On conditions for global existence of solutions of ordinary differential, stochastic differential and parabolic equations on manifolds

Yuri E. Gliklikh & Lora A. Morozova
Mathematics Faculty & Research Institute of Mathematics
Voronezh State University
Universitetskaya pl. 1
394006 Voronezh, Russia
e-mail: yuri@yeg.vsu.ru

Introduction

In this paper we consider some questions connected with global in time existence of solutions of various differential equations (ordinary, stochastic and parabolic). The main goal is to obtain necessary and sufficient conditions.

At the moment plenty of sufficient conditions for global existence of solutions can be found in the literature. We would like to point out the so-called conditions with one-sided estimates for ordinary differential equations, similar conditions with Lyapunov functions for parabolic equations and a certain very general condition of the same nature from [1] for stochastic differential equations. It is shown in this paper that after some modification and transition to extended phase space conditions of this sort become necessary and sufficient or close to them. We deal with the general case of equations on finite-dimensional manifolds.

In §1 we consider the case of ordinary differential equations. We obtain a necessary and sufficient condition, similar to those with one-sided estimates, that is based on the same idea as the necessary and sufficient condition of two-sided sort from [3] (see also §1 in [4]).

In previous publications [5] and [6] we obtained necessary and sufficient conditions for a stochastic flow to be well-posed on the entire half-line and also to belong to a certain sort of functional space L^1 (we call this property L^1-completeness of the flow). The content of §2 here is in some sense complementary to those publications. We reformulate the problem in terms of Feller evolution families (semigroups) generated by the flow. We introduce the notion of complete Feller evolution family, that in fact corresponds to an L^1-complete flow, and derive new necessary and sufficient conditions for existence of the former. As a corollary this allows us to obtain also some results for stochastic flows and generalized solutions of parabolic equations. The necessary and sufficient condition is similar to that from §1 and rather close to the sufficient condition $IX.6A$ of [1].

The research is partially supported by Grant 99-00559 from INTAS, by Grant UR.04.01.008 of the Program "Universities of Russia", by Grant 03-01-00112 by RFBR and by U.S. CRDF – RF Ministry of Education Award VZ-010-0.

The authors are grateful to K.D. Elworthy for very useful discussions.

1 The case of ordinary differential equations

Let M be a smooth manifold with dimension $n < \infty$.

Consider a certain smooth jointly in $t \in R$, $m \in M$ vector field $X = X(t, m)$ on M. Its coordinate representation in a chart with respect to local coordinates $(q^1, \dots, q^n)$ takes the form $X = X^1 \frac{\partial}{\partial q^1} + ... + X^n \frac{\partial}{\partial q^n}$. The vector field X can be also considered as the first order differential operator on C^1-functions on M. For a function f the value of the above operator is given as $Xf = X^1 \frac{\partial f}{\partial q^1} + ... + X^n \frac{\partial f}{\partial q^n}$. The function Xf is called also the derivative of f along vector field X.

Definition 1. *A curve $m(t)$ on M is called integral curve of X, if at any t the vector $X_{m(t)}$ is eq1ual to the derivative $\dot{m}(t)$.*

Thus, the integral curves of X are defined by the ordinary differential equation

$$\dot{m}(t) = X(t, m(t)).$$

Let $\gamma(t)$ be an integral curve of X such that $\gamma(0) = m$. It is well known that Xf is represented in terms of $\gamma(t)$ as follows: $Xf(m) = \frac{d}{dt} f(\gamma(t))|_{t=0}$.

Definition 2. *A vector field X is called complete if all its integral curves are well-posed for $t \in (-\infty, +\infty)$.*

Denote by $m(s) : M \to M$, $s \in R$ the flow of X. For any point $x \in M$ and time instant t the orbit $m(s)(t, m) = m_{t,m}(s)$ of the flow is the solution of equation

$$\dot{m}_{t,m}(s) = X(s, m_{t,m}(s)), \tag{1}$$

with the initial condition

$$m_{t,m}(t) = m. \tag{2}$$

Consider extended phase space $M^+ = R \times M$ with the natural projection $\pi^+ : M^+ \to M$, $\pi^+(t, m) = m$. At any point $(t, m) \in M^+$ introduce the vector field $X^+_{(t,m)} = (1, X(t, m))$. It is clear that its coordinate representation is given in the form $X^+ = \frac{\partial}{\partial t} + X^1 \frac{\partial}{\partial q^1} + ... + X^n \frac{\partial}{\partial q^n}$. Hence the corresponding differential operator on the space of C^1-smooth functions on M^+ takes the form $\frac{\partial}{\partial t} + X$.

Definition 3. *A function f on a topological space $\mathbf{X}$ is called proper if the preimage of any relatively compact set in R is a relatively compact set in $\mathbf{X}$.*

Recall that in any finite-dimensional space (in particular, in R) a set is relatively compact if and only if it is bounded.

In what follows we shall mainly deal with proper functions on smooth manifolds.

Theorem 4. *A smooth vector field X on a finite-dimensional manifold M is complete if and only if there exists a smooth proper function $\varphi : M^+ \to R$ such that the absolute value of derivative $|X^+\varphi|$ of φ along X^+ is uniformly bounded, i.e., $|X^+\varphi| = |(\frac{\partial}{\partial t} + X)\varphi| \le C$ at any $(t, m) \in M^+$ for a certain constant $C > 0$ that does not depend on (t, m).*

Proof.

Sufficiency.

Consider the flow $m^+(s) : M^+ \to M^+$, $s \in R$. Its orbits $m^+(s)(t, m) = m^+_{(t,m)}(s)$ are the solutions of equation

$$\dot{m}^+_{(t,m)}(s) = X^+(m^+_{(t,m)}(s))$$

with initial conditions

$$m^+_{(t,m)}(t) = (t, m).$$

Consider the derivative $X^+\varphi$ of φ along X^+. At $(t, m) \in M^+$ we get the equality

$$X^+\varphi(t, m) = \frac{d}{ds}\varphi(m^+_{(t,m)}(s))|_{s=t},$$

(see above) and under the hypothesis of our theorem

$$\left| \frac{d}{ds}\varphi(m^+_{(t,m)}(s))|_{s=t} \right| \le C. \tag{3}$$

Represent the values of φ along the orbit $m^+_{(t,m)}(s)$ as follows:

$$\varphi(m^+_{(t,x)}(s)) - \varphi(t,m) = \int_0^s \frac{d}{d\tau}\varphi(m^+_{(t,m)}(\tau))d\tau.$$

From the last equality and from inequality (3) we evidently obtain that if s belongs to a finite interval, the values $\varphi(m^+_{(t,x)}(s))$ are bounded in R. Then since φ is proper this means that the set $m^+_{(t,m)}(s)$ is relatively compact in M^+.

Recall that by the classical solution existence theorem the domain of any solution of ODE is an open interval in R. In particular for $s > 0$ the solution of above Cauchy problem is well-posed for $s \in [t, \varepsilon)$. If $\varepsilon > 0$ is finite, then the corresponding values of the solution belong to a relatively compact set in M and so the solution is well-posed for $s \in [t, \varepsilon]$. The same arguments are valid also for $s < 0$. Thus the domain is both open and closed and so it coincides with the entire real line $(-\infty, \infty)$.

Taking into account the construction of vector field X^+, we can represent the integral curves $m^+_{(t,m)}(s)$ in the form $m^+_{(t,m)}(s) = (s, m_{t,m}(s))$. Hence from global existence of integral curves of X^+ we obviously obtain the global existence of integral curves of X. So, the vector field X is complete.

Necessity.

Let the vector field X be complete. Thus all orbits $m_{t,m}(s)$ of the flow $m(s)$ are well-posed on the entire real line. Specify a certain countable locally-finite cover $\mathcal{V} = \{V_i\}_{i\in N}$ of M where all V_i are open and relatively compact. Such a cover does exist because any manifold is paracompact by definition and the finite-dimensional manifold M is locally compact. Introduce the functions $\psi_i : M \to R$ by the formula

$$\psi_i(m) = \begin{cases} i \ \text{ if } \ m \in V_i \\ 0 \ \text{ if } \ m \notin V_i \end{cases}.$$

Denote by $\{\phi_i\}_{i=1}^{\infty}$ the smooth partition of unity subordinated to the above cover and define the function ψ on M of the form: $\psi(m) = \sum_{i=1}^{\infty} \psi_i(m)\phi_i(m)$. It is clear that $\psi(m)$ is smooth and proper by the construction. The construction of function $\psi(m)$ is taken from [7].

Introduce the function $\Phi : M^+ \to R$ as follows. For any point $(t,m) \in M^+$ set $\Phi(t,m) = \psi(m_{t,m}(0))$. By the construction the function Φ is constant along any orbit of the flow $m^+(s)$. Indeed, for $m^+(s)(t,m) = (s, m_{t,m}(s))$ the equality $m_{s,m_{t,m}(s)}(0) = m_{t,m}(0)$ holds.

Consider the function $\varphi : M^+ \to R$, $\varphi(t,m) = \Phi(t,m) + t$. Obviously φ is smooth and proper. Consider $X^+\varphi$. By the construction of the vector field X^+ and of the function φ we get

$$X^+\varphi = X^+(\Phi(t,m)) + X^+t = 0 + 1 = 1.$$

Thus we have proven the necessity part of our theorem for $C = 1$. This completes the proof. □

2 The case of stochastic differential and parabolic equations

In this section we introduce the notion of complete Feller evolution family and, combining the necessary and sufficient condition for completeness of vector field from §1 and Elworthy's sufficient condition for completeness of a stochastic flow $IX.6A$ from [1], we prove some necessary and sufficient conditions for existence of complete Feller evolution families.

Let M be a finite-dimensional manifold. Consider a Stochastic Dynamical System (SDS) on M (see [1]) with the generator $\mathcal{A}(t,x)$ that is an elliptic (but not necessarily strongly elliptic) operator on the space of smooth enough functions on M. In local coordinates the SDS is described in terms of a stochastic differential equation with smooth coefficients in Itô or in Stratonovich form. Since the coefficients are smooth we can pass from Stratonovich to Itô equation and vice versa.

Denote by $\xi(s) : M \to M$ the stochastic flow of the above-mentioned SDS. For any point $x \in M$ and time instant t the orbit $\xi_{t,x}(s)$ of this flow is the unique solution of the above equation with initial conditions $\xi_{t,x}(t) = x$. As the coefficients of equation are smooth, this is a strong solution and a Markov diffusion process given on a certain random time interval. Below we denote the probability space, where the flow is defined, by $(\Omega, \mathcal{F}, \mathrm{P})$ and suppose that it is complete. We also deal with separable realizations of all processes.

Specify $T \in (0, \infty)$.

Definition 5. *The flow $\xi(s)$ is complete on $[0,T]$ if $\xi_{t,x}(s)$ exists for any couple (t,x) and for all $s \in [t,T]$.*

Definition 6. *The flow $\xi(s)$ is complete if it is complete on any interval $[0,T] \subset R$.*

Consider the space of bounded measurable functions on M with the norm $\|f\| = \sup_{x\in M} |f(x)|$. If the flow $\xi(t)$ is complete, it possible to construct on this space the evolution family $S(t,s)$ (the semigroup, if $\mathcal{A}$ is autonomous) defined by the formula

$$[S(t,s)f](x) = Ef(\xi_{t,x}(s)), \tag{4}$$

where E is the mathematical expectation.

Definition 7. *An evolution family is called Feller one if for any $t \geq 0$, $s \geq t$ operators $S(t,s)$ are contracting and send any continuous bounded positive function into a continuous bounded positive one.*

It is a well-known fact that (4) is a Feller evolution family.

Consider the following Cauchy problem for the parabolic PDE on M

$$\frac{\partial u}{\partial s} = \mathcal{A}u,$$

$$u(0,x) = u_0(x). \tag{5}$$

If Feller evolution family (4) exists, the function

$$u(s,x) = [S(0,s)u_0](x) = Eu_0(\xi_{0,x}(s)) \tag{6}$$

is a generalized solution of (5) (see, e.g., [1] and [8]). If it is smooth enough, it is a classical solution of (5).

On the other hand, starting from Cauchy problem (5) with $\mathcal{A}$ from a broad class of operators, one can construct an SDS whose stochastic flow determines generalized solutions (6) of (5). We refer the reader, e.g., to [8] for details.

Thus, if we find conditions for existence of Feller evolution family, this will give us conditions for global existence of solutions of a stochastic differential equation, describing the trajectories $\xi_{t,x}(s)$, and of generalized solutions of Cauchy problem (5).

Definition 8. *If the flow $\xi(s)$ is complete so that formula (4) is well-posed, we say that operator $\mathcal{A}$ generates Feller evolutions family $S(t,s)$.*

Below we shall find necessary and sufficient conditions for $\mathcal{A}$ to generate a Feller evolution family of some special sort, called complete Feller evolution family.

Definition 9. *Evolution family $S(t,s)_{s\geq t\geq 0}$ is called complete Feller one, if:*

(i) the operators $S(t,s)$ form Feller evolution family on the space of bounded continuous functions on the manifold M;

And for any $0 < T < \infty$

(ii) there exists a smooth proper positive function $v^T : M \to R_+$ such that $S(t,T)v^T$ is well-posed, i.e., $[S(t,T)v^T](x) < \infty$ for all $x \in M$, $t \in [0,T]$;

(iii) for any $K > 0$ there exists a compact set $C_{K,T} \subset M$, depending on K and T, such that from the inequality $[S(t,T)v^T](x) < K$ it follows that $x \in C_{K,T}$;

(iv) the map $(t,x) \longmapsto [S(t,T)v^T](x)$ is C^1-smooth in t and C^2-smooth in x.

Consider the direct products $\tilde{M} = [0,\infty) \times M$ and $M^T = [0,T] \times M$. Let $\pi^T : M^T \to M$ be the natural projection, $\pi^T(t,x) = x$. On the manifold $\tilde{M}$ consider diffusion processes $\eta_{(t,x)}(s) = (s, \xi_{t,x}(s))$ satisfying the conditions $\eta_{(t,x)}(t) = (t,x)$. These processes have the same infinitesimal operator that on the space of smooth functions on M^T coincides with $\mathcal{A}^T$ defined by the formula

$$\mathcal{A}^T = \frac{\partial}{\partial t} + \mathcal{A} \ . \tag{7}$$

It is obvious that if $\xi_{t,x}(s)$ exists for all initial data t, x and for all $s \in [t,\infty)$, $\eta_{(t,x)}(s)$ also exists for $s \in [t,\infty)$ and for all initial points $(t,x) \in M^T$. Then the Feller evolution $\tilde{S}(t,s)_{s\geq t\geq 0}$ family

$$[\tilde{S}(t,s)g](t,x) = Eg(\eta_{(t,x)}(s)), s \geq t$$

on the space of continuous bounded functions on $\tilde{M}$ is well-posed. Notice that $\tilde{S}(t,s)$ for $t \leq s \leq T$ is well-posed for functions $g : M^T \to R$.

Theorem 10. *Operator $\mathcal{A}$ generates complete Feller evolution family $S(t,s)_{s\geq t\geq 0}$ on M if and only if for any $0 \leq T < \infty$ there exists a smooth proper positive function $u^T : M^T \to R_+$ such that at any $(t,x) \in M^T$ the following conditions are satisfied:*

1) *$\mathcal{A}^T u^T \leq C^T$ where C^T is a certain positive constant, depending on T;*

2) $$[\tilde{S}(t,T)u^T](t,x) = Eu^T(\eta_{(t,x)}(T)) < \infty \text{ and}$$

$$|[\tilde{S}(t,T)u^T](t,x) - u^T(t,x)| < C_1$$

where $C_1 > 0$ is a certain constant depending on T;

3) *the function $[\tilde{S}(t,T)u^T](t,x)$ is C^1-smooth in t and C^2-smooth in x.*

Proof.

Sufficiency.

Let there exist a smooth proper positive function $u^T(t,x)$ on M^T such that $\mathcal{A}^T_{(t,x)}u^T \le C$ for all points of M^T. Then from the theorem from $IX.6A$ [1] it follows that for any $0 \le T < \infty$ the process $\eta_{(t,x)}(s) = (s, \xi_{t,x}(s))$ exists for all initial points $(t,x) \in M^T$ and all $s \in [t,T]$. Since it is valid for any $0 \le T < \infty$, this means that the flow $\xi(s)$ is complete. Then there exists the Feller evolution family $S(t,s)_{s\ge t\ge 0}$, acting on the space of continuous bounded functions on M by the formula $[S(t,s)f](x) = Ef(\xi_{t,x}(s))$.

Consider the function

$$v^T(x) = u^T(T,x). \tag{8}$$

By the construction it is obviously smooth and positive. Show that it is proper. Consider an arbitrary compact $D \subset R_+$. One can easily see that $(v^T)^{-1}(D) \subset \pi^T((u^T)^{-1})(D)$. Then from properness of u^T and from continuity of the map π^T it follows that the set $\pi^T((u^T)^{-1})(D)$ is compact.

Lemma 11. $[S(t,T)v^T](x) = [\tilde{S}(t,T)u^T](t,x)$ *for any* $t \in [0,T]$ *and* $x \in M$.

Proof of Lemma 11. Consider $[S(t,T)v^T](x)$. Taking into account the construction of $S(t,T)$ and the equality $v^T = u^T(T,x)$ we get:

$$[S(t,T)v^T](x) = Ev^T(\xi_{t,x}(T)) = Eu^T(T,\xi_{t,x}(T)).$$

On the other hand, from the construction of diffusion process $\eta_{(t,x)}(s) = (s,\xi_{t,x}(s))$ it follows that

$$Eu^T(T,\xi_{t,x}(T)) = Eu^T(\eta_{(t,x)}(T))$$

and by the definition of $\tilde{S}(t,T)$ we get

$$Eu^T(\eta_{(t,x)}(T)) = [\tilde{S}(t,T)u^T](t,x). \qquad \square$$

From Lemma 11 and from condition 3) of the Theorem we immediately obtain that the map $(t,x) \longmapsto [S(t,T)v^T](x)$ is smooth. Hence condition (iv) of Definition 9 is fulfilled.

Show that $[S(t,T)v^T](x)$ is bounded. From condition 2) of the theorem and from Lemma 11 we get $|[S(t,T)v^T](x) - u^T(t,x)| < C_1$. This means that

$$-C_1 + u^T(t,x) < [S(t,T)v^T](x) < C_1 + u^T(t,x).$$

Hence, $[S(t,T)v^T](x) < \infty$.

Suppose that $[S(t,T)v^T](x) < K$. Then from Lemma 11 we get $Eu^T(\eta_{(t,x)}(T)) < K$. Taking into account condition 2), we see that:

$$|Eu^T(\eta_{(t,x)}(T)) - u^T(t,x)| < C_1,$$

i.e.,

$$-C_1 + Eu^T(\eta_{(t,x)}(T)) < u^T(t,x) < C_1 + Eu^T(\eta_{(t,x)}(T)).$$

Recall that the function u^T is positive, hence,

$$0 < u^T(t,x) < C_1 + K.$$

Thus, the values $u^T(t,x)$ belong to the compact $[0, C_1+K] \subset R_+$ and if $[S(t,T)v^T](x) < K$, $x \in \pi^T((u^T)^{-1}([0, C_1+K]))$ while the last set is compact since u^T is proper and the map π^T is continuous.

So, conditions $(i)-(iv)$ of Definition 9 are satisfied, i.e., $S(t,s)_{s\ge t\ge 0}$ is a complete Feller evolution family.

Necessity.

Let $S(t,s)_{s\ge t\ge 0}$ be a complete Feller evolution family. For any $0 \le T < \infty$ denote by $v^T : M \to R_+$ the smooth proper positive function from Definition 9. Construct the function $u^T : M^T \to R$ by the formula

$$u^T(t,x) = [S(t,T)v^T](x) = Ev^T(\xi_{t,x}(T).$$

This function is C^1-smooth in t and C^2-smooth in x by condition (iv) of Definition 9. It is also obvious that the function $u^T(t,x)$ is positive.

Show that $\mathcal{A}^T u^T = 0$. To prove this, we modify some technical approaches of Chapter VIII [2].

Consider the sets $\tilde{W}_n = (v^T)^{-1}([0,n])$, $n \in N$. Since the function v^T is proper, the sets $\tilde{W}_n$ are compact. Moreover, it is easy to see that the family of compacts $\tilde{W}_n$ form a cover of the manifold M such that $\tilde{W}_n \subset \tilde{W}_{n+1}$ for any n. For $x \in \tilde{W}_n$ denote by $\tilde{\tau}_n$ the first exit time of $\xi_{t,x}(s)$ from $\tilde{W}_n$.

Consider

$$Eu^T((t+\triangle t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\triangle t)\wedge\tilde{\tau}_n)) = Eu^T(\eta_{(t,x)}((t+\triangle t)\wedge\tilde{\tau}_n)).$$

From the construction of $\eta_{(t,x)}(s)$ it follows that if $\tilde{\tau}_n$ is the first exit time of $\xi_{t,x}(s)$ from the compact $\tilde{W}_n$, $\tilde{\tau}_n$ is also the first exit time of $\eta_{(t,x)}(s)$ from the compact $[0,T]\times\tilde{W}_n$ on the manifold M^T.

Since the processes are considered up to the first exit times from compacts, we may use the Itô formula and the fact that in this case the expectation of Itô integral on the interval $[t, (t+\triangle t)\wedge\tilde{\tau}_n]$ equals zero. Thus we obtain:

$$Eu^T((t+\triangle t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\triangle t)\wedge\tilde{\tau}_n)) =$$

$$u^T(t,x) + E\int_t^{(t+\triangle t)\wedge\tilde{\tau}_n} \mathcal{A}^T u^T(\eta_{(t,x)}(s))ds. \tag{9}$$

Notice that $Eu^T((t+\triangle t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\triangle t)\wedge\tilde{\tau}_n)) = u^T(t,x)$. Indeed, by the construction of function u^T

$$Eu^T((t+\triangle t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\triangle t)\wedge\tilde{\tau}_n)) =$$

$$= E(Ev^T(\xi_{(t+\triangle t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\triangle t)\wedge\tilde{\tau}_n)}(T))) =$$

$$= E(Ev^T(\xi_{t,x}(T))) = Ev^T(\xi_{t,x}(T)) = u^T(t,x).$$

Then from (9) we get:

$$0 = Eu^T((t+\triangle t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\triangle t)\wedge\tilde{\tau}_n)) - u^T(t,x) =$$

$$= E\int_t^{(t+\triangle t)\wedge\tilde{\tau}_n} \mathcal{A}^T u^T(\eta_{(t,x)}(s))ds. \tag{10}$$

Multiply both sides of (10) by $\frac{1}{\triangle t}$ and find the limit as $\triangle t \to 0$. We obtain:

$$0 = \lim_{\triangle t\to 0}\frac{1}{\triangle t}E\int_t^{(t+\triangle t)\wedge\tilde{\tau}_n} \mathcal{A}^T u^T(\eta_{(t,x)}(s))ds.$$

Taking into account (7) one can easily transform the last equality to the form:

$$\lim_{\triangle t\to 0}\frac{1}{\triangle t}E\int_t^{(t+\triangle t)\wedge\tilde{\tau}_n}(\frac{\partial u^T(s,\xi_{t,x}(s))}{\partial s} + \mathcal{A}u^T(s,\xi_{t,x}(s)))ds = 0.$$

The function u and its derivatives are considered here on the compact set $[0,T]\times\tilde{W}_n$ and so they are bounded. Hence we can apply Lebesgue's theorem to get to the limit under the mathematical expectation and also to obtain that there exists a value $s' \in [t,(t+\triangle t)\wedge\tilde{\tau}_n]$ such that:

$$\int_t^{(t+\triangle t)\wedge\tilde{\tau}_n}(\frac{\partial u^T(s,\xi_{t,x}(s))}{\partial s} + \mathcal{A}u^T(s,\xi_{t,x}(s)))ds =$$

$$= \frac{\partial u^T(s',\xi_{t,x}(s'))}{\partial s} + \mathcal{A}u^T(s',\xi_{t,x}(s'))((t+\triangle t)\wedge\tilde{\tau}_n - t).$$

One can easily see that

$$(t+\triangle t)\wedge\tilde{\tau}_n - t = ((t+\triangle t)-t)\wedge(\tilde{\tau}_n - t) = \triangle t\wedge(\tilde{\tau}_n - t).$$

As a result we obtain

$$\lim_{\triangle t\to 0}\frac{1}{\triangle t}E(\frac{\partial u^T(s',\xi_{t,x}(s'))}{\partial s} + \mathcal{A}u^T(s',\xi_{t,x}(s'))(\triangle t\wedge(\tilde{\tau}_n - t)) =$$

$$= E\lim_{\triangle t\to 0}(\frac{\partial u^T(s',\xi_{t,x}(s'))}{\partial s} + \mathcal{A}u^T(s',\xi_{t,x}(s'))\frac{\triangle t\wedge(\tilde{\tau}_n - t)}{\triangle t} = 0. \tag{11}$$

Notice that $\tilde{\tau}_n - t > 0$ a.s. by the definition of first exit time. Hence $\tilde{\tau}_n - t$ is bounded and does not depend on $\triangle t$. So,

$$\lim_{\triangle t\to 0}\frac{\tilde{\tau}_n - t}{\triangle t} = \infty,$$

From the last equality it obviously follows that

$$\lim_{\triangle t\to 0}\frac{1}{\triangle t}(\triangle t\wedge(\tilde{\tau}_n - t)) = 1\wedge\lim_{\triangle t\to\infty}\frac{\tilde{\tau}_n - t}{\triangle t} = 1.$$

Since $s' \in [t,(t+\triangle t)\wedge\tilde{\tau}_n]$ and since we can apply the Lebesgue's theorem, $s' \to t$ $\triangle t \to 0$.

Thus equality (11) takes the form:

$$\frac{\partial u^T(t,\xi_{t,x}(t))}{\partial t} + \mathcal{A}u^T(t,\xi_{t,x}(t)) = 0.$$

This means that

$$\mathcal{A}^T u^T(t,x) = 0.$$

Lemma 12. *The function u^T on M^T is proper.*

Proof of Lemma 12. Suppose that u^T is not proper. Then there exists a sequence $(t_k, x_k) \in M^T$ such that $0 < u^T(t_k, x_k) < K$ for all k, where $0 < K < \infty$ is a certain real number, and $v^T(x_k) \to \infty$ as $k \to \infty$. Since v^T is proper, this means that x_k leaves any specified compact in M. But if $0 < u^T(t_k, x_k) < K$, by the construction of function u^T we get $[S(t_k, T)v^T](x_k) < K$ and so by condition (iii) of Definition 9 x_k must belong to a certain compact $C_{K,T}$. □

Lemma 13. *For any $t \in [0,T]$, $x \in M$ the equality $Eu^T(\eta_{(t,x)}(T)) = Ev^T(\xi_{t,x}(T))$ takes place.*

Proof of Lemma 13. Recall that $\eta_{(t,x)}(s) = (s, \xi_{t,x}(s))$ and so

$$Eu^T(\eta_{(t,x)}(T)) = Eu^T(T, \xi_{t,x}(T))).$$

By the construction of u^T

$$Eu^T(T, \xi_{t,x}(T))) = E(Ev^T(\xi_{T,\xi_{t,x}(T)}(T))),$$

Taking into account the properties of mathematical expectation and the evolution property of $\xi_{t,x}(s)$ we obtain

$$E(Ev^T(\xi_{T,\xi_{t,x}(T)}(T))) = Ev^T(\xi_{t,x}(T)). \qquad \square$$

From the construction of $S(t,s)$ and $\tilde{S}(t,s)$ it follows that

$$[\tilde{S}(t,T)u^T](t,x) = Eu^T(\eta_{(t,x)}(T)) =$$

$$= Ev^T(\xi_{t,x}(T)) = [S(t,T)v^T](x) = u^T(t,x).$$

Then from (iv) of Definition 9 we derive that $[\tilde{S}(t,T)u^T](t,x)$ is C^1-smooth in t and C^2-smooth in x. Condition 3) is fulfilled.

Notice in addition that $|[\tilde{S}(t,T)u^T](t,x) - u^T(t,x)| = 0$, i.e., it is less than any positive constant. This means that Condition 2) is fulfilled.

This completes the proof of necessity and of the Theorem. □

Remark. The similarity between the assertions of Theorems 4 and 10 becomes more clear if one passes from Cauchy problem (5) to the corresponding abstract Cauchy problem, i.e., to the ordinary first order differential equation in Banach space. Then the assertion of Theorem 10 is very close to reformulation of Theorem 4 for solutions of the abstract Cauchy problem (i.e., for generalized solutions of (5)). Notice also that Theorem 10 gives a necessary and sufficient condition for special type of completeness of the flow $\xi(s)$ that in [5] and [6] was called the L^1-completeness.

Corollary 14. *Operator $\mathcal{A}$ generates complete Feller evolution family $S(t,s)_{s\geq t\geq 0}$ on M if and only if for any $0 \leq T < \infty$ there exists a smooth positive proper function $u^T : M^T \to R_+$ such that at any point $(t,x) \in M^T$ the following conditions are satisfied:*

1) *$\mathcal{A}^T u^T \leq C^T$, where $C^T \geq 0$ is a certain constant, depending on T;*

2) $$\tilde{S}(t,T)u^T](t,x) = Eu^T(\eta_{(t,x)}(T)) < \infty \textit{ and}$$

$$[\tilde{S}(t,T)u^T](t,x) = u^T(t,x);$$

3) *the function* $[\tilde{S}(t,T)u^T](t,x)$ *is* C^1*-smooth in* t *and* C^2*-smooth in* x.

Proof. Notice that in the proof of Necessity in Theorem 10 we first proved the equality $[\tilde{S}(t,T)u^T](t,x) = u^T(t,x)$, i.e., condition 2) of Corollary 14, from which we derived that Condition 2) of Theorem 10 was satisfied. Thus we need only to modify the proof of Sufficiency under the assumption that Condition 2) of Theorem 10 is replaced by that of Corollary 14.

The proof that the Feller evolution family $S(t,s)$ on the space of continuous bounded functions on M exists is absolutely the same as for conditions of Theorem 10

Construct $v^T(x) = u^T(T,x)$ and show that $[S(t,T)v^T](x)$ is bounded. From Condition 2) of the Corollary and from Lemma 11 we obtain $[\tilde{S}(t,T)u^T](t,x) - u^T(t,x) = [S(t,T)v^T](x) - u^T(t,x) = 0$. Hence

$$[S(t,T)v^T](x) = u^T(t,x). \tag{12}$$

Thus, $[S(t,T)v^T](x) < \infty$.

From equality (12) it also follows that the map $(t,x) \longmapsto [S(t,T)v^T](x)$ is smooth.

Suppose that $[S(t,T)v^T](x) < K$. Then from (12), since u^T is positive, we get $0 < u^T(t,x) < K$. Thus, the values $u^T(t,x)$ belong to the compact set $[0,K] \subset R_+$. Hence from $[S(t,T)v^T](x) < K$ it follows that $x \in \pi^T((u^T)^{-1}([0,K]))$ while the last set is compact since u^T is proper and π^T is continuous.

So, Conditions $(i) - (iv)$ of Definition 9 are satisfied and $S(t,s)_{s\geq t\geq 0}$ is a complete Feller evolution family. □

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