INVARIANT MEASURES OF GENERALIZED STOCHASTIC POROUS MEDIUM EQUATIONS

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In this paper, we extend the results of the recent works [1], [2] on existence of infinitesimally invariant measures for the stochastic porous medium equation. The corresponding partial differential equation (see [3]) is

$$\partial x(u,t)/\partial t = \Delta\Psi(x)(u,t) + \Phi(x)(u,t), \tag{1}$$

where Ψ and Φ are certain functions, e.g., polynomials. The associated stochastic partial differential equation is heuristically written as

$$dx_t = \sqrt{2}dW_t + [\Delta\Psi(x_t) + \Phi(x_t)]\,dt. \tag{2}$$

However, the rigorous interpretation is not obvious in case of nonlinear functions Ψ and Φ. One of the possible approaches to this problem is to consider the associated infinite dimensional elliptic operator L on a suitable domain, find an infinitesimally invariant measure μ for L, and construct a Markovian semigroup on $L^2(\mu)$ having μ as in invariant measure such that the generator of the semigroup extends L, and finally construct a strong Markov process with continuous paths that solves the martingale problem corresponding to (1). In the case $\Psi(s) = s^m + \alpha s$, $\alpha > 0$, and $\Phi = 0$, where m is an odd number, this programme has been fulfilled in [2] and existence of an infinitesimally invariant measure for $\alpha = 0$ and $m = 3$ has been proved in [1]. Here we consider more general Ψ and nonzero Φ. In addition to greater generality of our assumptions, a novelty of this paper is that it provides constructive finite dimensional approximations of the invariant measure. The existence result is an application of a result of our earlier work [4].

Let $D \subset \mathbb{R}^d$ be a bounded open domain with a smooth boundary and let $\{e_n\}$ be the orthonormal basis in $L^2(D)$ formed by the eigenfunctions of the Laplacian Δ with Dirichlet boundary conditions. Thus, $\Delta e_i = \lambda_i e_i$ and we assume that $\lambda_1 \le \lambda_2 \le \cdots$. The inner product and norm in $L^2(D)$ are denoted by $(x,y)_2$ and $\|x\|_2$. Let $H_0^{2,1}(D)$ be the classical Sobolev space obtained as the completion of $C_0^\infty(D)$ with respect to the Sobolev norm $\|x\|_{2,1} = \|x\|_2 + \||\nabla x|\|_2$.

Let $r > 1$ and let $\zeta_r(s) := |s|^r \operatorname{sgn} s$, $s \in \mathbb{R}^1$. If r is an odd number, then $\zeta_r(s) = s^r$. Let Ψ be a C^1-function with $\Psi(0) = 0$ such that for some positive numbers κ_0, C_0, and κ_1 one has

$$\kappa_0|s|^{r-1} \le \Psi'(s) \le C_0 + \kappa_1|s|^{r-1} \quad \text{for all } s \in \mathbb{R}^1,$$

and let Φ be a continuous function satisfying the following condition: $|\Phi(s)| \le C + \delta|s|^r$, where $0 < \delta < 4\kappa_0\lambda_1(r+1)^{-2}$ and C is a constant. For example, it suffices that $|\Phi(s)| \le \kappa_2 + \kappa_3|s|^q$, where $q \in (0,r)$, $\kappa_2, \kappa_3 \in (0,+\infty)$.

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We study the existence of infinitesimally invariant measures for the infinite dimensional elliptic operator L informally given by

$$Lf := \Delta_Q f + \langle b, \nabla f\rangle, \quad b(x) = \Delta\Psi(x) + \Phi(x)$$

on smooth cylindrical functions defined on $X := L^2(D)$ or on the negative Sobolev space $H := H^{2,-1}(D)$. A rigorous interpretation is this. Let $\mathcal{FC}_0^\infty$ be the linear span of the class of all functions f on X of the form $f(x) = f_0(x_1, \ldots, x_n)$, $f_0 \in C_0^\infty(\mathbb{R}^n)$, $x_i = (x, e_i)_2$. Let

$$b_i(x) := \int_D \Big[\Psi(x(u))\Delta e_i(u) + \Phi(x(u))e_i(u)\Big]\, du, \quad x \in L^r(D).$$

Let $q_i > 0$ be such that $S := \sum_{i=1}^{\infty} q_i < \infty$. The operator

$$Lf := \sum_{i=1}^{\infty} \big[q_i \partial_{e_i}^2 f + b_i \partial_{e_i} f\big], \quad f \in \mathcal{FC}_0^\infty,$$

where ∂_{e_i} stands for the partial derivative along e_i, is well defined (for every $f \in \mathcal{FC}_0^\infty$, Lf is just a finite sum). The second order part of L can be regarded as trace $(QD^2 f)$, where Q is the operator on X defined by $Qe_i = q_i e_i$. The operator Q is the covariance of the Wiener process W_t in (2).

We shall say that a Borel probability measure μ on X is infinitesimally invariant for L if $\mu(L^r(D)) = 1$, $b_i \in L^1(\mu)$ for all i and

$$\int_X Lf(x)\,\mu(dx) = 0 \quad \forall f \in \mathcal{FC}_0^\infty. \tag{3}$$

We write this symbolically as $L^*\mu = 0$.

Let E_n denote the linear space spanned by $e_1, \ldots, e_n$ and let

$$L_n f := \sum_{i=1}^{n} \big[q_i \partial_{e_i}^2 f + b_i \partial_{e_i} f\big].$$

The orthogonal projection in $L^2(D)$ to E_n is denoted by P_n.

Lemma 1. *Let ψ be a C^1-function on the real line such that $\psi(0) = 0$.*

(i) *Suppose that $|\psi'(s)| \geq C|s|^{r-1}$, where $C > 0$ and $r \geq 1$. Let $f \in L^2(D)$ and $\psi \circ f \in H_0^{2,1}(D)$. Then $|f|^r \mathrm{sgn} f \in H_0^{2,1}(D)$.*

(ii) *Suppose that $|\psi'(s)| \leq C'|s|^{r-1}$, where $C' > 0$ and $r \geq 1$. Let $f \in L^2(D)$ and $|f|^r \mathrm{sgn} f \in H_0^{2,1}(D)$. Then $\psi \circ f \in H_0^{2,1}(D)$.*

Proof. (i) The inverse function to ψ will be denoted by η. Then $\zeta_r \circ x = \zeta_r \circ \eta \circ \psi \circ x$. The function η is continuous, strictly increasing and differentiable outside the origin. Now it suffices to observe that the function $h := \zeta_r \circ \eta$ is Lipschitzian and $h(0) = 0$. Indeed,

$$|h'(s)| = |\zeta_r'(\eta(s))\eta'(s)| = |\zeta_r'(\eta(s))/\psi'(\eta(s))| \leq r/C.$$

Assertion (ii) is proved analogously. □

We observe that the assumption $\psi(0) = 0$ is only needed to ensure the zero boundary condition; in the case of $H^{2,1}(D)$, the same reasoning applies without that assumption. In place of the continuous differentiability of ψ one can require that it is Lipschitzian (then the estimate on $|\psi'|$ should hold a.e.).

Note that the inclusion $\zeta_r \circ x \in H_0^{2,1}(D)$ implies the inclusion $|x|^r \in H_0^{2,1}(D)$, but obviously is not equivalent to the latter.

Theorem 1. (i) *Under the above assumptions, there exists a Borel probability measure μ on X that is infinitesimally invariant for L and is concentrated on the set of functions x such that $\zeta_{(r+1)/2} \circ x \in H_0^{2,1}(D)$ and*

$$\int_X \int_D |\nabla(\zeta_{(r+1)/2} \circ x)(u)|^2 \, du \, \mu(dx) < \infty. \tag{4}$$

(ii) *If, in addition, $\sum_{i=1}^{\infty} q_i \sup_{u \in D} e_i(u)^2 =: M < \infty$, then there exists a Borel probability μ that is infinitesimally invariant for L and concentrated on the set of functions x such that $\Psi \circ x \in H_0^{2,1}(D)$ and consequently $\zeta_r \circ x \in H_0^{2,1}(D)$ and one has*

$$\int_X \int_D |\nabla(\Psi \circ x)(u)|^2 \, du \, \mu(dx) < \infty \tag{5}$$

and

$$\int_X \int_D |\nabla(\zeta_r \circ x)(u)|^2 \, du \, \mu(dx) < \infty. \tag{6}$$

Finally, (4) *holds. So* (6) *remains valid for ζ_s in place of ζ_r with any s between $(r+1)/2$ and r.*

Proof. (i) We verify that the hypotheses of Theorem 5.1 in [4] are satisfied. Those hypotheses are:

(a) existence of functions $V\colon X \to [0, +\infty]$ and $\Theta\colon X \to [0, +\infty]$ such that the sets $\{\Theta \le c\}$ are compact and the restrictions of V to the subspaces E_n are compact C^2-functions,

(b) the continuity of the functions b_i on the sets $\{\Theta \le c\}$ and subspaces E_n,

(c) estimates $L_n V(x) \le C - \kappa\Theta(x)$ for all $x \in E_n$ and $|b_i(x)| \le C_i + \delta_i(\Theta(x))\Theta(x)$ for $x \in \{\Theta < \infty\}$, where $\kappa > 0$ and $\lim_{s \to +\infty} \delta_i(s) = 0$. For all $x \in X$, let

$$V(x) := \int_D x(u)^2 \, du,$$

$$\Theta(x) := \int_D |\nabla(\zeta_{(r+1)/2} \circ x)(u)|^2 \, du,$$

where $\Theta(x) := +\infty$ if $\zeta_{(r+1)/2} \circ x \notin H_0^{2,1}(D)$. By the Sobolev embedding theorem, the sets $\{\Theta \le c\}$ are compact in X and the functions b_i are continuous on them (in the topology of X), hence also on E_n. Indeed, given a sequence of functions $x_j \in X$ such that $\zeta_{(r+1)/2} \circ x_j \in H_0^{2,1}(D)$ and $\|\zeta_{(r+1)/2} \circ x_j\|_{2,1}^2 \le c$, one can find a subsequence $\zeta_{(r+1)/2} \circ x_{j_k}$ that converges in $L^2(D)$. Therefore, the sequence x_{j_k} converges in $L^2(D)$ to some function x. Clearly, $\|\zeta_{(r+1)/2} \circ x\|_{2,1}^2 \le c$. The continuity of b_i on $\{\Theta \le c\}$ is seen by the same reasoning. In addition, V is a positive definite quadratic form on the spaces E_n. We have for all $x \in E_n$

$$\sum_{i=1}^{n} x_i b_i(x) = \sum_{i=1}^{n} [\lambda_i x_i (\Psi \circ x, e_i)_2 + x_i (\Phi \circ x, e_i)_2] = (\Delta x, \Psi \circ x)_2 + (x, \Phi \circ x)_2.$$

Let us pick $\alpha > 1$ and $\kappa > 0$ such that $\alpha\delta + \kappa = 4\kappa_0 \lambda_1 (r+1)^{-2}$. One can find $C_\alpha > 0$ such that $|x(u)\Phi(x(u))| \le C_\alpha + \alpha\delta |x(u)|^{r+1}$. Taking into account the estimate

$$\frac{(r+1)^2}{4} \int_D |x(u)|^{r-1} |\nabla x(u)|^2 \, du = \int_D \left|\nabla |x|^{(r+1)/2}(u)\right|^2 du \ge \lambda_1 \int_D |x(u)|^{r+1} \, du,$$

we obtain for all $x \in E_n$

$$
\begin{aligned}
L_n V(x) &= 2\sum_{i=1}^n q_i + 2\sum_{i=1}^n x_i b_i(x) \\
&= 2\sum_{i=1}^n q_i + 2\int_D \big[\Delta x(u)\,\Psi(x(u)) + x(u)\Phi(x(u))\big]\,du \\
&= 2\sum_{i=1}^n q_i - 2\int_D \Psi'(x(u))|\nabla x(u)|^2\,du + 2\int_D x(u)\Phi(x(u))\,du \\
&\le 2S - 2\kappa_0 \int_D |x(u)|^{r-1}|\nabla x(u)|^2\,du + 2C_\alpha|D| + 2\delta\alpha\int_D |x(u)|^{r+1}\,du \\
&\le \kappa' - \kappa\Theta(x),
\end{aligned}
$$

where $\kappa' \ge 0$ and $|D|$ is the measure of D. Finally, taking into account that e_i is a bounded function, $|b_i(x)|$ can be estimated by $\alpha_i + \beta_i \int_D |x(u)|^r\,du$ with some positive numbers α_i and β_i. It remains to observe that for all $x \in \{\Theta < \infty\}$ one has

$$
\int_D |x(u)|^r\,du \le \Big(\int_D |x(u)|^{r+1}\,du\Big)^{r/(r+1)} \le \lambda_1^{-r/(r+1)}\Big(\int_D \big|\nabla|x|^{(r+1)/2}(u)\big|^2\,du\Big)^{r/(r+1)}.
$$

Therefore, we obtain

$$
|b_i(x)| \le \alpha_i' + \beta_i'\Theta(x)^{-1/(r+1)}\Theta(x), \quad x \in \{\Theta < \infty\}.
$$

Thus, all the hypotheses of the theorem cited are satisfied. Therefore, we obtain a probability measure μ on X satisfying equation (3) such that $\zeta_{(r+1)/2}\circ x \in H_0^{2,1}(D)$ for μ-a.e. x and the function $\|\zeta_{(r+1)/2}\circ x\|_{2,1}^2$ is μ-integrable. In addition, μ is the weak limit of a subsequence of the sequence of probability measures μ_n on E_n satisfying $L_n^*\mu_n = 0$ and

$$
\sup_n \int_{E_n}\int_D |\nabla(\zeta_{(r+1)/2}\circ x)(u)|^2\,du\,\mu_n(dx) =: K < \infty. \tag{7}
$$

In particular,

$$
\sup_n \int_{E_n}\int_D |x(u)|^{r-1}\,du\,\mu_n(dx) =: K_1 < \infty. \tag{8}
$$

(ii) Now we suppose that $\sum\limits_{i=1}^\infty q_i \sup\limits_{u\in D} e_i(u)^2 =: M < \infty$. First we consider the case where the function $|\Phi|$ is bounded. Let μ be the measure constructed in (i). We show that $\zeta_r\circ x \in H_0^{2,1}(D)$ for μ-a.e. x and the function $\|\zeta_r\circ x\|_{2,1}^2$ is μ-integrable. We may assume that the whole sequence $\{\mu_n\}$ converges weakly to μ. It is important that this sequence is uniformly tight on the space $L^r(D)$, hence converges weakly to μ also on that space. This is obvious from (7) and the Sobolev embedding theorem. Set

$$
\Xi(t) := \int_0^t \Psi(s)\,ds.
$$

Let n be fixed and let

$$
V_n(x) := \int_D \Xi(x(u))\,du, \quad x \in E_n.
$$

Let $\Lambda := \sqrt{-\Delta}$. We observe that for any $u \in H_0^{2,1}(D)$, one has

$$
(u, \Delta P_n u)_2 = -(\Lambda u, \Lambda P_n u)_2 = -(P_n\Lambda u, P_n\Lambda u)_2,
$$

since $P_n = P_n^2$ and P_n commutes with Λ. Therefore, for all $x \in E_n$ we have

$$
\begin{aligned}
L_n V_n(x) &= \sum_{i=1}^n q_i \int_D \Psi'(x(u)) e_i(u)^2\,du + \int_D \Psi(x(u)) \Delta P_n(\Psi\circ x)(u)\,du \\
&+ \big(P_n(\Phi\circ x), P_n(\Psi\circ x)\big)_2 \\
&= \sum_{i=1}^n q_i \int_D \Psi'(x(u)) e_i(u)^2\,du - \|P_n\Lambda(\Psi\circ x)\|_2^2 + \big(P_n(\Phi\circ x), P_n(\Psi\circ x)\big)_2 \\
&\le C_0 M + \kappa_1 M \int_D |x(u)|^{r-1}\,du - \|P_n\Lambda(\Psi\circ x)\|_2^2 \\
&+ \frac{\lambda_1}{2}\|P_n(\Psi\circ x)\|_2^2 + \frac{1}{2\lambda_1}\|\Phi\circ x\|_2^2 \\
&\le C_0 M + \kappa_1 M \int_D |x(u)|^{r-1}\,du - \frac{1}{2}\|P_n\Lambda(\Psi\circ x)\|_2^2 + \frac{1}{2\lambda_1}\|\Phi\circ x\|_2^2.
\end{aligned}
$$

It is easily seen (see, e.g., the proofs of Lemma 1.2 in [5] and Theorem 4.1 in [4]) that this estimate along with (8) yields

$$
\int_{E_n} \|P_n\Lambda(\Psi\circ x)\|_2^2\,\mu_n(dx) \le 2C_0M + 2\kappa_1 M K_1 + \frac{1}{\lambda_1}\int_{E_n} \|\Phi\circ x\|_2^2\,\mu_n(dx).
$$

Then for every N we have

$$
\int_X \|P_N\Lambda(\Psi\circ x)\|_2^2\,\mu(dx) \le 2C_0M + 2\kappa_1 M K_1 + \frac{1}{\lambda_1}\int_X \|\Phi\circ x\|_2^2\,\mu(dx). \tag{9}
$$

Indeed, if $n \ge N$, then $\|P_N\Lambda(\Psi\circ x)\|_2^2 \le \|P_n\Lambda(\Psi\circ x)\|_2^2$. Hence

$$
\int_{E_n} \|P_N\Lambda(\Psi\circ x)\|_2^2\,\mu_n(dx) \le 2C_0M + 2\kappa_1 M K_1 + \frac{1}{\lambda_1}\int_{E_n} \|\Phi\circ x\|_2^2\,\mu_n(dx).
$$

We observe that

$$
g_N(x) := \|P_N\Lambda(\Psi\circ x)\|_2^2 = \sum_{i=1}^N \lambda_i (\Psi\circ x, e_i)_2^2
$$

is a continuous function on $L^r(D)$. Hence by the weak convergence we arrive at (9). By Fatou's lemma we obtain $\|\Lambda(\Psi\circ x)\|_2^2 < \infty$ for μ-a.e. x and

$$
\int_X \|\Lambda(\Psi\circ x)\|_2^2\,\mu(dx) \le 2C_0M + 2\kappa_1 M K_1 + \frac{1}{\lambda_1}\int_X \|\Phi\circ x\|_2^2\,\mu(dx).
$$

By using the estimates $|\Phi(s)| \le C + 4r\lambda_1(r+1)^{-2}|\Psi(s)|$ and $4r(r+1)^{-2} < 1$ for $r > 1$ along with the inequality $\lambda_1\|\Psi\circ x\|_2^2 \le \|\Lambda(\Psi\circ x)\|_2^2$, we obtain

$$
\int_X \|\Lambda(\Psi\circ x)\|_2^2\,\mu(dx) \le N(r, C, C_0, M, \kappa_1, K_1, \lambda_1). \tag{10}
$$

In the case where Φ is not bounded, we apply the above proved assertion to the functions Φ_j defined as follows: $\Phi_j(t) = \Phi(t)$ if $|\Phi(t)| \le j$, $\Phi_j(t) = j \,\mathrm{sgn}\,\Phi(t)$ if $|\Phi(t)| > j$. Due to (10) the obtained measures μ_j form a uniformly tight sequence. We take for μ a limit point of $\{\mu_j\}$ in the weak topology. It is clear that for the new measures μ_j one has the uniform estimate (7), which yields (4) (we do not claim that this measure μ coincides with the measure constructed in (i)). Estimate (6) follows by Lemma 1. □

For example, if $\Psi(t) = t^m$, where m is an odd number, then one can take for Φ any polynomial of degree m with a sufficiently small leading coefficient (the smallness of which depends on λ_1, in particular, one can take $\Phi(x) = x^m$ provided λ_1 is sufficiently large).

Now we can pass from cylindrical functions to C_b^2 functions, however, defined on larger spaces such as H. Note that by Lemma 1 and assertion (ii) of Theorem 1 we have $\Delta\Psi \circ x \in H$ for μ-a.e. x. By finite dimensional approximations we obtain the following.

Corollary 1. *In the situation of assertion* (ii) *of Theorem* 1*, the constructed measure μ satisfies the equation $L^*\mu = 0$ on H with respect to the class $C_b^2(H)$.*

Corollary 2. *Suppose that in either assertion of Theorem* 1 *one has $\Psi'(0) > 0$. Then $\mu(H_0^{2,1}(D)) = 1$ and*

$$\int_X \int_D |\nabla x(u)|^2 \, du \, \mu(dx) < \infty.$$

Proof. Due to our assumption, in a neighborhood of zero $\Psi'(s) \geq \alpha$ for some constant $\alpha > 0$. Now the assertion follows by the same reasoning as in the theorem with the function

$$\Theta(x) = \int_D \left[|\nabla x(u)|^2 + |\nabla(\zeta_{(r+1)/2} \circ x)(u)|^2 \right] du.$$

□

It is very likely that as in [1], [2], for any $k \in \mathbb{N}$, $k \geq 2$, one has $x^k \in H_0^{2,1}(D)$ for μ-a.e. x and $\|x^k\|_{2,1}^2$ is μ-integrable. If we formally consider the Lyapunov functions $\int_D x^{2k}(t) \, dt$, then we obtain for Θ the function $\int_D |x(u)|^{r+2k-2} |\nabla x(u)|^2 \, du$. However, we have not managed to justify this (except for the case $k = r$ in assertion (ii) above).

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