Invariance implies Gibbsian: some new results

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Abstract

We investigate stationary distributions of stochastic gradient systems in Riemannian manifolds and prove that, under certain assumptions, such distributions are symmetric. These results are extended to countable products of finite dimensional manifolds and applied to Gibbs distributions in the case where the single spin spaces are Riemannian manifolds. In particular, we obtain a new result concerning the question whether all invariant measures are Gibbsian. Actually, we consider a more general object: weak elliptic equations for measures, which, on the one hand, yields the results obtained stronger than the above mentioned statements, and, on the other hand, enables us to give simpler proofs of more general than previously known facts. Applications to concrete models of lattice systems over $\mathbb{Z}^d$ with not necessarily compact spin space are presented (also in the case $d \geq 3$).

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1 Introduction

Let M be a Riemannian manifold, let Z be a vector field on M, and let μ be a stationary probability distribution of the diffusion ξ_t in M with generator

$$Lf = L_Z f := \Delta f + \langle Z, \nabla f \rangle.$$

Suppose that $Z = \nabla U$ for some function U on M. We are looking for conditions on U such that μ has the form $C \exp[U] \mathrm{d}\lambda_M$, where λ_M is the Riemannian volume element and $C > 0$ is a normalization constant. An infinite dimensional analogue of this problem, also addressed to in this work, is usually stated as the question "when every stationary distribution is Gibbsian". Considerable progress in this direction was achieved in [24] (the case of the torus) and [18], [19], [20], [21] (the case of linear spin spaces). As we shall see, our problem can be effectively investigated by the method of elliptic equations for measures, developed recently in a series of articles [4], [6], [8], [9], [10], [11], [12], and [14]. The idea is to study the elliptic equation

$$L^*\mu = 0 \tag{1.1}$$

understood in the following weak form: $|Z| \in L^1_{loc}(\mu)$ and

$$\int L\varphi \,\mathrm{d}\mu = 0, \quad \forall\, \varphi \in C_0^\infty(M), \tag{1.2}$$

where $C_0^\infty(M)$ is the class of all infinitely differentiable functions with compact support in M (if M is compact we set $C_0^\infty(M) := C^\infty(M)$). Under very general assumptions, any stationary distribution μ of the diffusion generated by L_Z, satisfies this equation. Conversely, under reasonable assumptions, any probability measure μ solving the above elliptic equation is a stationary distribution of the associated diffusion. Various results about properties of solutions of (1.1) can be found in the above cited works. We first briefly summarize some of those results important for this article (cf. Theorem 2.1 below). Our results in finite dimensions (Theorems 2.3 and 2.4 below) state that if $Z = \nabla U$, then, under certain assumptions, μ is proportional to $\exp[U]\lambda_M$. Clearly, if $Z = \nabla U$, then we have the locally finite measure $\mathrm{d}\mu = \exp[U]\,\mathrm{d}\lambda_M$ satisfying the equation $L_Z^*\mu = 0$. In the general case, this measure may be infinite, e.g., if $Z = 0$ and if M has infinite volume. Under some additional conditions, our results provide a positive answer to the following question: is $\exp[U]$ integrable if there exists a probability measure μ satisfying the equation $L_Z^*\mu = 0$? In general, this is not true even if $Z = 0$ (see Remark 2.5). However, this question is still open for $M = \mathbb{R}^d$ and other manifolds with bounded geometry.

The main goal of this paper is to obtain analogous results in the infinite dimensional case, i.e. M is replaced by $\prod_{i\in\mathbb{Z}^d} M^i$ with manifolds M^i as above. These results are contained in Theorems 3.5, 3.6, and 3.7 below. In particular, we study elliptic equations for stationary distributions of infinite stochastic gradient systems. An advantage of the above approach is that the infinite dimensional version of elliptic equation (1.1) makes sense even in the case when there is no associated diffusion. It has been proved in [11], [14] that the existence of solutions to elliptic equations for measures can be proved under broader assumptions than the existence of associated diffusions. This is especially relevant in the case of Gibbs measures, when the study of an associated process is essentially a tool for constructing Gibbs measures. In this situation, the infinitesimal invariance is more intrinsic than a diffusion semigroup. Our applications concern finite range interactions, although more general models can be investigated along the same lines. Theorem 3.5 is a generalization of a well-known result of Holley and Stroock [24]. This generalization is in two directions: more general (not necessarily compact) single spin spaces M^i are considered and the assumptions on the vector fields are considerably weaker (even in the case of a torus). In the case $M^i = \mathbb{R}^d$ for all $i \in \mathbb{Z}^d$, Theorem 3.5 also provides a generalization of a well-known result, due to Fritz [18]. Our proof employs some ideas from [24], [18], [19]. It should be also noted that Theorem 3.5 is an essential improvement of Theorem 7.8 in our work [14]. The principal novelty (in addition to weaker assumptions) is that now in order to conclude that any infinitesimally invariant measure is Gibbsian we do not require the existence of some Gibbs measure as we did in [14], which is of particular importance in the case of non compact spin spaces. In Theorem 3.6, we consider the case when there exists a reference measure satisfying the logarithmic Sobolev inequality. When the reference measure is Gibbsian this case has been intensively studied by many authors (see, e.g., [5], [15], [17], [26], [28], [29], [31], [32], [33], [34], [35], [36], [37], where many additional references can be found). It is known that in this case, under rather broad assumptions, such a measure is a unique Gibbs state and in fact a unique stationary distribution. For finite range interactions we yet broaden the corresponding assumptions. What we actually prove is essentially the following result: if a measure μ satisfies our elliptic equation whose drift Z (which now is not necessarily of gradient type) is sufficiently close (in the sense of (3.14) to the logarithmic gradient of another measure γ which satisfies the log-Sobolev inequality, then μ has a density with respect to γ and this density belongs to some Sobolev class. In addition, we obtain an estimate of the density which yields the equality $\mu = \gamma$ if Z coincides with the logarithmic gradient of μ. In the case when γ is a Gibbs measure, this result means that any infinitesimally invariant measure (i.e., any solution of the corresponding elliptic equation) coincides with γ provided it satisies (3.14). In Example 4.2 the latter holds for all infinitesimally invariant measures, so we have that all of them coincide if there is one of them satisfying a log-Sobolev inequality. Finally, Theorem 3.7 extends a result of Ramirez [30, Theorem 4]: the torus is replaced by a more general manifold and conditions on the vector fields are much weaker. We emphasize that we make no assumptions about compactness of the manifolds we deal with and consider also non translation invariant interactions of finite range. The latter assumption can be weakened as the reader will see, but we deliberately do not use the broadest possible assumptions under which our techniques work in order to single out the main ideas. It is worth noting that the proofs of our main results are extremely short (up to justification of certain integrations by part based on our previous work). The concrete applications to lattice systems over $\mathbb{Z}^d$ mentioned in the abstract are presented in Section 4.

2 Finite dimensional case

Let M be a complete Riemannian manifold of dimension d. Let λ_M denote the Riemannian volume on M and let $B(x,r)$ be the closed geodesic ball with center x and radius r. We denote by $H^{\alpha,1}_{loc}(\lambda_M)$, where $\alpha \geq 1$, the Sobolev class of all functions on M that, together with the generalized gradients, are locally in $L^\alpha(M, \lambda_M)$. Let $H^{2,1}(\lambda_M)$ be the closure of $C_0^\infty(M)$ with respect to the Sobolev norm $\|\cdot\|_{H^{2,1}}$ given by

$$\|\psi\|^2_{H^{2,1}} = \int_M |\psi|^2 \, \mathrm{d}\lambda_M + \int_M |\nabla\psi|^2 \, \mathrm{d}\lambda_M.$$

In some of the results below we make use of the following condition **(C)** on the manifold M:

(C): the Ricci curvature of M is bounded below and $\inf_{x\in M} \lambda_M\big(B(x,r)\big) > 0 \; \forall\, r > 0$.

Let $L^2(\mu, Vec(M))$ denote the Hilbert space of all μ-square integrable vector fields on M equipped with the inner product

$$(\pi_1, \pi_2)_2 := \int \langle \pi_1, \pi_2 \rangle \, \mathrm{d}\mu.$$

We shall employ the following statements implied by [7, Theorem 1] and [14, Corollary 2.3].

Theorem 2.1. *Let μ be a locally finite measure on M such that $L_Z^*\mu = 0$. We have*

(i) *μ has a density f_μ with respect to λ_M. If there is $p > 1$ such that $|Z| \in L^p_{loc}(\lambda_M)$ then $f_\mu \in H^{1,1}_{loc}(\lambda_M)$.*

(ii) *If either $|Z| \in L^\alpha_{loc}(\mu)$ or $|Z| \in L^\alpha_{loc}(\lambda_M)$ with some $\alpha > d$, then $f_\mu \in H^{\alpha,1}_{loc}(\lambda_M)$. In particular, f_μ has a Hölder continuous version. Moreover, if $\mu \geq 0$, then the continuous version of f_μ is strictly positive in every connected component where it is not identically zero.*

(iii) *If M satisfies condition* **(C)**, *μ is a probability measure and $|Z| \in L^2(\mu)$, then $\sqrt{f_\mu} \in H^{2,1}(\lambda_M)$ and one has*

$$\int_M \frac{|\nabla f_\mu|^2}{f_\mu} \, \mathrm{d}\lambda_M \leq \int_M |Z|^2 \, \mathrm{d}\mu. \tag{2.1}$$

In addition, the vector field $\nabla f_\mu / f_\mu$ coincides with the orthogonal projection of Z to the space $\Gamma(\mu)$ defined as the closure of $\{\nabla\varphi, \varphi \in C_0^\infty(M)\}$ in the Hilbert space $L^2(\mu, Vec(M))$.

We note that without condition **(C)** Theorem 2.1 (iii) does not hold, cf. Remark 2.5 (ii) below.

Let us observe that if $f \in H^{1,1}_{loc}(\lambda_M)$, then $\nabla f = 0$ a.e. on the set $\{f = 0\}$. We shall set throughout

$$\frac{\nabla f}{f} = 0 \quad \text{on the set } \{f = 0\}.$$

According to a classical theorem of Kolmogorov [25], if μ solves (1.1) with smooth Z and L is symmetric on $L^2(\mu)$, then Z coincides with the logarithmic gradient of μ, i.e., $\mu = \exp[U] \, \mathrm{d}\lambda_M$ and $Z = \nabla U$. In the next proposition we extend this to our setting of general (possibly locally unbounded) drifts and show that once L is symmetric on $L^2(\mu)$, then Z coincides μ-a.e. with the logarithmic gradient of μ. However, in this more general situation, there might be no function U with the above property. We shall see that if μ has a strictly positive continuous density, then Z is the gradient of some function U.

Lemma 2.2. (i) *Suppose that μ is a Borel probability measure on M with a density $f_\mu \in H^{1,1}_{loc}(\lambda_M)$ and let*

$$Z(x) = \frac{\nabla f_\mu(x)}{f_\mu(x)}, \tag{2.2}$$

where we set $\nabla f_\mu(x)/f_\mu(x) = 0$ if $f_\mu(x) = 0$. Then one has

$$\int \psi L_Z \varphi \, \mathrm{d}\mu = -\int \langle \nabla\varphi, \nabla\psi \rangle \, \mathrm{d}\mu, \quad \forall\, \varphi, \psi \in C_0^\infty(M), \tag{2.3}$$

and

$$\int \psi L_Z\varphi \,\mathrm{d}\mu = \int \varphi L_Z\psi \,\mathrm{d}\mu, \quad \forall\, \varphi,\psi \in C_0^\infty(M),$$

In particular, if $|\nabla f_\mu/f| \in L^2_{loc}(M,\mu)$, *then* $\big(L_Z, C_0^\infty(M)\big)$ *is symmetric on* $L^2(\mu)$.

(ii) *Conversely, let* μ *be a Borel probability measure on* M *such that* $|Z| \in L^p_{loc}(M,\mu)$ *for some* $p>1$ *and*

$$\int \psi L_Z\varphi \,\mathrm{d}\mu = \int \varphi L_Z\psi \,\mathrm{d}\mu, \quad \forall\, \varphi,\ \psi \in C_0^\infty(M).$$

Then μ *satisfies* (1.1), *has a density* $f_\mu \in H^{1,1}_{loc}(\lambda_M)$, *and equality* (2.2) *holds* μ*-a.e.*

Proof. Assertion (i) follows by the integrating by parts. (ii) Let $\varphi,\psi \in C_0^\infty(M)$ be such that $\psi = 1$ on the support of φ. Then $\varphi L_Z\psi = 0$, which yields (1.2) by the symmetry of L_Z. According to Theorem 2.1, μ has a density $f_\mu \in H^{1,1}_{loc}(\lambda_M)$. Therefore, for all $\varphi,\psi \in C_0^\infty(M)$, one has by (2.3) and the integration by parts formula

$$\begin{aligned} &-\int \langle\nabla\psi,\nabla\varphi\rangle f_\mu \mathrm{d}\lambda_M = \int (\psi L_Z\varphi)\, f_\mu \,\mathrm{d}\lambda_M \\ &= -\int \langle\nabla\varphi,\nabla\psi\rangle f_\mu \,\mathrm{d}\lambda_M - \int \langle\nabla\varphi,\nabla f_\mu\rangle\psi \,\mathrm{d}\lambda_M + \int \langle Z,\nabla\varphi\rangle\psi f_\mu \,\mathrm{d}\lambda_M. \end{aligned}$$

Thus

$$\int \langle\nabla f_\mu - f_\mu Z, \nabla\varphi\rangle\psi \,\mathrm{d}\lambda_M = 0.$$

Hence $\langle\nabla f_\mu - f_\mu Z,\nabla\varphi\rangle = 0$ a.e. since $\psi \in C_0^\infty(M)$ is arbitrary. Moreover, we can take φ such that $\varphi(x) = x_i\ (1\le i\le d)$ in a local chart to conclude that $\nabla f_\mu - f_\mu Z = 0$ a.e. Therefore $Z = \nabla f_\mu/f_\mu$ μ-a.e. if we set $\nabla f_\mu/f_\mu = 0$ on $\{f_\mu = 0\}$. □

Theorem 2.3. *Suppose that* $Z = \nabla U$, *where* $U \in H^{\alpha,1}_{loc}(M)\cap L^\infty_{loc}(\lambda_M)$ *for some* $\alpha > d$. *Assume that there exists a nonnegative function* $V \in C^2(M)$ *such that the sets* $\{V \le c\}$ *are compact for every* $c < \sup\limits_M V$ *and cover* M *and there exists a compact set* K *such that* $LV \le -1$ *outside* K. *Then* $\exp[U]$ *is* λ_M*-integrable. If* M *is connected, then the normalized measure* $\mathrm{d}\mu = C\exp[U]\,\mathrm{d}\lambda_M$ *is a unique probability measure solving* (1.1).

Proof. We can find strictly increasing numbers c_k such that the sets $B_k := \{V < c_k\}$ cover M and the sets $\{V = c_k\}$ have zero measures. We shall consider the measures $\mu_k := I_{B_k}\exp U\, d\lambda_M$ and show that these measures are uniformly bounded. Suppose not. The measures μ_k satisfy the equality

$$\int_{B_k} L\varphi\, d\mu_k = 0 \tag{2.4}$$

for all $\varphi \in C^2(M)$ with compact support in B_k, which follows by the integration by parts formula. Let $B = B_{k_0}$ be such that $LV \le -1$ outside B. Such k_0 exists, since K is covered by the open sets $\{V < c_k\}$, hence admits a finite subcover. Further we consider only $k > k_0$. Set $S = \int_B |LV|\exp[U]\,\mathrm{d}\lambda_M$. Clearly, $S < \infty$, since $|Z| \in L^\alpha(B,\lambda_M)$ and U is locally bounded. We observe that

$$\int_{B_k} |LV|\, d\mu_k \le 2S \quad \forall\, k \ge k_0. \tag{2.5}$$

Indeed, let $k > k_0$ be fixed and let $c_{k_0} < c < s < c' < c_k$. There exists a function $\theta \in C^2(\mathbb{R}^1)$ such that $\theta' \geq 0$, $\theta'' \leq 0$, $\theta(t) = t$ if $t \leq c$, $\theta(t) = s$ if $t \geq c'$. We have

$$L(\theta \circ V) = \theta'(V)LV + \theta''(V)\langle \nabla V, \nabla V \rangle.$$

Hence $L(\theta \circ V) \leq 0$ outside B, $L(\theta \circ V) = LV$ on $\{V \leq c\}$, and $L(\theta \circ V) = 0$ on $\{V > c'\}$, since $\theta \circ V = s$ on $\{V > c'\}$. Since by (2.4) one has

$$\int_{\{V \leq c'\}} L(\theta \circ V)\, d\mu_k = \int_{\{V \leq c'\}} L(\theta \circ V - s)\, d\mu_k = 0,$$

and $L(\theta \circ V) \leq 0$ on $\{c < V \leq c'\}$, we obtain $\int_{\{V \leq c\}} L(\theta \circ V)\, d\mu_k \geq 0$, i.e.,

$$\int_{\{V \leq c\}} LV\, d\mu_k \geq 0.$$

Since this is true for every $c \in (c_{k_0}, c_k)$ and the set $\{V = c_k\}$ has measure zero, we conclude that

$$\int_{B_k} LV\, d\mu_k \geq 0.$$

Taking into account that $LV \leq -1$ outside B, we arrive at (2.5), which gives $\mu_k(M\backslash B) \leq 2S$. Therefore, $\exp[U] \in L^1(M, \lambda_M)$. The fact that the corresponding normalized measure is a unique solution of our elliptic equation follows from a general result [12, Corollary 3.4] valid for not necessarily gradient class drifts (in the manifold case, this result is proved in [13]), in our case μ is a reversible measure. By the way, the existence of solution is also known for more general drifts (see [10]), so the main point here is to show that this unique solution has the above form. □

If **(C)** holds, we do not know whether the assumption in Theorem 2.3 of the existence of a Lyapunov function can be replaced by that of the existence of a probability measure μ solving (1.1). This is not true without **(C)** (see Remark 2.5 below). However, as the next result shows, there is no problem if $|Z| \in L^2(M, \mu)$ and if **(C)** holds.

Theorem 2.4. *Let M be connected and satisfy condition* **(C)** *and let μ be a probability measure such that $L_Z^*\mu = 0$, where $|Z| \in L^2(\mu)$. Assume that $Z = \nabla U$, where $U \in H^{1,1}_{loc}(\lambda_M)$. Then $\exp[U] \in L^1(M, \lambda_M)$ and $\mathrm{d}\mu = \|\exp[U]\|^{-1}_{L^1(\lambda_M)} \exp[U]\, \mathrm{d}\lambda_M$.*

Proof. According to Theorem 2.1, $\mu = f\,\lambda_M$, where $\sqrt{f} \in H^{2,1}(\lambda_M)$. In addition, $\nabla f/f$ coincides with the projection of ∇U to $\Gamma(\mu)$ in $L^2\big(\mu, Vec(M)\big)$. Since $U \in H^{1,1}_{loc}(\lambda_M)$ and $|\nabla U|^2 \in L^2(\mu)$, it follows by [14, Lemma 2.1] that $\nabla U \in \Gamma(\mu)$. Then we obtain $\nabla U = \nabla f/f$ μ-a.e., because $\nabla U - \nabla f/f \perp \nabla\varphi$ for every $\varphi \in C_0^\infty(M)$, which follows at once from the elliptic equation. Now, by analogy with [9, Lemma 6.4], we prove that for some constant C one has $\log f = U + C$ a.e. Indeed, let K be a connected ball in M with the Riemannian volume $|K|$. We may assume that K is small enough so that it is contained in a local chart. Therefore, we may assume that we deal with $\mathbb{R}^d$ equipped with some Riemannian metric. Let us consider the functions $g_n = \log(f + 1/n)$. Since $f \in H^{1,1}_{loc}(\lambda_M)$, we obtain that $g_n \in H^{1,1}_{loc}(\lambda_M)$. Moreover, $\nabla g_n = \nabla f/(f+1/n)$. Therefore,

$g_n \in H^{2,1}_{loc}(\lambda_M)$ since $\sqrt{f} \in H^{2,1}(\lambda_M)$. Let us set $c_n = |K|^{-1} \int_K g_n \, \mathrm{d}\lambda_M$. By the Poincaré inequality we have

$$\int_K |g_n - c_n|^2 \, \mathrm{d}\lambda_M \leq \kappa \int_K |\nabla g_n|^2 \, \mathrm{d}\lambda_M \leq \kappa \int_K \frac{|\nabla f|^2}{f} \, \mathrm{d}\lambda_M < \infty \tag{2.6}$$

with some κ independent of n. This yields that $\sup_n |c_n| < \infty$. Indeed, otherwise there is a subsequence in $\{c_n\}$ tending to infinity. We may assume that $c_n \to \infty$. By the Fatou theorem, we obtain that $\limsup_{n\to\infty} g_n = \infty$ a.e., which contradicts the equality $\log f = \lim_{n\to\infty} g_n$ on the set $\{f > 0\}$. Applying again (2.6) we obtain that $\log f \in H^{2,1}_{loc}(\lambda_M)$ and $f > 0$ a.e. Therefore, $\nabla \log f = \nabla U$ a.e., whence our claim follows. □

Remark 2.5. (i) Let M be a connected complete Riemannian manifold with bounded below Ricci curvature and positive injectivity radius and let $Z = 0$. Then we arrive at the well-known fact (see [23, § 13] for more general results) that every positive integrable harmonic function h on M is constant. Indeed, we consider $\mu = h \, \mathrm{d}x$ and obtain $\nabla h/h = 0$ as the orthogonal projection of zero.

(ii) However, there exist connected complete Riemannian manifolds M possessing positive integrable harmonic functions h (see [16], [27]). Then the probability measure $\mu := Ch\mathrm{d}\lambda_M$ satisfies our elliptic equation with $Z = 0$ and does not have the form indicated in Theorem 2.4 (moreover, in this case, λ_M is infinite, which follows from [23, Theorem 7.3] or from the results in [12], [13]). Therefore, Theorem 2.4 and assertion (iii) of Theorem 2.1 may fail for general connected complete Riemannian manifolds.

3 Infinite dimensional case

Let $S = \mathbb{Z}^m$ and let $M^S = \prod_{i\in S} M^i$, where

($\mathbf{C}'$): the M^i's are connected Riemannian manifolds which satisfy condition ($\mathbf{C}$).

The points in M^S are denoted by $x = (x_i)_{i\in S}$. For every non empty $\Lambda \subset S$, let $x_\Lambda = (x_i)_{i\in\Lambda}$. The complement of Λ is denoted by Λ^c. Let $\mathcal{P}(M^S)$ be the set of all Borel probability measures on M^S. Given $\mu \in \mathcal{P}(M^S)$ and $\Lambda \subset S$, let μ_Λ be the projection of μ to M^Λ.

Let $\mathcal{FC}_0^\infty(M^S)$ stand for the union of all classes $C_0^\infty(M^\Lambda)$, where $\Lambda \subset S$ is finite.

Suppose that we are given a family $Z = (Z_i)_{i\in S}$ of Borel vector fields Z_i on M^S such that $Z_i(x) \in T_{x_i}M^i$, $i \in S, x \in M^S$. We shall say that Z is of finite range $R > 0$ if, for every $i \in S$, Z_i depends only on the coordinates x_j with $j \in i + \Lambda_1$, where

$$\Lambda_k := \{s = (s_1, \ldots, s_m) \in \mathbb{Z}^m \colon \ |s_j| \leq kR\}.$$

We say that a Borel probability measure μ on M^S has a partial logarithmic gradient β_i^μ along x_i if β_i^μ is a μ-measurable vector field on M^S such that $\beta_i^\mu(x) \in T_{x_i}M^i$, $|\beta_i^\mu| \in L^1_{loc}(\mu)$ and, for every compactly supported smooth vector field v on M^i and every $\psi \in \mathcal{FC}_0^\infty(M^S)$, one has

$$\int_{M^S} \langle \nabla_i \psi, v \rangle \, \mathrm{d}\mu = -\int_{M^S} \psi(\operatorname{div} v + \langle v, \beta_i^\mu \rangle) \, \mathrm{d}\mu, \tag{3.1}$$

where $\nabla_i\psi$ is the partial gradient with respect to x_i and $\operatorname{div} v$ is the usual divergence on M^i.

Note that if μ is the product of the measures μ_i on M^i with $\mu_i = f_i\, \mathrm{d}\lambda_{M^i}$, where $f_i \in H^{1,1}(\lambda_{M^i})$, then $\beta_i^\mu = \nabla_i f_i/f_i$.

The operator L_Z is defined on smooth cylindrical functions in the natural way, i.e., if f is a smooth function of x_Λ, where $\Lambda \subset S$ is a finite set, then

$$L_Z f := \Delta_\Lambda f + \sum_{i\in\Lambda}\langle\nabla_i f, Z_i\rangle,$$

where Δ_Λ is the Laplace–Beltrami operator on $\prod_{i\in\Lambda} M^i$. Given a measure $\mu \in \mathcal{P}(M^S)$, the equation

$$L_Z^*\mu = 0$$

with respect to $\mathcal{F}C_0^\infty$ is understood as follows: $|Z_i| \in L^1(\mu)$ for all $i \in S$ and

$$\int_{M^S} L_Z f\, \mathrm{d}\mu = 0 \quad \forall\, f \in \mathcal{F}C_0^\infty.$$

Remark 3.1. Let $\mu \in \mathcal{P}(M^S)$ be such that $L_Z^*\mu = 0$. If μ has the partial logarithmic derivative β^μ such that $\beta_i^\mu = Z_i$ for all $i \in S$, then obviously as in the finite dimensional case μ is symmetrizing for L_Z, i.e.,

$$\int_{M^S} L_Z f\, g\, \mathrm{d}\mu = \int_{M^S} f\, L_Z g\, \mathrm{d}\mu \quad \text{for all } f, g \in \mathcal{F}C_0^\infty.$$

In this case one says that μ is Gibbsian. For Z_i of the particular form as in Theorem 3.5 below the equivalence of this notion of Gibbsian and the more classical one defined by the Dobrushin–Lanford–Ruelle equations (cf. [22]) for the underlying specification was proved in this generality in [1] (see also [2], in particular, with respect to Example 4.2 below) in the case $M = \mathbb{R}^d$ and in [14] for general manifolds.

For any bounded set $\Lambda \subset \mathbb{Z}^m$, let U_Λ be a function depending only on x_Λ and as a function of x_Λ be in $H^{2,1}_{loc}(\lambda_{M^\Lambda})$ and locally bounded. For example, it suffices that $U_\Lambda \in \mathrm{Lip}_{loc}(M^\Lambda)$, i.e., be locally Lipschitzian. Set

$$W_k := -\sum_{\Lambda\subset\Lambda_k} U_\Lambda.$$

It follows that W_k is a function of x_{Λ_k}. Let us assume that, for every k, the function $\exp[W_k]$ is integrable over M^{Λ_k}. We shall deal with the corresponding normalized probability measure

$$\mathrm{d}\nu_k := c_k \exp[W_k]\, \mathrm{d}\lambda_k,$$

where λ_k stands for $\lambda_{M^{\Lambda_k}}$, the Riemannian volume measure on M^{Λ_k}.

Given a Borel probability measure μ on M^S such that $|Z_i| \in L^1(\mu)$, let $\mathbb{E}^\mu_{\Lambda_k} Z_i$ be the conditional expectation of Z_i with respect to the σ-field generated by x_{Λ_k} and the measure μ, i.e., $\mathbb{E}^\mu_{\Lambda_k} Z_i(x) \in T_{x_i}M^i$ depends only on x_{Λ_k} and, for any compactly supported smooth vector field v on M^{Λ_k}, one has

$$\int \langle\mathbb{E}^\mu_{\Lambda_k} Z_i, v\rangle\, \mathrm{d}\mu = \int \langle Z_i, v\rangle\, \mathrm{d}\mu.$$

It is easily verified that if $L_Z^*\mu = 0$ with respect to $\mathcal{F}C_0^\infty$, then one has $L^*_{\mathbb{E}^\mu_{\Lambda_k} Z_{\Lambda_k}}\mu_k = 0$, where μ_k stands for μ_{Λ_k}. By Theorem 2.1(i), μ_k is absolutely continuous with respect to ν_k. Let $f_k := \frac{\mathrm{d}\mu_k}{\mathrm{d}\nu_k}$.

We shall use the following simple technical lemmas analogous to [14, Lemmas 7.1 and 7.2] (those lemmas do not apply directly, since in the present situation, ν_k may not coincide with the projection of ν_{k+1}) and inspired by similar results in [24], [18], [19] in more special cases.

Lemma 3.2. *Assume that* $(\mathbf{C}')$ *holds and that* Z *is of finite range* R. *Let* $\mu \in \mathcal{P}(M^S)$ *be such that* $|\nabla_i W_k|, |Z_i| \in L^2(\mu)$ *for all* $i \in \Lambda_k$ *and* $k \in \mathbb{N}$. *Suppose that* $L_Z^*\mu = 0$ *with respect to* $\mathcal{F}C_0^\infty$. *Then* $|\nabla f_k|/f_k \in L^2(\mu_k)$ *and*

$$\begin{aligned}\int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k}\,\mathrm{d}\nu_k &= \sum_{i\in\Lambda_{k-1}} \int_{M^{\Lambda_k}} \langle Z_i - \nabla_i W_k, \nabla_i f_k\rangle\,\mathrm{d}\nu_k \\ &\quad + \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^S} \Big\langle Z_i - \nabla_i W_k, \frac{\nabla_i f_k}{f_k}\Big\rangle\,\mathrm{d}\mu \end{aligned}\tag{3.2}$$

and

$$\begin{aligned}\sum_{i\in\Lambda_{k-1}} \int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k}\,\mathrm{d}\nu_k &= \sum_{i\in\Lambda_{k-1}} \int_{M^{\Lambda_k}} \langle Z_i - \nabla_i W_k, \nabla_i f_k\rangle\,\mathrm{d}\nu_k \\ &\quad + \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \Big\langle \frac{\nabla_i f_k}{f_k}, Z_i - \nabla_i W_{k+1} - \frac{\nabla_i f_{k+1}}{f_{k+1}}\Big\rangle\,\mathrm{d}\mu_{k+1}.\end{aligned}\tag{3.3}$$

Proof. First note that $\mathrm{d}\mu_k = g_k\,\mathrm{d}\lambda_k$, where $g_k = c_k f_k \exp[W_k]$. According to Theorem 2.1(iii) we have $g_k \in H^{1,1}_{loc}(\lambda_k)$ and $|\nabla g_k/g_k| \in L^2(\mu_k)$. Since $|\nabla W_k| \in L^2(\mu_k)$ by our hypothesis, we obtain that $f_k \in H^{1,1}_{loc}(\lambda_k)$ and $|\nabla f_k/f_k| \in L^2(\mu_k)$, in particular, $\sqrt{f_k} \in H^{2,1}_{loc}(\lambda_k)$. Let Δ_{Λ_k} denote the Laplacian on M^{Λ_k}. For any $\varphi \in C_0^\infty(M^{\Lambda_k})$ we have

$$\int_{M^{\Lambda_k}} (\Delta_{\Lambda_k}\varphi)\, f_k\,\mathrm{d}\nu_k + \sum_{i\in\Lambda_k}\int_{M^S}\langle Z_i, \nabla_i\varphi\rangle\,\mathrm{d}\mu = 0. \tag{3.4}$$

Approximating f_k by $\big(n\wedge\sqrt{f_k}\big)^2 \in H^{2,1}(\nu_k), n \in \mathbb{N}$, allows to integrate by parts, so using that Z_i, $i \in \Lambda_{k-1}$, depends only on x_{Λ_k}, we obtain

$$\begin{aligned}\int_{M^{\Lambda_k}} \langle\nabla\varphi, \nabla f_k\rangle\,\mathrm{d}\nu_k &= -\sum_{i\in\Lambda_k}\int_{M^{\Lambda_k}} \langle \nabla_i W_k, \nabla_i\varphi\rangle\, f_k\,\mathrm{d}\nu_k + \sum_{i\in\Lambda_k}\int_{M^S}\langle Z_i, \nabla_i\varphi\rangle\,\mathrm{d}\mu \\ &= \sum_{i\in\Lambda_{k-1}}\int_{M^{\Lambda_k}}\langle Z_i - \nabla_i W_k, \nabla_i\varphi\rangle\, f_k\,\mathrm{d}\nu_k + \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}}\int_{M^S}\langle Z_i - \nabla_i W_k, \nabla_i\varphi\rangle\,\mathrm{d}\mu.\end{aligned}\tag{3.5}$$

The first desired equality follows if we set $\nabla_i\varphi = \nabla_i f_k/f_k$, but this requires some justification. We observe that

$$\int_{M^S}\langle Z_i, \nabla_i\varphi\rangle\,\mathrm{d}\mu = \int_{M^{\Lambda_k}}\langle \mathbb{E}^\mu_{\Lambda_k} Z_i, \nabla_i\varphi\rangle f_k\,\mathrm{d}\nu_k.$$

We recall that $|\mathbb{E}^\mu_{\Lambda_k} Z_i| \in L^2(\mu_k)$, because $|Z_i| \in L^2(\mu)$ for all $i \in S$ by assumption. Since $|\nabla_i W_k|, |\nabla f_k/f_k| \in L^2(\mu_k)$, it suffices to show that there exists a sequence of functions

$\varphi_i \in C_0^\infty(M^{\Lambda_k})$ such that $|\nabla\varphi_i - \nabla f_k/f_k| \to 0$ in $L^2(\mu_k)$, i.e., $\nabla f_k/f_k \in \Gamma(\mu_k)$. Since we have $\frac{\nabla f_k}{f_k} = \frac{\nabla g_k}{g_k} - \nabla W_k$, it remains to note that $\frac{\nabla g_k}{g_k}, \nabla W_k \in \Gamma(\mu_k)$ by [14, Lemma 2.1 and Corollary 2.3]. Equality (3.3) is proved in a similar manner taking into account that, for every $i \in \Lambda_k \backslash \Lambda_{k-1}$, one has

$$\int_{M^S} \left[\Delta_i\varphi + \langle Z_i, \nabla_i\varphi\rangle\right] \mathrm{d}\mu = \int_{M^{\Lambda_{k+1}}} \left[\Delta_i\varphi + \langle Z_i, \nabla_i\varphi\rangle\right] f_{k+1}\, \mathrm{d}\nu_{k+1}$$
$$= \int_{M^{\Lambda_{k+1}}} \Big[-\langle\nabla_i\varphi, \nabla_i f_{k+1}\rangle + \langle\nabla_i\varphi, Z_i - \nabla_i W_{k+1}\rangle\, f_{k+1}\Big]\, \mathrm{d}\nu_{k+1}.$$

Combining this with (3.4) we arrive at the relationship

$$\begin{aligned}
\sum_{i\in\Lambda_{k-1}} \int_{M^{\Lambda_k}} \langle\nabla_i\varphi, \nabla_i f_k\rangle\, \mathrm{d}\nu_k &= -\sum_{i\in\Lambda_{k-1}} \int_{M^S} \big(\Delta_i\varphi + \langle\nabla_i\varphi, \nabla_i W_k\rangle\big)\, \mathrm{d}\mu \\
&= \sum_{i\in\Lambda_{k-1}} \int_{M^S} \langle Z_i - \nabla_i W_k, \nabla_i\varphi\rangle\, \mathrm{d}\mu \\
&\quad + \sum_{i\in\Lambda_k\backslash\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \langle\nabla_i\varphi, f_{k+1}(Z_i - \nabla_i W_{k+1}) - \nabla_i f_{k+1}\rangle\, \mathrm{d}\nu_{k+1} \\
&= \sum_{i\in\Lambda_{k-1}} \int_{M^S} \langle Z_i - \nabla_i W_k, \nabla_i\varphi\rangle\, \mathrm{d}\mu \\
&\quad + \sum_{i\in\Lambda_k\backslash\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \Big\langle\nabla_i\varphi, Z_i - \nabla_i W_{k+1} - \frac{\nabla_i f_{k+1}}{f_{k+1}}\Big\rangle\, \mathrm{d}\mu.
\end{aligned}$$

With the above justification, one can replace $\nabla_i\varphi$ by $\nabla_i f_k/f_k$ and hence the second desired equality follows. □

Lemma 3.3. *Assume that* $(\mathbf{C}')$ *holds.*

(i) *Let* $\mu_{k+1} \in \mathcal{P}(M^{\Lambda_{k+1}})$ *have a logarithmic derivative* $\beta_i^{\mu_{k+1}}$ *along* x_i *for some* $i \in \Lambda_k$ *and let* $|\beta_i^{\mu_{k+1}}| \in L^2(\mu_{k+1})$. *Let* μ_k *be the projection of* μ_{k+1} *to* M^{Λ_k}. *Then* $\beta_i^{\mu_k} := \mathbb{E}_{\Lambda_k}^{\mu_{k+1}} \beta_i^{\mu_{k+1}}$ *is the logarithmic derivative* $\beta_i^{\mu_k}$ *of* μ_k *along* x_i *and*

$$\int_{M^{\Lambda_k}} |\beta_i^{\mu_k}|^2\, \mathrm{d}\mu_{_k} \le \int_{M^{\Lambda_{k+1}}} |\beta_i^{\mu_{k+1}}|^2\, \mathrm{d}\mu_{k+1}. \tag{3.6}$$

(ii) *Let* $\mu \in \mathcal{P}(M^S)$. *Let* $\Lambda \subset \widetilde{\Lambda}$ *be two finite subsets of* S *and let* $\mathrm{d}\nu := \exp[W]\mathrm{d}\lambda_{M^\Lambda}$ *and* $\mathrm{d}\widetilde{\nu} := \exp[\widetilde{W}]\mathrm{d}\lambda_{M^{\widetilde{\Lambda}}}$ *be two probability measures on* M^Λ *and* $M^{\widetilde{\Lambda}}$, *respectively, with* W, $\widetilde{W}$ *locally bounded and* $W \in H^{2,1}_{loc}(\lambda_{M^\Lambda})$, $\widetilde{W} \in H^{2,1}_{loc}(\lambda_{M^{\widetilde{\Lambda}}})$. *If* $\mu_\Lambda = f\nu$ *and* $\mu_{\widetilde{\Lambda}} = \widetilde{f}\widetilde{\nu}$ *with* $f^{1/2} \in H^{2,1}(\nu)$, $\widetilde{f}^{1/2} \in H^{2,1}(\widetilde{\nu})$. *Then, for every* $i \in \Lambda$ *such that* $\nabla_i W = \nabla_i \widetilde{W}$ *one has*

$$\int_{M^\Lambda} \frac{|\nabla_i f|^2}{f}\, \mathrm{d}\nu \le \int_{M^{\widetilde{\Lambda}}} \frac{|\nabla_i \widetilde{f}|^2}{\widetilde{f}}\, \mathrm{d}\widetilde{\nu}. \tag{3.7}$$

Proof. The first claim is easily verified and yields (3.6). In order to prove (3.7), let us note that both integrals are finite according to our conditions. The left-hand side in (3.7) is equal to the square of the norm of $|\nabla_i f/f|$ in $L^2(\mu_\Lambda)$, hence coincides with the supremum of

$$\Big(\int_{M^\Lambda} \Big\langle \frac{\nabla_i f}{f}, v\Big\rangle \,\mathrm{d}\mu_\Lambda\Big)^2$$

over all smooth compactly supported vector fields v on M^Λ such that $v(x) \in T_{x_i}M^i$ and $\big\||v|\big\|_{L^2(\mu_\Lambda)} \le 1$. Given such a field, since v and $\nabla_i W = \nabla_i \widetilde{W}$ depend only on x_Λ, we have

$$\begin{aligned}\int_{M^\Lambda} \langle \nabla_i f, v\rangle \,\mathrm{d}\nu &= -\int_{M^\Lambda} (\operatorname{div} v + \langle v, \nabla_i W\rangle)\, f \,\mathrm{d}\nu \\ &= -\int_{M^{\tilde\Lambda}} (\operatorname{div} v + \langle v, \nabla_i \widetilde{W}\rangle)\, \tilde f \,\mathrm{d}\tilde\nu = \int_{M^{\tilde\Lambda}} \langle \nabla_i \tilde f, v\rangle \,\mathrm{d}\tilde\nu \\ &\le \Big(\int_{M^{\tilde\Lambda}} \frac{|\nabla_i \tilde f|^2}{\tilde f^2}\,\mathrm{d}\mu_{\tilde\Lambda}\Big)^{1/2} \Big(\int_{M^{\tilde\Lambda}} |v|^2 \,\mathrm{d}\mu_{\tilde\Lambda}\Big)^{1/2} \\ &\le \Big(\int_{M^{\tilde\Lambda}} \frac{|\nabla_i \tilde f|^2}{\tilde f^2}\,\mathrm{d}\mu_{\tilde\Lambda}\Big)^{1/2}.\end{aligned}$$

The justification of the above integration by parts (which is not needed if $f \in H^{2,1}_{loc}(\lambda_\Lambda)$ or if $\nabla_i W$ is locally bounded) is easy: it suffices to approximate f by $\big(n \wedge \sqrt{f}\big)^2 \in H^{2,1}_{loc}(\lambda_\Lambda)$, $n \in \mathbb{N}$, and the same for $\tilde f$. □

Lemma 3.4. *Suppose we have a sequence of nonnegative numbers T_j and a sequence of strictly positive numbers C_k with the following property: there exists $\eta \ge 0$ such that for each k*

$$\sum_{j=1}^{k} T_j \le \eta + \sqrt{C_k T_k}$$

and $\sum\limits_{k=1}^{\infty} C_k^{-1} = \infty$. Then $\sum\limits_{j=1}^{\infty} T_j \le \eta$.

Proof. Let us consider two functions g and ψ on $[1, +\infty)$ such that $g(t) = T_k$ and $\psi(t) = C_k$ if $t \in [k, k+1)$. By the above inequality we obtain that the function $G(t) := \int_1^t g(s)\,\mathrm{d}s$ satisfies the inequality $G(t) \le \eta + \sqrt{\psi(t)G'(t)}$. If there is $t_0 > 0$ such that $G(t_0) > \eta$, then letting $\xi(t) := G(t) - \eta$ we arrive at

$$\xi^2(t) \le \psi(t)\xi'(t), \quad \xi(t_0) > 0,\ t \ge t_0.$$

Thus,

$$0 \le \frac{1}{\xi(t)} \le \frac{1}{\xi(t_0)} - \int_{t_0}^{t} \frac{\mathrm{d}s}{\psi(s)}, \quad t > t_0$$

which is impossible since $\int_{t_0}^{\infty} \frac{\mathrm{d}s}{\psi(s)} \ge \sum_{k > t_0+1} C_k^{-1} = \infty$. □

We shall now consider the case where $Z_i := -\sum\limits_{\Lambda \ni i} \nabla_i U_\Lambda$, where $\{U_\Lambda\}$ has finite range R, i.e., $U_\Lambda = 0$ if $\operatorname{diam}(\Lambda) := \sup\big\{|s_j - s'_j|:\ s, s' \in \Lambda, 1 \le j \le m\big\} > R$. Suppose that $\mu \in \mathcal{P}(M^S)$ is such that $\nabla_i U_\Lambda \in L^2(\mu)$. Then one has

$$D_k^\mu := \sum_{i \in \Lambda_k \setminus \Lambda_{k-1}} \int_{M^S} \big|\mathbb{E}^\mu_{\Lambda_k} Z_i - \nabla_i W_k\big|^2 \,\mathrm{d}\mu = \sum_{i \in \Lambda_k \setminus \Lambda_{k-1}} \int_{M^S} \Big| \sum_{\Lambda \cap \Lambda_k^c \ne \emptyset} \mathbb{E}^\mu_{\Lambda_k} \nabla_i U_\Lambda \Big|^2 d\mu. \quad (3.8)$$

Theorem 3.5. *Assume* ($\mathbf{C}'$). *Consider* $Z_i := -\sum_{\Lambda\ni i} \nabla_i U_\Lambda$, *where* $\{U_\Lambda\}$ *has finite range* R. *Let* $\mu \in \mathcal{P}(M^S)$ *be such that* $\nabla_i U_\Lambda$, $|\nabla_i W_k| \in L^2(\mu)$ *for all* $i \in S$, $k \in \mathbb{N}$ *and finite* Λ *and let* $L_Z^*\mu = 0$ *with respect to* $\mathcal{FC}_0^\infty$. *Assume that* $D_k^\mu \le C_k$, *where* $C_k > 0$ *are numbers with*

$$\sum_{k=1}^{\infty} \frac{1}{C_k + C_{k+1}} = \infty. \tag{3.9}$$

Then β_i^μ *exists and coincides with* Z_i *for every* $i \in S$. *In particular,* μ *is Gibbsian.*

Proof. Let us use the notation from the above lemmas. In the present case one has $Z_i = \nabla_i W_k$ for $i \in \Lambda_{k-1}$. Then it follows from (3.3) that

$$\begin{aligned}\sum_{i\in\Lambda_{k-1}} \int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k}\, \mathrm{d}\nu_k &= -\sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \Big\langle \frac{\nabla_i f_k}{f_k}, \frac{\nabla_i f_{k+1}}{f_{k+1}} \Big\rangle\, \mathrm{d}\mu_{k+1} \qquad (3.10)\\ &\le \Big(\sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k^2}\, \mathrm{d}\mu_k \Big)^{1/2} \Big(\sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}^2}\, \mathrm{d}\mu_{k+1} \Big)^{1/2}.\end{aligned}$$

We observe that the first factor on the right in (3.10) is majorized by $\sqrt{C_k}$. Indeed, by (3.2), (3.8) and the equality $\nabla_i W_k = Z_i$ for every $i \in \Lambda_{k-1}$, we have

$$\begin{aligned}\int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k^2}\, \mathrm{d}\mu_k &= \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^S} \Big\langle \mathbb{E}_{\Lambda_k}^\mu Z_i - \nabla_i W_k, \frac{\nabla_i f_k}{f_k} \Big\rangle\, \mathrm{d}\mu\\ &\le \sqrt{C_k} \Big(\int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k^2}\, \mathrm{d}\mu_k \Big)^{1/2}.\end{aligned}$$

This implies $\big\| |\nabla f_k|/f_k \big\|_{L^2(\mu_k)} \le \sqrt{C_k}$. Let T_k be defined by

$$T_k := \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}}\, \mathrm{d}\nu_{k+1}. \tag{3.11}$$

Then we obtain by (3.10) and (3.7) that

$$\sum_{j=1}^{k-1} T_j \le \sqrt{C_k T_k}.$$

By the estimate $T_k \le \sqrt{C_{k+1} T_k}$, which follows from the estimate

$$T_k \le \big\| |\nabla f_{k+1}|/f_{k+1} \big\|_{L^2(\mu_{k+1})}^2 \le C_{k+1},$$

we obtain $\sum_{j=1}^{k} T_j \le \sqrt{C_k' T_k}$, where $C_k' := 2(C_k + C_{k+1})$. By Lemma 3.4 this yields that $T_k = 0$ for all k. Hence (3.10) implies $\nabla_i \sqrt{f_k} = 0$ ν_k-a.e. for all $i \in \Lambda_{k-1}$, i.e., f_k only depends on $x_{\Lambda_k\setminus\Lambda_{k-1}}$. Therefore

$$\beta_i^{\mu_k} = \frac{\nabla_i f_k}{f_k} + \nabla_i W_k = \nabla_i W_k = Z_i \quad \mu_k\text{-a.e. for all } i \in \Lambda_{k-1}.$$

It follows that β_i^μ exists and $\beta_i^\mu = Z_i$. Indeed, for v and ψ as in (3.1) and $i \in S$ choose $k \in \mathbb{N}$ such that $\psi \in C_0^\infty(M^{\Lambda_k})$, $i \in \Lambda_{k-1}$. Then

$$\begin{aligned}\int_{M^S} \langle \nabla_i \psi, v\rangle \,\mathrm{d}\mu = \int_{M^{\Lambda_k}} \langle \nabla_i \psi, v\rangle \,\mathrm{d}\mu_k = -\int_{M^S} \psi(\operatorname{div} v + \langle v, \beta_i^{\mu_k}\rangle)\,\mathrm{d}\mu_k \\ = -\int_{M^S} \psi(\operatorname{div} v + \langle v, Z_i\rangle)\,\mathrm{d}\mu_k = -\int_{M^S} \psi(\operatorname{div} v + \langle v, Z_i\rangle)\,\mathrm{d}\mu,\end{aligned}$$

where the last step follows since Z_i is a function of x_{Λ_k} for all $i \in \Lambda_{k-1}$. □

We observe that condition (3.9) is fulfilled if

$$Z_i(x) = \nabla_i W_k(x_{\Lambda_k}) + Z_i^{(k)}(x), \qquad i \in \Lambda_k,$$

where

$$\sup_{i \in \Lambda_k \setminus \Lambda_{k-1}} |Z_i^{(k)}(x)|^2 \le c' k^{2-m}, \qquad \forall\, k \ge 1, \tag{3.12}$$

for some $c' > 0$. Indeed, in this case the cardinality of $\Lambda_k \setminus \Lambda_{k-1}$ is estimated by $c(m)k^{m-1}$, hence $D_k^\mu \le c'' k$. For example, (3.9) is fulfilled if $m \le 2$ and one has (3.12) with $\sup_{i,k} |Z_i^{(k)}| < \infty$ or if $m \le 2$ and the Z_i's are uniformly bounded. In the case $M = \mathbb{R}^d$, in many concrete models the following two conditions are satisfied: one has an estimate $|Z_i^{(k)}(x)| \le Q(|x_{i+\Lambda_R}|)$ for some polynomial Q and, for every p, the functions $|x_j|^p$ have uniformly bounded integrals with respect to any stationary measure μ. This yields (3.9) if $m = 2$ (cf. Example 4.2 below). Certain a priori estimates which can be used for the verification of the square integrability of $\nabla_i W_k$ with respect to μ are obtained in [14]. This assumption is certainly much less restrictive than the main assumption (3.9), but yet is a restriction. Thus, in the case of the two dimensional lattice, the above theorem gives broad sufficient conditions for the reversibility of every stationary measure of the stochastic system associated with a Gibbs measure (see Section 4 below).

In the next theorem we consider the situation when the fields Z_i are sufficiently close to the partial logarithmic gradients β_i^γ of some measure γ which, in addition, satisfies the logarithmic Sobolev inequality. It turns out that μ admits a density with respect to γ. In particular, if $Z_i = \beta_i^\gamma$, i.e., γ is Gibbsian with respect to Z, then γ is a unique solution of our elliptic equation (that is, any stationary distribution is Gibbs). As noted in the introduction, this phenomenon has already been discovered in a number of special situations. We introduce the following conditions:

(C1) The projection γ_k of γ to M^{Λ_k}, where Λ_k is the same as above, has a density $\exp(G_k)$ with respect to the Riemannian volume such that $G_k \in H_{loc}^{2,1}(\lambda_k) \cap L_{loc}^\infty(\lambda_k)$. Set $\beta^{\gamma_k} := (\beta_i^{\gamma_k})_{i \in \Lambda_k}$, $\beta_i^{\gamma_k} := \nabla_i G_k$, where we fix some Borel version.

(C2) The measure γ satisfies the logarithmic Sobolev inequality, i.e., there exists $\kappa \in (0, +\infty)$ such that

$$\int \varphi^2 \log \varphi^2 \,\mathrm{d}\gamma \le \kappa \int |\nabla \varphi|^2 \,\mathrm{d}\gamma \quad \forall\, \varphi \in \mathcal{F}C_0^\infty(M^S) \text{ with } \|\varphi\|_{L^2(\gamma)} = 1, \tag{3.13}$$

and $H^{2,1}(\gamma)$, the completion of $\mathcal{F}C_0^\infty$ with respect to the Sobolev norm defined by

$$\|\varphi\|_{H^{2,1}(\gamma)}^2 = \int \varphi^2 \,\mathrm{d}\gamma + \sum_{i \in S} \int |\nabla_i \varphi|^2 \,\mathrm{d}\gamma,$$

embeds into $L^2(\gamma)$ (which is, e.g., trivially the case if γ has a partial logarithmic derivative β_i^γ for all $i \in S$). The spaces $H^{2,1}(\gamma_\Lambda)$ are defined analogously. We say that a locally γ_k-integrable vector field v on M^{Λ_k} has divergence $\operatorname{div}^{\gamma_k} v$ with respect to γ_k if $\operatorname{div}^{\gamma_k} v \in L^1_{loc}(\gamma_k)$ and

$$\int_{M^{\Lambda_k}} \langle v, \nabla\varphi\rangle \,\mathrm{d}\gamma_k = -\int_{M^{\Lambda_k}} \varphi \operatorname{div}^{\gamma_k} v \,\mathrm{d}\gamma_k \quad \forall\, \varphi \in C_0^\infty(M^{\Lambda_k}).$$

Theorem 3.6. *Assume* $(\mathbf{C}')$. *Let* $\gamma \in \mathcal{P}(M^S)$ *be such that* $(\mathbf{C1})$ *and* $(\mathbf{C2})$ *are fulfilled. Let* $\mu \in \mathcal{P}(M^S)$ *satisfy* $L_Z^*\mu = 0$ *with respect to* $\mathcal{FC}_0^\infty$, *where* $Z = (Z_i)_{i\in S}$ *is of finite range* R, $|Z_i| \in L^2(\mu)$, $|\beta_i^{\gamma_k}| \in L^2(\mu)$, $i \in \Lambda_k$, *and let* $C_k > 0$ *be such that*

$$\sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_k}} |\mathbb{E}^\mu_{\Lambda_k} Z_i - \beta_i^{\gamma_k}|^2 \,\mathrm{d}\mu_k \le C_k, \quad \sum_{k=1}^\infty \frac{1}{C_k + C_{k+1}} = \infty. \tag{3.14}$$

Suppose that, for all $i \in \Lambda_{k-1}$, *one has* $|Z_i| \in L^2_{loc}(\gamma_k)$ *and that* $\operatorname{div}^{\gamma_k}(\beta_i^{\gamma_k} - Z_i)$ *exists, is in* $L^1(\mu_k)$, *and*

$$\eta := \sup_k \int_{M^{\Lambda_k}} \Big| \sum_{i\in\Lambda_{k-1}} \operatorname{div}^{\gamma_k}(\beta_i^{\gamma_k} - Z_i)\Big| \,\mathrm{d}\mu_k < \infty. \tag{3.15}$$

Then $\mu = f^2\,\gamma$, *where* $f \in H^{2,1}(\gamma)$. *If* $Z_i = \beta_i^\gamma$ *for all* i, *then* $\mu = \gamma$.

Proof. By the same reasoning as in Theorem 3.5, we have that $f_k := \mathrm{d}\mu_k/\mathrm{d}\gamma_k$ exists and $\sqrt{f_k} \in H^{2,1}(\gamma_k)$. Let $\pi_i := Z_i - \beta_i^{\gamma_k}$, $i \in \Lambda_{k-1}$. We observe that

$$\sum_{i\in\Lambda_{k-1}} \int_{M^{\Lambda_k}} \langle \pi_i, \nabla_i f_k\rangle \,\mathrm{d}\gamma_k = -\int_{M^{\Lambda_k}} \sum_{i\in\Lambda_{k-1}} \operatorname{div}^{\gamma_k}\pi_i\, f_k \,\mathrm{d}\gamma_k \le \eta. \tag{3.16}$$

This is done exactly as in [14, Theorem 7.6], where the assumptions on γ_k were even weaker (note that $\langle \pi_i, \nabla_i f_k\rangle \in L^1(\gamma_k)$, since $|\pi_i|\sqrt{f_k}, |\nabla_i f_k|/\sqrt{f_k} \in L^2(\gamma_k)$). By (3.2) and (3.16) we obtain the estimate

$$\sum_{j=1}^{k-1} T_j \le \eta + \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int \langle \mathbb{E}^\mu_{\Lambda_k} Z_i - \beta_i^{\gamma_k}, \frac{\nabla_i f_k}{f_k}\rangle \,\mathrm{d}\mu \le \eta + \sqrt{C_k T_k},$$

where T_k is defined by (3.11) with γ_{k+1} in place of ν_{k+1}. As in the previous theorem this yields that $\sum_{j=1}^\infty T_j \le \eta$. Due to the log-Sobolev inequality we obtain that the sequence f_k is uniformly γ-integrable, hence $\mu = f^2\,\mathrm{d}\gamma$. In addition, one has $f \in H^{2,1}(\gamma)$. Finally, if $Z_i = \beta_i^\nu$ for all i, then $\eta = 0$, hence $T_j = 0$, which implies by the log-Sobolev inequality that f is a constant (cf. the proof of [14, Corollary 7.7]). ☐

Note that the first estimate in (3.14) holds if

$$\sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^S} |Z_i - \beta_i^{\gamma_k}|^2 \,\mathrm{d}\mu \le C_k.$$

Finally, we present the following result which considerably improves [14, Theorem 7.4] and hence also [30, Theorem 4] where $M^i = \mathbb{S}^1$ and $\gamma^{(i)} = \lambda_{\mathbb{S}^1}$ are considered.

Theorem 3.7. *Let* $\gamma := \prod_{i\in S} \gamma^{(i)}$ *such that* $\gamma^{(i)} := \exp[W^{(i)}]\mathrm{d}\lambda_{M^i}$ *for every* i *is a probability measure on* M^i*, where* $W^{(i)} \in H^{2,1}_{loc}(\lambda_{M^i}) \cap L^\infty_{loc}(\lambda_{M^i})$*. Set* $\gamma_k := \prod_{i\in\Lambda_k} \gamma^{(i)}$*. Assume* $(\mathbf{C}')$ *and* $(\mathbf{C2})$*. Let* Z *be of finite range* R *and let* $\mu \in \mathcal{P}(M^S)$ *solve* $L_Z^*\mu = 0$ *with respect to* $\mathcal{F}C_0^\infty$*. Suppose that* $\sup_{i\in S} \|Z_i - \nabla_i W^{(i)}\|_\infty < \infty$ *and that there is* $\delta \geq 0$ *such that* $\mathrm{div}^{\gamma_{k+1}}(Z_i - \nabla_i W^{(i)}) \in L^1(\gamma_{k+1})$ *exists for all* $i \in \Lambda_k$ *and* $k \in \mathbb{N}$ *and*

$$\sum_{i\in\Lambda_k} \int_{M^{\Lambda_k}} \left[\mathbb{E}^\gamma_{\Lambda_k} \mathrm{div}^{\gamma_{k+1}}(Z_i - \nabla_i W^{(i)})\right] \mathrm{d}\mu_k \geq -\delta \tag{3.17}$$

for all $k \geq 1$*, where we assume also the existence of the integrals. Then* $\mathrm{d}\mu = f^2 \mathrm{d}\gamma$ *with* $f \in H^{2,1}(\gamma)$ *and* $\int_{M^S} |\nabla f|^2 \,\mathrm{d}\gamma \leq \delta/4$*.*

Proof. By (3.2), where $W_k = \sum_{i\in\Lambda_k} W^{(i)}$ (note that $\nabla_i W_k = \nabla_i W^{(i)}$, since $W^{(i)}$ is a function of x_i), using the same notation as in the proof of the previous theorem we have

$$\begin{aligned}
I &:= \int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k} \,\mathrm{d}\gamma_k = \sum_{i\in\Lambda_k} \int_{M^{\Lambda_{k+1}}} \langle Z_i - \nabla_i W^{(i)}, \nabla_i f_k\rangle \,\mathrm{d}\gamma_{k+1} \\
&\quad + \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \Big\langle Z_i - \nabla_i W^{(i)}, \frac{\nabla_i f_k}{f_k}\Big\rangle (f_{k+1} - f_k) \,\mathrm{d}\gamma_{k+1} \\
&= -\sum_{i\in\Lambda_k} \int_{M^{\Lambda_k}} \left[\mathbb{E}^\gamma_{\Lambda_k} \mathrm{div}^{\gamma_{k+1}}(Z_i - \nabla_i W^{(i)})\right] f_k \,\mathrm{d}\gamma_k \\
&\quad + \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \Big\langle Z_i - \nabla_i W^{(i)}, \frac{\nabla_i f_k}{f_k}\Big\rangle (f_{k,i} - f_k) \,\mathrm{d}\gamma_{k+1},
\end{aligned} \tag{3.18}$$

where

$$f_{k,i} := \int f_{k+1} \prod_{j\in\Lambda_{k+1}\setminus(\Lambda_k\cup\Lambda_1(i))} \mathrm{d}\gamma^{(j)}, \quad \Lambda_1(i) := \{j :\ |i_l - j_l| \leq R, 1 \leq l \leq m\}.$$

Next, letting $c_1 := \sup_i \|Z_i - \nabla_i W^{(i)}\|_\infty$, we obtain

$$\begin{aligned}
I_i &:= \int_{M^S} \Big\langle Z_i - \nabla_i W^{(i)}, \frac{\nabla_i f_k}{f_k}\Big\rangle (f_{k,i} - f_k) \,\mathrm{d}\gamma \\
&\leq \frac{c_1^2}{2} \int_{M^S} \frac{|\nabla_i f_k|^2}{f_k^2} \Big(\sqrt{f_{k,i}} + \sqrt{f_k}\Big)^2 \mathrm{d}\gamma + \frac{1}{2} \int_{M^S} \Big(\sqrt{f_{k,i}} - \sqrt{f_k}\Big)^2 \mathrm{d}\gamma \\
&\leq 2c_1^2 \int_{M^S} \frac{|\nabla_i f_k|^2}{f_k} \,\mathrm{d}\gamma + \int_{M^S} \big(f_k - \sqrt{f_k f_{k,i}}\big) \,\mathrm{d}\gamma.
\end{aligned}$$

Noting that

$$\sqrt{f_k} \geq \int \sqrt{f_{k,i}} \prod_{j\in\Lambda_1(i)\setminus\Lambda_k} \mathrm{d}\gamma^{(j)},$$

and applying (3.7) and the Poincaré inequality following from **(C2)**, we obtain that, for some constant $c > 0$, one has

$$
\begin{aligned}
I_i &\leq 2c_1^2 \int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k} \, \mathrm{d}\gamma_k + \int_{M_k^\Lambda} \left[f_k - \left(\int \sqrt{f_{k,i}} \prod_{j \in \Lambda_1(i) \setminus \Lambda_k} \mathrm{d}\gamma^{(j)} \right)^2 \right] \mathrm{d}\gamma_k \\
&\leq 2c_1^2 \int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k^2} \, \mathrm{d}\gamma_k + c \int_{M_k^\Lambda} \frac{\nabla_{\Lambda_1(i)\setminus\Lambda_k} f_{k,i}|^2}{f_{k,i}} \, \mathrm{d}\gamma_k \\
&\leq 2c_1^2 \int_{M^{\Lambda_k}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}^2} \, \mathrm{d}\gamma_{k+1} + c \int_{M_k^\Lambda} \frac{\nabla_{\Lambda_1(i)\setminus\Lambda_k} f_{k+1}|^2}{f_{k+1}} \, \mathrm{d}\gamma_{k+1}.
\end{aligned}
$$

Combining this with (3.18) and (3.17), we obtain for some constant $c_2 > 0$

$$
I \leq \delta + c_2 \sum_{i \in \Lambda_{k+1} \setminus \Lambda_{k-1}} \int_{M^S} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}} \, \mathrm{d}\gamma. \tag{3.19}
$$

Letting

$$
T_j := \sum_{i \in \Lambda_{j+1} \setminus \Lambda_{j-1}} \int_{M^S} \frac{|\nabla_i f_{j+1}|^2}{f_{j+1}} \, \mathrm{d}\gamma,
$$

according to (3.7) and (3.19) we obtain

$$
\sum_{j=1}^{k-1} T_j \leq 2I \leq 2\delta + 2c_2 T_k.
$$

Then the remainder of the proof is the same as that of [14, Theorem 7.4], just note that in the present setting one has

$$
2 \sup_k \int_{M^S} \frac{|\nabla f_k|^2}{f_k} \, \mathrm{d}\gamma \leq \sum_{j=1}^{\infty} T_j.
$$

□

Note that (3.17) is fulfilled in the case where $\gamma^{(i)} = \lambda_{M^i}$, i.e., $W^{(i)} = 0$, and $\mathrm{div} Z_i = 0$.

4 Applications to lattice systems from statistical mechanics

In this section we consider two concrete examples illustrating Theorem 3.5.

Example 4.1. *Assume* ($\mathbf{C}'$). *Let* $\sup_i \dim M^i < \infty$ *and let the Ricci curvature of* M^i *be bounded below uniformly in* $i \in S$. *Let* ϱ_i *be the Riemannian distance function on* M^i *to a fixed point. Consider the following potential of a two body interaction:*

$$
U_\Lambda := \begin{cases} C\varrho_i^N & \textit{if } \Lambda = \{i\}, \\ U_{ij} & \textit{if } \Lambda = \{i,j\}, 0 < |i-j| \leq R, \\ 0 & \textit{otherwise,} \end{cases}
$$

where $C > 0$ and $N \geq 2$ are two constants, and $U_{ij} = U_{ji}$ is a continuously differentiable function of x_i and x_j. Assume that there exist two families of nonnegative numbers $\{c_i\colon\ i \in S\}$ and $\{c_{ij}\colon\ i,j \in S, |i-j| \leq R\}$ such that

$$\left|\sum_{j\colon\ 0<|i-j|\leq R} \nabla_i U_{ij}\right| \leq c_i + \sum_{j\colon\ |j-i|\leq R} c_{ij}\varrho_j^n, \quad i \in S$$

and

$$c := \sup_i \left[c_i + \sum_{j\colon\ |j-i|\leq R} c_{ij}\right] < \infty.$$

If either $N-1 > n$ or $N-1 = n$ but $c < CN$, and if

$$\sum_{k=1}^{\infty} \left\{ \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \left[c_i + \sum_{j\colon\ |i-j|\leq R} c_{ij}\right]\right\}^{-1} = \infty, \tag{4.1}$$

then there exists μ such that $L_Z^*\mu = 0$ and all such measures are Gibbsian. In particular, (4.1) holds if $c_i + c_{ij} \leq \dfrac{A\log(2+|i|)}{|i|^{m-2}}$ for some constant $A > 0$ and all i,j with $|i-j| \leq R$.

Proof. We have $\langle \nabla_i U_i, \nabla \varrho_i\rangle \geq CN\varrho_i^{N-1}$ and

$$\left|\sum_{j\colon\ 0<|j-i|\leq R} \nabla_i U_{ij}\right| \leq c + \sum_{j\colon\ |j-i|\leq R} c_{ij}\varrho_j^n$$

for all $i \in S$. If $n = N-1$ and $c < CN$, then [14, Theorem 5.5] applies ((5.4) in the statement of this theorem is a misprint; it should have been (5.1)). If $N-1 > n$ then for any $\varepsilon \in (0,1)$ there exists $c(\varepsilon) > 0$ such that

$$\sum_{j\colon\ |j-i|\leq R} c_{ij}\varrho_j^n \leq c(\varepsilon) + \varepsilon \sum_{j\colon\ |j-i|\leq R} c_{ij}\varrho_j^{N-1}.$$

Hence [14, Theorem 5.5] applies by taking small ε. Therefore, the solution to $L_Z^*\mu = 0$ exists and for any solution μ one has

$$\delta := \sup_i \int_{M^S} \varrho_i^n \,\mathrm{d}\mu < \infty.$$

Thus,

$$D_k^\mu \leq \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \left[c_i + \int_{M^S} \sum_{j\colon\ |i-j|\leq R} c_{ij}\varrho_j^n \,\mathrm{d}\mu\right] \leq \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \left[c_i + \delta \sum_{j\colon\ |i-j|\leq R} c_{ij}\right].$$

Then it follows from Theorem 3.5 that all solutions to $L_Z^*\mu = 0$ are Gibbsian provided (4.1) holds. Finally, if $c_i + c_{ij} \leq \frac{A\log(2+|i|)}{|i|^{m-2}}$ for all i,j with $|i-j| \leq R$, then

$$\sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \left[c_i + \sum_{j\colon\ |i-j|\leq R} c_{ij}\right] \leq Bk\log(2+k) := C_k$$

for some constant $B > 0$ and all $k \geq 1$. Obviously, one has $\sum_{k=1}^\infty (C_k + C_{k+1})^{-1} = \infty$. $\square$

Finally, let us consider an example where M is a flat space. The assumptions in this example are not as minimal as possible, but they apply to realistic models.

Example 4.2. *Let $M = \mathbb{R}^1$ and let $S = \mathbb{Z}^2$, $|i| := |i'| + |i''|$, $i = (i', i'') \in \mathbb{Z}^2$. Suppose that, for all $i, j \in S$, we are given continuously differentiable functions V_i and $W_{i,j}$ on $\mathbb{R}^1$ and $\mathbb{R}^2$, respectively, such that $W_{i,j} = W_{j,i}$, and $W_{i,j} = 0$ if $i = j$ or $|i - j| > R$, where $R \in \mathbb{N}$ is a fixed number. We shall impose the following standard assumptions (cf. [22] and the references therein as well as [3], [2]) on the interaction $W_{i,j}$, self-interaction potentials V_i, respectively: there exist numbers $K_0 \geq 0$, $C > 0$, $K > 0$, $\alpha \geq 2$ such that*

$$|W_{i,j}(x_i, x_j)| \leq K_0(1 + |x_i|^\alpha + |x_j|^\alpha),$$

$$|\partial_{x_i} W_{i,j}(x_i, x_j)| \leq K_0(1 + |x_i|^{\alpha-1} + |x_j|^{\alpha-1}),$$

$$x_i V_i'(x_i) \geq K|x_i|^\alpha - C.$$

So, in our above setting, we have $U_\Lambda = 0$ unless Λ consists of at most two points and for $x = (x_i)_{i \in S} \in M^S$ one has

$$U_\Lambda(x) = \begin{cases} V_i(x_i) & \text{if } \Lambda = \{i\}, \\ W_{i,j}(x_i, x_j) & \text{if } \Lambda = \{i, j\}, i \neq j. \end{cases}$$

Hence

$$W_k(x) = -\sum_{\Lambda \subset \Lambda_k} U_\Lambda(x) = -\sum_{i \in \Lambda_k} V_i(x_i) - \sum_{i,j \in \Lambda_k} W_{i,j}(x_i, x_j).$$

Finally, assume that $K > 12K_0(R+1)^{2+p}$ for some $p > 2$ and that each V_i has a polynomially bounded derivative. Set

$$Z_i(x) := -V_i'(x_i) - \sum_{j \in \mathbb{Z}^2} \partial_{x_i} W_{i,j}(x_i, x_j).$$

Then, there exist solutions to $L_Z^* \mu = 0$ in the class of probability measures on the Banach space

$$X_0 := \Big\{ (x_i)_{i \in \mathbb{Z}^2} \colon \sum_{i \in \mathbb{Z}^2} q_i |x_i|^\alpha < \infty \Big\}, \quad \text{where } q_i := |i|^{-p},\ q_0 = 1,$$

and all such measures are symmetrizing. In addition, if one solution of $L_Z^* \mu = 0$ satisfies a log-Sobolev inequality, then there exists exactly one such solution. Analogous assertions are true if $M = \mathbb{R}^m$ (then V_i' and $x_i V_i'(x_i)$ are replaced by $\nabla V_i(x_i)$ and $\langle x_i, \nabla V_i(x_i) \rangle$).

Proof. The existence part follows from [11, Example 6.12], where we take $q = (q_i)_{i \in \mathbb{Z}^2}$ with $q_i := |i|^{-p}$, $q_0 = 1$. Indeed, letting $J_{i,j} := K_0$ if $0 < |i - j| \leq R$ and $J_{i,j} := 0$ otherwise, we obtain an infinite symmetric matrix that generates a bounded linear operator J on the weighted space $l^1(q)$ of all sequences (x_i) with $\|(x_i)\|_{l^1(q)} := \sum_{i \in \mathbb{Z}^2} q_i |x_i| < \infty$. Note that, for any $i \in \mathbb{Z}^2$ with $|i| \geq R + 1$, one has

$$\sum_{j \in \mathbb{Z}^2} J_{i,j} q_j = K_0 \sum_{0 < |i-j| \leq R} q_j \leq K_0 (2R+1)^2 \Big(\frac{|i|}{|i| - R} \Big)^p \frac{1}{|i|^p} \leq 4K_0(R+1)^{2+p} q_i.$$

If $|i| \le R$, then $\sum\limits_{j\in\mathbb{Z}^2} J_{i,j}q_j \le K_0(2R+1)^2 \le 4K_0(R+1)^{2+p}q_i$, since $q_i \ge R^{-p}$. It follows by these estimates that the operator norm of J on the space $l^1(q)$ is majorized by the number $4K_0(R+1)^{2+p}$. Then $K > 3\|J\|_{L(l^1(q))}$, hence [11, Example 6.12] directly applies and yields the existence of a probability measure μ on the Banach space X_0 such that, for every $i \in \mathbb{Z}^2$, the function Z_i is the logarithmic derivative of μ along x_i (i.e., μ is Gibbsian). Moreover, according to [11, Example 4.6], for an arbitrary probability measure μ on X_0 having Z_i as the logarithmic derivative along x_i for each $i \in \mathbb{Z}^2$, for every $r > 0$, there exists numbers $B_r > 0$ such that

$$\int |x_i|^r \,\mu(dx) \le B_r, \quad \forall\, i \in \mathbb{Z}^2. \tag{4.2}$$

In particular, $|Z_i|, |\nabla_i W_k| \in L^2(\mu)$ for all $i \in \Lambda_k$, $k \in \mathbb{N}$. Estimate (4.2) also enables us to show that $D_k^\mu \le M(R,\alpha,K_0)k$, where $M(R,\alpha,K_0)$ is independent of k, which implies (3.9), where $W_k := -\sum\limits_{i\in\Lambda_k} V_i - \sum\limits_{i,j\in\Lambda_k} W_{i,j}$. Indeed, if $i \in \Lambda_k\backslash\Lambda_{k-1}$, then, since

$$Z_i(x) = -\partial_{x_i}V_i(x_i) - \sum_{j:\ |i-j|\le R} \partial_{x_i}W_{i,j}(x_i,x_j),\ \partial_{x_i}W_k(x) = -\partial_{x_i}V_i(x_i) - \sum_{j\in\Lambda_k}\partial_{x_i}W_{i,j}(x_i,x_j),$$

we obtain

$$\begin{aligned}\left|\mathbb{E}^\mu_{\Lambda_k} Z_i - \partial_{x_i}W_k\right| &= \Big|\sum_{j\notin\Lambda_k:\ |i-j|\le R} \mathbb{E}^\mu_{\Lambda_k}\partial_{x_i}W_{i,j}\Big| \\ &\le K_0 \sum_{j:\ |i-j|\le R} (1 + |x_i|^{\alpha-1} + \mathbb{E}^\mu_{\Lambda_k}|x_j|^{\alpha-1}).\end{aligned}$$

Since the cardinality of the set $\{j:\ |j-i| \le R\}$ is $(2R+1)^2$, we obtain by (4.2) and the contraction property of the conditional expectation

$$\int \left|\mathbb{E}^\mu_{\Lambda_k} Z_i - \partial_{x_i}W_k\right|^2 d\mu \le 3K_0^2(2R+1)^4(2B_{2\alpha-2}+1).$$

Taking into account that the cardinality of $\Lambda_k\backslash\Lambda_{k-1}$ is majorized by $2(2R+1)^2k$, we arrive at the desired estimate. So, Theorems 3.5 and 3.6, respectively, imply the two assertions. □

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