde{\mu}_\epsilon})\big)$. It follows from [27, Chapter VI, Exercise 1.1] that $(\mathcal{E}_\epsilon,D(\mathcal{E}_\epsilon))$ is a Dirichlet form. Let $\mathcal{H}_\epsilon$ (respectively $\widetilde{\mathcal{H}}_\epsilon$) denote the generator of the form $(\mathcal{E}_\epsilon,D(\mathcal{E}_\epsilon))$ (respectively $(\mathcal{E}^\Gamma_{\widetilde{\mu}_\epsilon},D(\mathcal{E}^\Gamma_{\widetilde{\mu}_\epsilon}))$) on $L^2(\Gamma;\mu_\epsilon)$ (respectively $L^2(\Gamma;\widetilde{\mu}_\epsilon)$). Then, it follows from the definition of $(\mathcal{E}_\epsilon,D(\mathcal{E}_\epsilon))$ that

$$\mathcal{H}_\epsilon=\mathcal{S}^{-1}_{\mathrm{out},\,\epsilon}\widetilde{\mathcal{H}}_\epsilon\mathcal{S}_{\mathrm{out},\,\epsilon}.\tag{7.4}$$

We have the following proposition (compare with [17, Theorem 4.1]).

Proposition 7.2 *Let the potential ϕ fulfill conditions* (S), (UI) *with $z=1$,* (A1) *and either* (A2) *or* (A3) *and let $\mu\in\mathcal{G}(1,\phi)$ be the Gibbs measure constructed as a limit of finite* *volume Gibbs measures with empty boundary conditions. For* $\omega \in \Gamma_\epsilon$*, let* $\mathbf{Q}^\epsilon_\omega := \mathbf{P}_{S^{-1}_{\mathrm{in},\,\epsilon} S^{-1}_{\mathrm{out},\,\epsilon}\omega}$*. Then, for all (*$\mu_\epsilon$*-versions) of* $F \in L^2(\Gamma_\epsilon;\mu_\epsilon)$ *and all* $t>0$*, the function*

$$\Gamma_\epsilon \ni \omega \mapsto p_\epsilon(t,F)(\omega) := \int_{\mathbf{\Omega}} F(\mathbf{X}_\epsilon(t))\, d\mathbf{Q}^\epsilon_\omega$$

is a μ_ϵ*-version of* $\exp(-t\mathcal{H}_\epsilon)F$*. The process*

$$\mathbf{M}_\epsilon := (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_{t/\epsilon^2})_{t\ge 0}, (\mathbf{\Theta}_{t/\epsilon^2})_{t\ge0}, (\mathbf{X}_\epsilon)(t))_{t\ge0}, (\mathbf{Q}^\epsilon_\omega)_{\omega\in\Gamma_\epsilon}) \tag{7.5}$$

is a diffusion process and thus up to μ_ϵ*-equivalence the unique process in this class which is properly associated with* $(\mathcal{E}_\epsilon, D(\mathcal{E}_\epsilon))$*. It has* μ_ε *as an invariant measure.*

Proof. By (7.4), to prove the first statement of the theorem it suffices to show that, for all ($\widetilde{\mu}_\epsilon$-versions of) $F \in L^2(\Gamma;\widetilde{\mu}_\epsilon)$ and all $t>0$, the function

$$\Gamma \ni \gamma \mapsto \int_{\mathbf{\Omega}} F(S_{\mathrm{in},\,\epsilon}(\mathbf{X}(\epsilon^{-2}t)))\, d\mathbf{P}_{S^{-1}_{\mathrm{in},\,\epsilon}\gamma},$$

is a $\widetilde{\mu}_\epsilon$-version of $\exp(-t\widetilde{\mathcal{H}}_\epsilon)F$. Analogously to $\mathcal{S}_{\mathrm{out},\,\epsilon}$, we define a unitary operator $\mathcal{S}_{\mathrm{in},\,\epsilon} : L^2(\Gamma;\widetilde{\mu}_\epsilon) \to L^2(\Gamma;\mu)$ through the transformation $S_{\mathrm{in},\,\epsilon}$ of Γ. We note that $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma)$ remains invariant under $\mathcal{S}_{\mathrm{in},\,\epsilon}$. A direct calculation shows that, for any $F, G \in \mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma)$,

$$\mathcal{E}^\Gamma_\mu(\mathcal{S}_{\mathrm{in},\,\epsilon}F, \mathcal{S}_{\mathrm{in},\,\epsilon}G) = \epsilon^2 \mathcal{E}^\Gamma_{\widetilde{\mu}_\epsilon}(F,G). \tag{7.6}$$

Since $\mathcal{S}^{-1}_{\mathrm{in},\,\epsilon} H^\Gamma_\mu \mathcal{S}_{\mathrm{in},\,\epsilon}$ is the generator of the closure of the bilinear form $(\mathcal{E}^\Gamma_\mu(\mathcal{S}_{\mathrm{in},\,\epsilon}\cdot, \mathcal{S}_{\mathrm{in},\,\epsilon}\cdot), \mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma))$ on $L^2(\gamma,\widetilde{\mu}_\epsilon)$, (7.6) implies that

$$\mathcal{S}^{-1}_{\mathrm{in},\,\epsilon} H^\Gamma_\mu \mathcal{S}_{\mathrm{in},\,\epsilon} = \epsilon^2 H^\Gamma_{\widetilde{\mu}_\epsilon}.$$

By using Theorem 6.1, we now easily obtain the first assertion.

The fact that $\mathbf{M}_\epsilon$ is a diffusion is straightforward to check. In particular, it then follows from [27, Chap. IV, Theorem 3.5] that $\mathbf{M}_\epsilon$ is properly associated with $(\mathcal{E}_\epsilon, D(\mathcal{E}_\epsilon))$. ■

7.2 Scaling limit of the Dirichlet form $\mathcal{E}^\Gamma_\mu$

We shall now show the convergence of the processes $\mathbf{M}_\epsilon$ to a generalized Ornstein–Uhlenbeck process in the sense of convergence of the corresponding Dirichlet forms $\mathcal{E}_\epsilon$. The limiting Dirichlet form will coincide, up to a constant factor, with the limiting Dirichlet form of [17].

We introduce the set $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\mathcal{D}')$ of all functions on $\mathcal{D}'$ of the form (3.1) where Γ is replaced by $\mathcal{D}'$. Thus, any function $F \in \mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma)$ is a restriction of some $\widetilde{F} \in \mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\mathcal{D}')$ to Γ. Notice that any function from $\mathcal{S}^{-1}_{\mathrm{out},\,\epsilon}\big(\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma)\big)$, defined on Γ_ϵ, may be extended to a function from $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\mathcal{D}')$, and so the set (of μ_ϵ-classes of) $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\mathcal{D}')$ is dense in $D(\mathcal{E}_\epsilon)$ with respect to the norm $\|\cdot\|_{\mathcal{E}_\epsilon} := \big(\|\cdot\|^2_{L^2(\mu_\epsilon)} + \mathcal{E}_\epsilon(\cdot)\big)^{1/2}$.

We next introduce a bilinear form $\mathcal{E}_{\nu_c}$ on $L^2(\mathcal{D}';\nu_c)$ as follows:

$$\begin{aligned}\mathcal{E}_{\nu_c}(F,G) &= \int_{\mathcal{D}'}\int_{\mathbb{R}^d} \partial_x F(\omega)(-\Delta_x)\partial_x G(\omega)\, m(dx)\nu_c(d\omega)\\ &= \int_{\mathcal{D}'}\int_{\mathbb{R}^d} \langle \nabla_x \partial_x F(\omega), \nabla_x \partial_x G(\omega)\, m(dx)\, \nu_c(d\omega),\end{aligned}$$

where $F, G \in D(\mathcal{E}_{\nu_c}) := \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\mathcal{D}')$. Here, ∂_x denotes the derivative in direction ε_x, i.e.,

$$\partial_x F(\omega) = \frac{d}{dt} F(\omega + t\varepsilon_x)\big|_{t=0},$$

and ∇_x and Δ_x denote the gradient and the Laplacian in the x variable, respectively. One easily sees that, for $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\mathcal{D}')$ of the form (3.1),

$$\begin{aligned}\mathcal{E}_{\nu_c}(F) := \mathcal{E}_{\nu_c}(F,F) = \sum_{i,j=1}^{N} \int_{\mathcal{D}'} & \partial_i g_F(\langle \varphi_1,\omega\rangle,\dots,\langle\varphi_N,\omega\rangle)\\ & \times \partial_j g_F(\langle \varphi_1,\omega\rangle,\dots,\langle\varphi_N,\omega\rangle)\, \nu_c(d\omega) \int_{\mathbb{R}^d} \langle \nabla\varphi_i(x), \nabla\varphi_j(x)\rangle\, m(dx). \quad (7.7)\end{aligned}$$

By using the integration by parts formula on Gaussian space (e.g. [8, Ch. 6, Theorems 6.1.2 and 6.1.3]), we conclude that

$$\mathcal{E}_{\nu_c}(F,G) = \int_{\mathcal{D}'} (\mathcal{H}_{\nu_c}F)(\omega) G(\omega)\, \nu_c(d\omega),$$

where

$$\begin{aligned}(\mathcal{H}_{\nu_c}F)(\omega) := -\sum_{i,j=1}^{N} & \partial_i\partial_j g_F(\langle \varphi_1,\omega\rangle,\dots,\langle\varphi_N,\omega\rangle) \int_{\mathbb{R}^d} \langle \nabla\varphi_i(x), \nabla\varphi_j(x)\rangle\, m(dx)\\ & - c^{-1}\sum_{j=1}^{N} \partial_j g_F(\langle \varphi_1,\omega\rangle,\dots,\langle\varphi_N,\omega\rangle)\langle \Delta\varphi_j,\omega\rangle.\end{aligned}$$

Hence, the bilinear form $\mathcal{E}_{\nu_c}$ is closable on $L^2(\mathcal{D}',\nu_c)$. Moreover, it is well known (e.g. [8, Ch. 6, Theorem 6.1.4]) that the operator $\mathcal{H}_{\nu_c}$ is essentially self-adjoint on $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\mathcal{D}')$. We preserve the same notation for its closure. The operator $\mathcal{H}_{\nu_c}$ generates an infinite-dimensional Ornstein–Uhlenbeck semigroup $(\exp(-t\mathcal{H}_{\nu_c}))_{t\ge 0}$ in $L^2(\mathcal{D}',\nu_c)$. This semigroup is associated to a generalized Ornstein–Uhlenbeck process $(\mathbf{N}(t))_{t\ge 0}$ on $\mathcal{D}'$, see e.g. [8, Chapter 6, Section 1.5]. This process informally satisfies the stochastic differential equation (1.7).

Theorem 7.1 *Let the conditions of Proposition* 7.2 *be fulfilled. Then, the bilinear forms* $\mathcal{E}_\epsilon$ *converge to the bilinear form* $\mathcal{E}_{\nu_c}$ *in the following sense: for all* $F, G \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\mathcal{D}')$,

$$\mathcal{E}_\varepsilon(F,G) \to \mathcal{E}_{\nu_c}(F,G) \quad \text{as } \epsilon \to 0.$$

Remark 7.1 Proposition 7.2 and Theorem 7.1 tell us that the scaled process $(\mathbf{X}_\epsilon(t))_{t\ge 0}$ converges to the Ornstein–Uhlenbeck process $(\mathbf{N}(t))_{t\ge 0}$ in the sense of the convergence of their respective Dirichlet forms on $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\mathcal{D}')$. In particular, equation (1.7) allows us to identify c^{-1} as the bulk diffusion coefficient corresponding to the initial process $(\mathbf{X}(t))_{t\ge 0}$.

Proof of Theorem 7.1. Due to the polarization identity, it suffices to show that, for each $F\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\mathcal{D}')$,

$$\mathcal{E}_\epsilon(F)\to\mathcal{E}_{\nu_c}(F)\quad\text{as }\epsilon\to 0.$$

Let $F\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\mathcal{D}')$ be of the form (3.1). Then, by (7.3),

$$\begin{aligned}
\mathcal{E}_\epsilon(F)&=\int_\Gamma\widetilde{\mu}_\epsilon(d\gamma)\int_{\mathbb{R}^d}\epsilon^{-d}m(dx)\,|\nabla_x(\mathcal{S}_{\mathrm{out},\epsilon}F)(\gamma+\varepsilon_x)|^2\\
&=\int_\Gamma\widetilde{\mu}_\epsilon(d\gamma)\int_{\mathbb{R}^d}\epsilon^{-d}m(dx)\,\big|\nabla_x\big(g_F(\langle\varphi_1,\epsilon^{d/2}(\gamma+\varepsilon_x)-\rho\epsilon^{-d/2}\rangle,\\
&\qquad\dots,\langle\varphi_N,\epsilon^{d/2}(\gamma+\varepsilon_x)-\rho\epsilon^{-d/2}\rangle)\big)\big|^2\\
&=\int_\Gamma\widetilde{\mu}_\epsilon(d\gamma)\int_{\mathbb{R}^d}\epsilon^{-d}m(dx)\sum_{i,j=1}^N\partial_i g_F(\langle\varphi_1,\epsilon^{d/2}\gamma-\rho\epsilon^{-d/2}\rangle+\epsilon^{d/2}\varphi_1(x),\\
&\dots,\langle\varphi_N,\epsilon^{d/2}\gamma-\rho\epsilon^{-d/2}\rangle+\epsilon^{d/2}\varphi_N(x))\,\partial_j g_F(\langle\varphi_1,\epsilon^{d/2}\gamma-\rho\epsilon^{-d/2}\rangle+\epsilon^{d/2}\varphi_1(x),\\
&\qquad\dots,\langle\varphi_N,\epsilon^{d/2}\gamma-\rho\epsilon^{-d/2}\rangle+\epsilon^{d/2}\varphi_N(x))\langle\epsilon^{d/2}\nabla\varphi_i(x),\epsilon^{d/2}\nabla\varphi_j(x)\rangle\\
&=\sum_{i,j=1}^N\int_{\mathcal{D}'}\mu_\epsilon(d\omega)\int_{\mathbb{R}^d}m(dx)\,\partial_i g_F(\langle\varphi_1,\omega\rangle+\epsilon^{d/2}\varphi_1(x),\dots,\langle\varphi_N,\omega\rangle+\epsilon^{d/2}\varphi_N(x))\\
&\times\partial_j g_F(\langle\varphi_1,\omega\rangle+\epsilon^{d/2}\varphi_1(x),\dots,\langle\varphi_N,\omega\rangle+\epsilon^{d/2}\varphi_N(x))\langle\nabla\varphi_i(x),\nabla\varphi_j(x)\rangle.
\end{aligned}\tag{7.8}$$

Let $\hat{\mu}_\epsilon$, resp. $\hat{\nu}_c$ denote the measure on $\mathbb{R}^N$ obtained as the image of μ_ϵ, resp. ν_c under the mapping

$$\mathcal{D}'\ni\omega\mapsto(\langle\varphi_1,\omega\rangle,\dots,\langle\varphi_N,\omega\rangle)\in\mathbb{R}^N.$$

Then, it follows from Proposition 7.1 that $\hat{\mu}_\epsilon$ converges weakly on $\mathbb{R}^N$ to $\hat{\nu}_c$. Since the functions $\partial_i g_F$, $i=1,\dots,N$, are continuous and bounded on $\mathbb{R}^N$, we therefore get from (7.7):

$$\begin{aligned}
&\sum_{i,j=1}^N\int_{\mathbb{R}^N}\partial_i g_F(x_1,\dots,x_N)\,\partial_j g_F(x_1,\dots,x_N)\,\hat{\mu}_\epsilon(dx_1,\dots,dx_N)\\
&\qquad\times\int_{\mathbb{R}^d}\langle\nabla\varphi_i(x),\nabla\varphi_j(x)\rangle\,m(dx)\to\mathcal{E}_{\nu_c}(F)\quad\text{as }\epsilon\to 0.
\end{aligned}\tag{7.9}$$

Choose $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$ such that $\operatorname{supp}\varphi_i\subset\Lambda$ for all $i=1,\dots,N$. Then, by (7.8) and (7.9), it suffices to show that, for any $i,j\in\{1,\dots,N\}$,

$$\int_{\mathbb{R}^N}\hat{\mu}_\epsilon(dx_1,\dots,dx_N)\int_\Lambda m(dx)\,\big|\partial_i g_F(x_1+\epsilon^{d/2}\varphi_1(x),\dots,x_N+\epsilon^{d/2}\varphi_N(x))$$

$$\times\, \partial_j g_F(x_1+\epsilon^{d/2}\varphi_1(x),\dots,x_N+\epsilon^{d/2}\varphi_N(x)) \\ -\, \partial_i g_F(x_1,\dots,x_N)\,\partial_j g_F(x_1,\dots,x_N)\Big| \to 0 \quad \text{as } \epsilon\to 0. \quad (7.10)$$

Set $\Lambda_n{:=}(-n,n)^N$, $n\in\mathbb{N}$. Since $\hat\nu_c(\mathbb{R}^N)=1$, we get $\hat\nu_c(\Lambda_n)\to 1$ as $n\to\infty$. Hence, for any fixed $\delta>0$, there exists $n_0\in\mathbb{N}$ such that $\hat\nu_c(\Lambda_{n_0})\ge 1-\delta$. Since Λ_{n_0} is open and $\hat\mu_\epsilon$ converges weakly to $\hat\nu_c$, we conclude from the Portemanteau theorem that

$$\liminf_{\epsilon\to 0}\hat\mu_\epsilon(\Lambda_{n_0})\ge\hat\nu_c(\Lambda_{n_0})\ge 1-\delta.$$

Hence, there exists $\epsilon_0>0$ such that, for each $\epsilon<\epsilon_0$, $\hat\mu_\epsilon(\Lambda_{n_0})\ge 1-2\delta$, so that $\hat\mu_\epsilon(\Lambda_{n_0}^c)\le 2\delta$. From here

$$\int_{\Lambda_{n_0}^c}\hat\mu_\epsilon(dx_1,\dots,dx_N)\int_\Lambda m(dx)\,\big|\partial_i g_F(x_1+\epsilon^{d/2}\varphi_1(x),\dots,x_N+\epsilon^{d/2}\varphi_N(x)) \\ \times\,\partial_j g_F(x_1+\epsilon^{d/2}\varphi_1(x),\dots,x_N+\epsilon^{d/2}\varphi_N(x))-\partial_i g_F(x_1,\dots,x_N)\,\partial_j g_F(x_1,\dots,x_N)\big| \\ \le 4\delta m(\Lambda)\max_{i=1,\dots,N}\ \sup_{(x_1,\dots,x_N)\in\mathbb{R}^N}|\partial_i g_F(x_1,\dots,x_N)|^2 \quad (7.11)$$

for $\epsilon<\epsilon_0$.

Let $\alpha{:=}\max_{i=1,\dots,N}\sup_{x\in\mathbb{R}^d}|\varphi_i(x)|$. Since the function $\partial_i g_F\,\partial_j g_F$ is uniformly continuous on the compact set $[-n_0-\alpha,n_0+\alpha]^N$, we conclude that there exists $\epsilon_1>0$, $\epsilon_1\le\epsilon_0$, such that

$$\big|\partial_i g_F(x_1+\epsilon^{d/2}\varphi_1(x),\dots,x_N+\epsilon^{d/2}\varphi_N(x))\,\partial_j g_F(x_1+\epsilon^{d/2}\varphi_1(x),\dots,x_N+\epsilon^{d/2}\varphi_N(x)) \\ -\,\partial_i g_F(x_1,\dots,x_N)\,\partial_j g_F(x_1,\dots,x_N)\big|<\delta$$

for all $(x_1,\dots,x_N)\in\Lambda_{n_0}$ and $\epsilon<\epsilon_1$. Therefore,

$$\int_{\Lambda_{n_0}}\hat\mu_\epsilon(dx_1,\dots,dx_N)\int_\Lambda m(dx)\,\big|\partial_i g_F(x_1+\epsilon^{d/2}\varphi_1(x),\dots,x_N+\epsilon^{d/2}\varphi_N(x)) \\ \times\,\partial_j g_F(x_1+\epsilon^{d/2}\varphi_1(x),\dots,x_N+\epsilon^{d/2}\varphi_N(x)) \\ -\,\partial_i g_F(x_1,\dots,x_N)\,\partial_j g_F(x_1,\dots,x_N)\big|\le m(\Lambda)\delta \quad (7.12)$$

for each $\epsilon<\epsilon_1$. Finally, (7.11) and (7.12) imply (7.10). ■

7.3 Tightness

In this subsection, we shall discuss the problem of convergence in law of the processes $\mathbf{M}_\epsilon$ as $\epsilon\to 0$.

For $\epsilon>0$ the law of the scaled equilibrium process is the probability measure on $C([0,\infty),\Gamma_\epsilon)$ given by

$$\mathbf{P}_\epsilon{:=}\mathbf{Q}_{\mu_\epsilon}\circ\mathbf{X}_\epsilon^{-1},$$

where

$$\mathbf{Q}_{\mu_\epsilon}{:=}\int_{\Gamma_\epsilon}\mathbf{Q}_\omega^\epsilon\,\mu_\epsilon(d\omega),$$

(cf. Proposition 7.2). Since $C([0,\infty),\Gamma_\epsilon)$ is a Borel subset of $C([0,\infty),\mathcal{D}')$ (under the natural embedding) with compatible measurable structure, we can consider $\mathbf{P}_\epsilon$ as a measure on the (common for all $\epsilon>0$) space $C([0,\infty),\mathcal{D}')$.

For $n\in\mathbb{Z}$, we define a weighted Sobolev space $\mathcal{H}_n$ as the closure of $\mathcal{D}$ with respect to the Hilbert norm

$$\|f\|_n^2=\langle f,f\rangle_n:=\int_{\mathbb{R}^d} A^n f(x)f(x)\,m(dx),\qquad f\in\mathcal{D},$$

where

$$Af(x):=-\Delta f(x)+|x|^2f(x),\qquad x\in\mathbb{R}^d.$$

We identify $\mathcal{H}_0=L^2(\mathbb{R}^d;m)$ with its dual and obtain

$$\mathcal{D}\subset S(\mathbb{R}^d)\subset\mathcal{H}_n\subset L^2(\mathbb{R}^d;m)\subset\mathcal{H}_n\subset S'(\mathbb{R}^d)\subset\mathcal{D}',\qquad n\in\mathbb{N}.$$

Here, as usual $S'(\mathbb{R}^d)$ denotes the space of tempered distributions which is the the topological dual of $S(\mathbb{R}^d)$, the Schwartz space of smooth functions on $\mathbb{R}^d$ decaying faster than any polynomial. Of course, $\mathcal{H}_{-n}$ is the topological dual of $\mathcal{H}_n$ with respect to $\mathcal{H}_0$. For each $n\in\mathbb{Z}$, the embedding $\mathcal{H}_n\hookrightarrow\mathcal{H}_{n-d}$ is of Hilbert–Schmidt type.

Theorem 7.2 *Let the conditions of Proposition* 7.2 *be satisfied. Then, there exists $k\in\mathbb{N}$, $k\ge d+1$, such that the family of probability measures $(\mathbf{P}_\epsilon)_{\epsilon>0}$ can be restricted to the space $C([0,\infty),\mathcal{H}_{-k})$. Furthermore, $(\mathbf{P}_\epsilon)_{\epsilon>0}$ is tight on $C([0,\infty),\mathcal{H}_{-k})$.*

Proof. The proof of this theorem is analogous to the proof of [17, Theorem 6.1].

Consider the diffusion process $\mathbf{M}_\epsilon$, $\epsilon>0$, on the state space $\mathcal{D}'$. Considering its distribution on $C([0,\infty),\mathcal{D}')$, we may regard its canonical realization (7.5). So, in particular $\mathbf{\Omega}=C([0,\infty),\mathcal{D}')$, $\mathbf{X}(t)(\omega)=\omega(t)$, $t\ge0$, $\mathbf{\Theta}_t(\omega)=\omega(t+\cdot)$, and $\mathbf{P}_\epsilon=\int_{\Gamma_\epsilon}\mathbf{Q}_\omega^\epsilon\,\mu_\epsilon(d\omega)$. Fix $T>0$. Below, we canonically project the process onto $\mathbf{\Omega}_T:=C([0,T],\mathcal{D}')$ without expressing this explicitly. We define the time reversal $r_T(\omega):=\omega(T-\cdot)$.

Let $f\in\mathcal{D}$. It is easy to show that $\langle f,\cdot\rangle\in D(\mathcal{E}_\epsilon)$. By the Lyons–Zheng decomposition, cf. [25, 15, 26], we have, for all $0\le t\le T$:

$$\langle f,\mathbf{X}(t)\rangle-\langle f,\mathbf{X}(0)\rangle=\frac12\,\mathbf{M}_t(\epsilon,f)+\frac12\,\big(\mathbf{M}_{T-t}(\epsilon,f)(r_T)-\mathbf{M}_T(\epsilon,f)(r_T)\big),$$

$\mathbf{P}_\epsilon$-a.e., where $(\mathbf{M}_t(\epsilon,f))_{0\le t\le T}$ is a continuous $(\mathbf{P}_\epsilon,(\mathbf{F}_{t/\epsilon^2})_{0\le t\le T})$-martingale and $(\mathbf{M}_t(\epsilon,f)(r_T))_{0\le t\le T}$ is a continuous $(\mathbf{P}_\epsilon,(r_T^{-1}(\mathbf{F}_{t/\epsilon^2}))_{0\le t\le T})$-martingale. Moreover, by (7.8),

$$\langle\mathbf{M}(\epsilon,f)\rangle_t=2t\int_{\mathbb{R}^d}|\nabla f(x)|^2\,m(dx),$$

as e.g. follows from [15, Theorem 5.2.3 and Theorem 5.1.3(i)] (see also a remark in the proof of [17, Theorem 6.1]). Hence, by the Burkholder–Davies–Gundy inequality and since $\mathbf{P}_\epsilon\circ r_T=\mathbf{P}_\epsilon$, we can find $C>0$ such that, for all $f\in\mathcal{D}$, $0\le s\le t\le T$,

$$\mathbb{E}_{\mathbf{P}_\epsilon}\big[|\langle f,\mathbf{X}(t)\rangle-\langle f,\mathbf{X}(s)\rangle|^4\big]$$

$$\le \mathbb{E}_{\mathbf{P}_\epsilon}\big[|\mathbf{M}_t(\epsilon,f)-\mathbf{M}_s(\epsilon,f)|^4\big]+\mathbb{E}_{\mathbf{P}_\epsilon}\big[|\mathbf{M}_{T-t}(\epsilon,f)(r_T)-\mathbf{M}_{T-s}(\epsilon,f)(r_T)|^4\big]$$
$$\le C(t-s)^2\|\,|\nabla f|\,\|_0^4. \tag{7.13}$$

Now, we can use (7.13) to define $\langle f,\mathbf{X}(t)\rangle-\langle f,\mathbf{X}(s)\rangle$ for $f\in S(\mathbb{R}^d)$ via an approximation as an element of $L^4(\mathbf{\Omega},\mathbf{P}_\epsilon)$. Then, the estimate (7.13) holds true for $f\in S(\mathbb{R}^d)$.

We can choose $\alpha>0$ and $k\in\mathbb{N}$ large enough, so that

$$\forall f\in S(\mathbb{R}^d):\qquad \|\,|\nabla f|\,\|_0^4\le\alpha\|f\|_{k-2d}^4. \tag{7.14}$$

Let $(e_i)_{i=0}^\infty$ be the sequence of Hermite functions forming an orthonormal basis of $\mathcal{H}_{k-2d}$. For $i\in\mathbb{N}$, let a_i denote the eigenvalue of the operator A belonging to the eigenvector e_i. Then, $(a_i^{k-d}e_i)_{i\in\mathbb{N}}$ forms an orthonormal basis in $\mathcal{H}_{-k}$. Hence, by (7.13) and (7.14),

$$\begin{aligned}
&\big(\mathbb{E}_{\mathbf{P}_\epsilon}\big[\|\mathbf{X}(t)-\mathbf{X}(s)\|_{-k}^4\big]\big)^{1/2}\\
&\quad=\left(\mathbb{E}_{\mathbf{P}_\epsilon}\left[\left(\sum_{i=0}^\infty a_i^{2k-2d}\big((e_i,\mathbf{X}(t))_{-k}-(e_i,\mathbf{X}(s))_{-k}\big)^2\right)^2\right]\right)^{1/2}\\
&\quad=\left(\mathbb{E}_{\mathbf{P}_\epsilon}\left[\left(\sum_{i=0}^\infty a_i^{-2d}\big(\langle e_i,\mathbf{X}(t)\rangle-\langle e_i,\mathbf{X}(s)\rangle\big)^2\right)^2\right]\right)^{1/2}\\
&\quad\le\left(\sum_{i=0}^\infty a_i^{-2d}\right)^{1/2}\left(\sum_{i=0}^\infty a_i^{-2d}\,\mathbb{E}_{\mathbf{P}_\epsilon}\left[(\langle e_i,\mathbf{X}(t)\rangle-\langle e_i,\mathbf{X}(s)\rangle)^4\right]\right)^{1/2}\\
&\quad\le C'(t-s),
\end{aligned}\tag{7.15}$$

where the constant $C':=(\alpha C)^{1/2}\sum_{i=0}^\infty a_i^{-2d}$ is finite, since A^{-d} is a Hilbert–Schmidt operator.

Since, by Proposition 7.1, $\mu_\epsilon\to\nu_c$ as $\epsilon\to0$, now the tightness of $(\mathbf{P}_\epsilon)_{\epsilon>0}$ on $C([0,\infty),\mathcal{H}_{-k})$ follows by standard arguments. ■

It follows from Theorem 7.2 that there exists at least one accumulation point $\widetilde{\mathbf{P}}$ of $(\mathbf{P}_\epsilon)_{\epsilon>0}$ on $C([0,\infty),\mathcal{H}_{-k})$, i.e., $\mathbf{P}_{\epsilon_n}\to\widetilde{\mathbf{P}}$ weakly for some subsequence $\epsilon_n\to0$. However, it is still an open question whether the measures $\mathbf{P}_\epsilon$ converge to the law $\mathbf{P}$ of the Ornstein–Uhlenbeck process $(\mathbf{N}(t))_{t\ge0}$, i.e., whether the measure $\widetilde{\mathbf{P}}$ must always coincide with $\mathbf{P}$.

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