On L^p-semigroups, capacities and balayage

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1. Introduction

Problems arising in L^p-potential theory are typically of nonlinear nature. In this article we present some results jointly obtained with N. Jacob [10] concerning a new approach to this problem. It is based on the theory of monotone operators due to Browder and Minty. We will show that this general technique gives a optimally suited frame for the nonlinear situation in L^p-potential theory.

Among different approaches to the construction of a stochastic process starting from a given operator the L^2-approach has turned out to be one of the most successful. In particular the theory of Dirichlet forms leads to comprehensive results in very general situations (see [6], [16]). However, a certain weakness of this apprach lies in the fact that one has to take into account exceptional sets and a process constructed by this method is determined only for starting points outside an exceptional set. The exceptional sets themselves are given by the sets of capacity zero, so the potential theory of the operator under consideration comes into play.

A possible remedy to this difficulty is to refine the potential theory and to replace the L^2-setting by an L^p-theory having in mind that an L^p-approach should give better regularity results. This led to the notion of (r,p)-capacities, see V.G Maz'ya, V.P. Havin [18] and D.R. Adams, L.I. Hedberg [1] as a standard reference. In the context of Dirichlet forms the concept of (r,p)-capacities was first introduced by P. Malliavin [17] and subsequently studied by M. Fukushima and H. Kaneko [4, 5, 14] and T. Kazumi, I. Shigekawa [15].

It turns out that by choosing the parameter p (or r) sufficiently large in many cases the exceptional sets disappear, i.e. every nonempty set has strictly positive (r,p)-capacity. Consequently, in this case it is possible to construct by Dirichlet form an associated process starting at every point.

2. (r,p)-capacities

The classical capacity corresponding to the Laplace operator is the Newtonian capacity or as a slightly modified version the 1-capacity, which for an open set $G \subset \mathbb{R}^n$ is defined by the minimization problem

$$\operatorname{cap}(G) := \inf\{\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \int_{\mathbb{R}^n} |u|^2\,dx, \quad u \in H^{1,2}(\mathbb{R}^n),\ u \geq 1 \text{ on } G \text{ a.e.}\}.$$

It is well-known that this problem has a unique minimizer u_G which is called the equilbrium potential of G.

In order to define analogous capacities in an L^p-setting one replaces for $1 \leq p < \infty$ and $r \geq 0$ the Sobolev space $H^{1,2}(\mathbb{R}^n)$ by the Bessel potential spaces

$$\begin{aligned} H^{r,p}(\mathbb{R}^n) &= (\mathrm{Id} - \Delta)^{-r/2}(L^p(\mathbb{R}^n)) \\ &= \{f \in L^p(\mathbb{R}^n) : f = G_r * g,\ g \in L^p(\mathbb{R}^n)\} \end{aligned}$$

and the corresponding norm

$$\|f\|_{r,p} = \|g\|_{L^p}.$$

Here G_r is the Bessel potential kernel given by its Fourier transform $\widehat{G_r}(\xi) = (1 + |\xi|^2)^{-r/2}$. The (r,p)-capacity then is defined as above by

$$\mathrm{cap}_{r,p}(G) := \inf\{\|u\|_{r,p}^p \quad u \in H^{r,p}(\mathbb{R}^n),\ u \geq 1 \text{ on } G \text{ a.e.}\},$$

which of course reduces to the initial case for $p = 2$ and $r = 1$.

The idea can be carried over also to the investigation of Lévy processes with characteristic exponent Ψ, i.e. Ψ is a continuous negative definite function or to Lévy-type processes generated by pseudo differential operators

$$-p(x,D)u(x) = \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\hat{u}(\xi)\, d\xi,$$

where the symbol $p(x,\xi)$ is assumed to satisfy certain estimates in terms of the fixed continuous negative definite reference function Ψ (see W. Hoh and N. Jacob [11, 7, 12, 8, 9]). In this case the correct function spaces are modified so-called Ψ-Bessel potential spaces

$$H_p^{\Psi,r}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \|u\|_{H_p^{\Psi,r}}, \infty\}$$

with norm $\|u\|_{H_p^{\Psi,r}} = \|F^{-1}((1+\Psi)^{r/2} \cdot Fu)\|_{L^p}$ (F denotes the Fourier transform). These spaces where studied in great detail by W. Farkas, N. Jacob, and R.L. Schilling [2, 3].

Our starting point will be as in the considerations of (r,p)-capacities for Dirichlet forms an L^p-semigroup. Let X be separable metric space equipped with a Radon measure μ and let for some $1 < p < \infty$

$$T_t^{(p)} : L^p(X) \to L^p(X), \quad t \geq 0,$$

be a strongly continuous, positivity preserving contraction semigroup on $L^p(X)$ with L^p-generator $A^{(p)}$. In particular we do not assume that $T_t^{(p)}$ is sub-Markovian. Even more important, since we are interested also in non-symmetric situations, we do neither assume that any symmetry is involved nor that the adjoint semigroup $T_t^{(p)*}$ on $L^{p'}(X)$, $\frac{1}{p} + \frac{1}{p'} = 1$, is sub-Markovian.

In order to define the appropriate function spaces we need the fractional power $(\mathrm{Id} - A^{(p)})^{-r/2}$ which can be defined directly in terms of the semigroup by the Gamma-transform

$$V_r^{(p)} u = \frac{1}{\Gamma(\frac{r}{2})} \int_0^\infty t^{\frac{r}{2}-1} e^{-t} T_t^{(p)} u \, dt.$$

Then $V_r^{(p)} : L^p(X) \to L^p(X)$ is injective, we denote its inverse by $T_{r,p}$. Define the function space

$$\mathcal{F}_{r,p} = V_r^{(p)}(L^p(X))$$

and the norm

$$\|u\|_{\mathcal{F}_{r,p}} = \|T_{r,p} u\|_{L^p}.$$

For an explicit investigation of the corresponding integral kernels in concrete situations we refer to N. Jacob and R.L. Schilling [13]

3. Monotone operators

Let Y be a reflexive separable Banach space with dual space Y^*.

Definition. Let $K \subset Y$ a be closed convex set and let $T : K \to Y^*$ be a (nonlinear) operator.

A. We call T monotone if $\langle Tu - Tv, u - v \rangle \geq 0$ for all $u, v \in K$.

B. The operator is called strictly monotone if $\langle Tu - Tv, u - v \rangle > 0$ for all $u, v \in K$ and $u \neq v$.

C. T is called uniformly monotone if there is a strictly increasing continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}$, $\gamma(0) = 0$ and $\lim\limits_{t\to\infty} \gamma(t) = \infty$, such that for all $u, v \in K$

$$\langle Tu - Tv, u - v \rangle \geq \gamma(\|u - v\|_Y) \cdot \|u - v\|_Y$$

holds.

D. T is coercive with respect to K if there is an element $\varphi \in K$ such that $\lim_{\substack{\|u\|_Y \to \infty \\ u \in K}} \frac{\langle Tu - T\varphi, u - \varphi \rangle}{\|u - \varphi\|_Y} = \infty$.

Moreover we need

Definition. Let $T : Y \to Y^*$ be an operator.

A. T is called hemicontinuous if for all $u, v \in Y$ and $h \in Y$ the function

$$s \mapsto \langle T(u + sv), h \rangle$$

is continuous on $[0, 1]$.

B. T is called demicontinuous if

$$u_n \to u \text{ in } Y \Rightarrow Tu_n \rightharpoonup Tu \text{ in } Y^*.$$

Now the main theorem on monotone operators states (see E. Zeidler [19]):

Theorem (Browder–Minty). Let $T : Y \to Y^*$ be a monotone, coercive, and hemicontinuous operator.

A. For every $f \in Y^*$ the set of solutions of

$$Tv = f$$

is non-empty, closed and convex.

B. If in addition T is strictly monotone, then the solution is unique and the inverse operator $T^{-1} : Y^* \to Y$ is strictly monotone, demicontinuous and bounded.

C. If T is even uniformly monotone, then T^{-1} is continuous.

4. Application to (r, p)-capacities

In order to define (r, p)-capacities in our general setting we have to consider a minimization problem for the functional

$$E_{r,p}(u) := \frac{1}{p}\|u\|^p_{\mathcal{F}_{r,p}} = \frac{1}{p}\int_X |T_{r,p}u|^p \mu(dx).$$

First note that the functional $E_{r,p} : \mathcal{F}_{r,p} \to \mathbb{R}$ is strictly convex and coercive, i.e. $\frac{E_{r,p}(u)}{\|u\|_{\mathcal{F}_{r,p}}} \to \infty$ as $\|u\|_{\mathcal{F}_{r,p}} \to \infty$. Moreover, $E_{r,p}$ is Gâteaux differentiable and we can explicitly calculate the Gâteaux derivative

$$\mathcal{A}_r^{(p)} : \mathcal{F}_{r,p} \to \mathcal{F}^*_{r,p}$$

at $u \in \mathcal{F}_{r,p}$:

$$\mathcal{A}_r^{(p)} u := T^*_{r,p}(|T_{r,p}u|^{p-2} \cdot T_{r,p}u).$$

Note that $\mathcal{A}_r^{(p)}$ is a nonlinear operator unless $p = 2$. We can prove the following inequality:

$$\langle \mathcal{A}_r^{(p)} u - \mathcal{A}_r^{(p)} v, u - v\rangle \geq 2^{2-p}\|u - v\|^p_{\mathcal{F}_{r,p}}$$

In particular this implies that on every closed convex subset of $\mathcal{F}_{r,p}$ the operator $\mathcal{A}_r^{(p)}$ is uniformly monotone and coercive.

Since $E_{r,p}$ is strictly convex, coercive and by definition continuous it is clear by the general theory of coercive functionals (see E. Zeidler [19] Theo. 25 D) that for every open subset $G \subset X$ $E_{r,p}$ attains a unique minimum on the closed convex subset $\{u \in \mathcal{F}_{r,p} : u \geq 1 \text{ on } G \text{ a.e.}\}$. Therefore, the unique minimizer e_G again defines an (r, p)-equilibrium potential and the (r, p)-capacity is given by

$$\mathrm{cap}_{r,p}(G) = E_{r,p}(e_G).$$

Analogously, for $h \in \mathcal{F}_{r,p}$ one can consider the closed convex set $\{u \in \mathcal{F}_{r,p} : u \geq h \text{ on } G \text{ a.e.}\}$ and obtains as the unique minimizer the balayaged function h_G.

In accordance with Dirichlet forms we introduce the notion of a mutual energy on $\mathcal{F}_{r,p} \times \mathcal{F}_{r,p}$:

$$\mathcal{E}_r^{(p)}(u, v) := \langle \mathcal{A}_r^{(p)} u, v \rangle,$$

which is again nonlinear with respect to the first argument. We now can find a better description of the minimizer in terms of a variational inequality which, as one would expect, involves the derivative $\mathcal{A}_r^{(p)}$ of the functional $E_{r,p}$:

Proposition. Let $K \subset \mathcal{F}_{r,p}$ be closed and convex. The unique minimizer of $E_{r,p}$ on K satisfies

$$\mathcal{E}_r^{(p)}(u, \varphi - u) = \langle \mathcal{A}_r^{(p)} u, \varphi - u \rangle \geq 0 \text{ for all } \varphi \in K.$$

This in particular implies that for an equilibrium potential e_G the variational inequality

$$\mathcal{E}_r^{(p)}(u_G, \psi) \geq 0 \text{ for all } \psi \in \mathcal{F}_{r,p},\ \psi|_G \geq 0$$

holds. In analogy to Dirichlet forms we call a function $u \in \mathcal{F}_{r,p}$ a potential if

$$\mathcal{E}_r^{(p)}(u, \psi) = \langle \mathcal{A}_r^{(p)} u, \psi \rangle \geq 0 \text{ for all } \psi \in \mathcal{F}_{r,p},\ \psi \geq 0.$$

Especially, equilibrium potentials are potentials in this sense.

In other words a potential is a function $u \in \mathcal{F}_{r,p}$ having the property that $\mathcal{A}_r^{(p)} u$ is a positive element in $\mathcal{F}_{r,p}^*$ (in a distributional sense). But, since $\mathcal{A}_r^{(p)}$ is an uniformly monotone operator that satisfies all assumptions of the Browder-Minty theorem, we know that it is invertible. We denote its inverse by

$$U_r^{(p)} = (\mathcal{A}_r^{(p)})^{-1} : \mathcal{F}_{r,p}^* \to \mathcal{F}_{r,p}$$

and thus have shown:

$u \in \mathcal{F}_{r,p}$ is a potential if and only if $u = U_r^{(p)} f$ for some positive $f \in \mathcal{F}_{r,p}^*$.

Again an explicit calculation is possible:

$$U_r^{(p)} f = V_r^{(p)}(|V_r^{(p)*} f|^{p'-2} \cdot V_r^{(p)*} f).$$

This operator $U_r^{(p)}$ is a well-known object called the nonlinear potential operator and has been investigated before for instance by V.G. Maz'ya, V.P. Havin [18] and D.R Adams, L.I. Hedberg [1]. Note that under reasonable assumptions the positive elements in $\mathcal{F}_{r,p}^*$ can be identified with measures on X (of finite energy), see T. Kazumi, I. Shigekawa [15]. In this sense we obtain a representation of the potentials which is completely analogous to the Riesz representation in classical potential theory.

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