Central Limit Theorem for a Probability Approximation of the Standard Map

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Abstract

Central Limit Theorem for a probability approximation of the standard map is proved.

Let $\varphi = \varphi(u)$ be a smooth function having the period 1 with respect to u, $\varphi(u) = \varphi(u+1)$. We consider a map $A_\varphi : (u, z) \to (u', z')$ of the cylinder $C = \{u, z : 0 \le u < 1, -\infty < z < \infty\}$ having the standard type

$$A_\varphi : \begin{cases} u' = u + z' \bmod 1 \\ z' = z + \varphi(u) \end{cases}, \tag{1}$$

which in the special case $\varphi = \varphi_*(u) = h \sin(2\pi u)$ reduces to the standard map

$$A_* : \begin{cases} u' = u + z' \bmod 1 \\ z' = z + h \sin(2\pi u) \end{cases}. \tag{2}$$

The maps A_φ, and especially A_*, have been studied in many papers (see [1] - [4], [6] - [8], [10]). Some conjectures connected with these maps have been formulated and are at present time very popular.

Conjecture *For sufficiently large value of the parameter h there exists a set $Q \subset C$ having positive Lebesgue measure such that if $z_n = z_n(u, z)$ is the coordinate of the point $(u_n, z_n) = A_*^n(u, z)$ (A_*^n is the n-th power of A_*) then the quantity $I_n = \int_Q z_n^2(u, z) du dz$ behaves like cn as $n \to \infty$, where c is a non-vanishing constant.*

The meaning of this conjecture is that for initial data from Q the change of coordinate z_n behaves like a path of random walk. As indicated in [10] at the present time we are far from its proof. In [7] we proposed a probability approximation of the standard map, for which this conjecture can be proved.

The aim of present paper is to formulate and to prove the Central Limit Theorem for probability approximation of standard map. The essence of our approach is that instead of iterations of the map A_* we consider consecutive actions of two maps A_* and A_φ with $\varphi \neq \varphi_*$ such that in every step we apply one of these maps with probability $p \geq 0$ and other map with probability $q = 1 - p$. If the functions φ_* and φ are close in some metric sense, then it is possible to say that such random dynamical system approximates the original deterministic system.

Let us introduce the following terminology and definitions:
mes is the Lebesgue measure on the cylinder C, i.e. $d\ mes = dudz$;
$S = \{0, 1\}$ is the set consisting of the two elements 0 and 1;
ν is a measure on S giving measure $p \geq 0$ to element 0 and measure $q = 1 - p$ to element 1;
Ω is space defined as the direct product of countable number of sets S, so that an arbitrary point of Ω is one-sided sequence $\omega = (\omega_1, \omega_2, \ldots)$ in which for any $n = 1, 2, \ldots$ the quantity ω_n takes one of two values 0 or 1;
b is the Bernoulli measure on Ω, i.e., the countable product of measures ν;
σ is the map of Ω into itself such that if $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, then $\sigma(\omega) = \omega' = (\omega'_1, \omega'_2, \ldots)$, where $\omega'_n = \omega_{n+1}$ $(n = 1, 2, \ldots)$;
$W = C \times \Omega$ is the direct product of the space C and Ω;
μ is the measure on W obtained by the product of the Lebesgue measure on C and the measure b, that is $d\mu = dudz \times db = d\ mes \times db$;
if Π is an Lebesgue measurable set on C having a finite measure mes, then we introduce the measure mes_Π on Π and the measure μ_Π on the set $\Pi \times \Omega$ so that $d\ mes_\Pi = \frac{d\ mes}{mes(\Pi)}$ and $d\mu_\Pi = \frac{d\mu}{mes(\Pi)} = d\ mes_\Pi \times db$;
$\psi(u, \omega)$ is the function on W having the form

$$\psi(u, \omega) = \begin{cases} \varphi_*(u), & \text{if} \quad \omega_1 = 0 \\ \varphi(u), & \text{if} \quad \omega_1 = 1 \end{cases};$$

B is the map of the space W into itself such that if $(u, z, \omega) \in W$, then $B(u, z, \omega) = (u', z', \omega')$, where $(u', z') = A_{\psi(u,\omega)}(u, z)$, $\omega' = \sigma(\omega)$.

Definition The map B of the space W into itself is called the *probabilistic analogue* of the standard map A_* of the cylinder C.

Consider the coordinate $z^{(n)} = z^{(n)}(u, z, \omega)$ of point $(u^{(n)}, z^{(n)}, \omega^{(n)}) = B^n(u, z, \omega)$, where $n \geq 0$ is an integer number. If Π is an Lebesgue measurable set on C having a finite measure *mes* and the point $(u, z) \in \Pi$, then we will regard the function $z^{(n)}(u, z, \omega)$ as the random variable $z^{(n)}_\Pi = z^{(n)}(u, z, \omega)$ on the set $\Pi \times \Omega$ with probability μ_Π.

Further we shall formulate and prove the Theorem, from which it follows that on the half-line $h > 0$ there exists an open set H having infinite Lebesgue measure such that for any $h \in H$ there exists a smooth function $\varphi(u)$ with period 1 that differs from $\varphi_*(u)$ on interval of arbitrarily small measure and concides with φ_* for the other points, such that there exists an open set $\Pi \subset C$ for which the random variables $z^{(n)}_\Pi$ satisfy the Central Limit Theorem.

In addition, there exist infinitely many such sets Π on C having the same measure and being mutually disjoint. If the function $\varphi(u)$ coincides with $\varphi_*(u)$, then the probabilistic analogue of the standard map coincides with the standard map. The difference of these functions on the set of small measure is a natural approximation from probabilistic point of view.

Theorem. *On the half-line $h > 0$ there exists an open set H having infinite Lebesgue measure on the straight line such that if $h \in H$ then there exists an integer $k = k(h) > 0$ for which the following assertion holds: for any ϵ such that $0 < \epsilon < 1$ there exist numbers α, β, a C^∞ function $\varphi(u)$ and an infinite number of mutually disjoint open sets $\Pi_m \subset C$ $(m \in \mathbf{Z})$ such that*

1) $0 < \alpha < \beta < 1, \beta - \alpha \leq \epsilon$;

2) if $u \in [0, \alpha] \cup [\beta, 1)$, then $\varphi(u) = \varphi_*(u)$ and if $u \in (\alpha, \beta)$, then $|\varphi(u) - \varphi_*(u)| \leq 2k$;

3) for any $m \in \mathbf{Z}$ the Lebesgue measure of set Π_m is equal to $\kappa = \int_{\Pi_0} dudz$, where $0 < \kappa < 1$, and if $(u, z) \in \Pi_m$, then for any $m \in \mathbf{Z}$ random variables $z^{(n)}_m = z^{(n)}_{\Pi_m}(u, z, \omega)$ satisfy the Central Limit Thoerem:

as $n \to \infty$ the probability distributions of the random variables

$$\frac{1}{k\sqrt{n(1 - p^2 - q^2 + 2pq)}}(z^{(n)}_m - kn(p - q)) \tag{3}$$

converge weakly to a normal distribution with density $(2\pi)^{-\frac{1}{2}} \exp(-\frac{x^2}{2})$.

Proof. Suppose that there exists u_0 satisfying the inequality $0 < u_0 < 1$ and an integer k such that

$$h \sin(2\pi u_0) = k > 0 \ , \tag{4}$$

$$-4 < 2\pi h \cos(2\pi u_0) = \rho < 0 \ , \tag{5}$$

$$\rho \neq -2 + 2\cos\frac{2\pi s}{q} \ , \quad (s = 0, \pm 1, \ldots, \pm q; q = 1, 2, 3, 4) \ , \tag{6}$$

and write

$$z_n^{(0)} = nk \ , n \in \mathbf{Z} \ . \tag{7}$$

By virtue of (2), (4) and (7) we have

$$A_*(u_0, z_n^{(0)}) = (u_0, z_{n+1}^{(0)}), n \in \mathbf{Z} \ . \tag{8}$$

Let

$$\delta = \frac{1}{2}\min(\epsilon, u_0, 1 - u_0) \ , \tag{9}$$

and define the numbers

$$\alpha = u_0 - \delta \ , \beta = u_0 + \delta \ , \alpha' = u_0 - \frac{\delta}{2} \ , \beta' = u_0 + \frac{\delta}{2} \ , \tag{10}$$

which by (9) satisfy the inequalities

$$0 < \alpha < \alpha' < u_0 < \beta' < \beta < 1 \ , \beta - \alpha \leq \epsilon \ . \tag{11}$$

Now consider the infinite-differentiable function $\varphi = \varphi(u)$ having period 1 with respect u such that the following conditions hold:

$$\varphi(u) = \varphi_*(u) \qquad \text{if} \qquad u \in [0, \alpha] \bigcup [\beta, 1) \ , \tag{12}$$

$$\varphi(u) = \varphi_*(u) - 2k \qquad \text{if} \qquad u \in [\alpha', \beta'] \ , \tag{13}$$

$$|\varphi(u) - \varphi_*(u)| \leq 2k \ , \qquad u \in [0, 1) \ . \tag{14}$$

According to (4) and (13) $\varphi(u_0) = -k$. Hence using of (1) and (7) we obtain:

$$A_\varphi(u_0, z_n^{(0)}) = (u_0, z_{n-1}^{(0)}) \ , n \in \mathbf{Z} \ . \tag{15}$$

For any $n \in \mathbf{Z}$ introduce the following two sequences of maps

$$A^{(n)} : (u, z) \to (u^*, z^*) = A_*(u + u_0, z + z_{n-1}^{(0)}) - A_*(u_0, z_{n-1}^{(0)}), \tag{16}$$

$$\hat{A}^{(n)} : (u,z) \to (\hat{u}^*, \hat{z}^*) = A_\varphi(u+u_0, z+z_{n+1}^{(0)}) - A_\varphi(u_0, z_{n+1}^{(0)}). \tag{17}$$

We remark that from the equalities (1), (2), (13), (16) and (17) it follows that if $\alpha' - u_0 \le u \le \beta' - u_0$, then for any $n \in \mathbf{Z}$ maps $A^{(n)}$ and $\hat{A}^{(n)}$ are the same as the map $\hat{A} : (u,z) \to (\hat{u}, \hat{z})$ having the form

$$\hat{A} : \begin{cases} \hat{u} = u + \hat{z} \\ \hat{z} = z + h\sin(2\pi(u_0+u)) - h\sin(2\pi u_0) \end{cases} \tag{18}$$

with $(0,0)$ as its fixed point. The map $\hat{A}$ preserves area and by (5) the trace $Sp(d\hat{A})$ of matrix

$$d\hat{A}(0,0) = \begin{pmatrix} 1 + 2\pi h\cos(2\pi u_0) & 1 \\ 2\pi\cos(2\pi u_0) & 1 \end{pmatrix},$$

defining the linear part of $\hat{A}$ at the point $(0,0)$, satisfies the inequality $|Sp(d\hat{A})| < 2$. Therefore the point $(0,0)$ is of elliptical type. Further we shall use the results of paper [6] (Theorem 1 and its proof) from which it follows that on the half-line $h > 0$ there exists an open set H, having infinite Lebesgue measure on the straight line, such that for any $h \in H$ there exists u_0 and integer k such that the equations (4), (5), (6) hold and the fix point $(0,0)$ of the map $\hat{A}$ is a point of general elliptical type, i.e., in its normal Birkhoff form (see [9]) $\theta' = \theta + a_0 + a_1\tau + \ldots, \tau' = \tau$ the coefficient $\alpha_1 \neq 0$, where θ, τ are certain polar coordinates with θ the angle and τ the radius. Therefore if $h \in H$, then Moser's theorem [5] can be applied to the map $\hat{A}$. According to this theorem in any neighborhood of the point $(0,0)$ there exists a continuous closed curve around the point $(0,0)$ which is invariant with respect to $\hat{A}$. Let us take the curve γ such that the points $(\tilde{u}, \tilde{z})$ lying inside the domain Γ bounded by this curve, satisfy the inequalities

$$\alpha' - u_0 < \tilde{u} < \beta' - u_0 \ , 2|\tilde{z}| < \epsilon \ , \tag{19}$$

and for any $m \in \mathbf{Z}$ we define the domain $\Pi_m \subset C$ as follows:

$$\Pi_m = \{u, z : u = \tilde{u} + u_0 \ , z = \tilde{z} + z_m^{(0)} \ , (\tilde{u}, \tilde{z}) \in \Gamma\} \ , \tag{20}$$

where $z_m^{(0)}$ are the quantities introduced in (7) for $m = n$. From the invariance of the curve γ with respect to $\hat{A}$, by virtue of (20), (19), (13), (7), (17) and definitions of maps $\hat{A}, A^{(n)}, \hat{A}^{(n)}$ in (18), (16) and (17) for any $m \in \mathbf{Z}$ we obtain the relations

$$A_*(\Pi_m) = \Pi_{m+1} \ , A_\varphi(\Pi_m) = \Pi_{m-1} \ . \tag{21}$$

According to (19), (20) and (7) the coordinate z_m of any point $(u_m, z_m) \in \Pi_m$ satisfies the inequality

$$|z_m - mk| < \frac{\epsilon}{2}\ . \tag{22}$$

We prove now that the domains $\Pi_m (m \in \mathbf{Z})$ satisfy the assertions of Theorem. By virtue of (22) for small ϵ the domains Π_m do not intersect each other for distinct n. Let $(u, z) \in \Pi_m$, $\omega \in \Omega$ and for all natural n we consider points $(u^{(n)}, z^{(n)}, \omega^{(n)}) = B^n(u, z, \omega)$. According to (21) and the definition of B there exists an integer m_n such that the point

$$(u^n, z^{(n)}) \in \Pi_{m_n}\ . \tag{23}$$

Therefore introducing $z^{(n)}$ by means of the formula $z^{(n)} = km_n + \xi_n$, and using (23) and (22) we have

$$|\xi_n| < \frac{\epsilon}{2}\ . \tag{24}$$

We note that the quantity $\eta_n = k(m_n - m)$ is the realization of the sum $\sum_{i=1}^n x_i$ of n independent random values $x_1, \ldots, x_n$ taking the values $x_i = k$ with probability p and the values $x_i = -k$ with probability q.

The mathematical expectation $E \stackrel{def}{=} E\ x_i$ of the variable x_i is equal to $E = k(p-q)$ and its variance $D \stackrel{def}{=} D\ x_i = k^2 - k^2(p-q)^2$, the normalized sums $\frac{1}{\sqrt{nD}}(\eta_n - nE) = \frac{1}{\sqrt{nD}}(\sum_{i=1}^n x_i - nE)$ satisfy the Central Limit Theorem ([11]):
as $n \to \infty$ the probability distributions of the random variables

$$\frac{1}{k\sqrt{n(1-p^2-q^2+2pq)}}(\eta_n - kn(p-q))$$

converge weakly to a normal distribution with density $(2\pi)^{-\frac{1}{2}} \exp(-\frac{x^2}{2})$. Now by the independence of η_n and ξ_n from equalities $z_m^{(n)} = km_n + \xi_n = \eta_n + km + \xi_n$ and from inequality (24) it follows that the same Central Limit Theorem is valid for random variables (3).

Theorem is proved.

The result of this paper was announced in [8].

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