Normalized solutions of Schrödinger equations with potentially bounded measures

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Abstract

Given a potentially bounded signed measure μ on a Brelot space $(X, \mathcal{H})$ with Green function G, it is well known that μ–harmonic functions (i. e., in the classical case, finely continuous versions of solutions to $\Delta u - u\mu = 0$) may be very discontinuous. In this paper it is shown that under very general assumptions on G (satisfied for large classes of elliptic second order linear differential operators) *normalized* perturbation, however, leads to a Brelot space $(X, \tilde{\mathcal{H}}^\mu)$ admitting a Green function $T^\mu(G)$ which is locally (or even globally) comparable with G and has all properties required of G before. In particular, iterated perturbation is possible. Moreover, intrinsic Hölder continuity of quotients of harmonic functions with respect to the local quasimetric $\rho := (G^{-1} + {}^*G^{-1})/2$ yields ρ–Hölder continuity for quotients of μ–harmonic functions as well.

1 Introduction

In [Han99a] the validity of Harnack inequalities for μ–harmonic functions was studied in a general framework covering uniformly elliptic operators as well as sub-Laplacians on stratified Lie algebras and sums of squares of smooth vector fields. Given a potentially bounded signed measure μ and an open set U admitting a locally bounded μ–harmonic function $h_0 > 0$, and Harnack inequalities hold for positive μ–harmonic functions on U (see [Han99a, Theorems 11.4 and 11.6]).

Moreover, it was shown that μ–harmonic functions can be very discontinuous. For example, given any bounded lower semi-continuous function φ there exists a bounded potential $p > 0$ and a potentially bounded signed measure $\mu \leq 0$ such that p is μ–harmonic and as discontinuous as φ, i. e., that $\limsup_{y\to x} p(y)-p(x) = \limsup_{y\to x} \varphi(y)-\varphi(x)$ for every x (see [Han99b] and [Han99a, Proposition 14.1]).

The question, however, if normalized μ–harmonic functions, i. e., quotients h/h_0 (h and h_0 μ–harmonic, $h_0 > 0$) are continuous or even Hölder continuous remained open.

In this paper we intend to show that even in the most general (not necessarily self-adjoint) setting considered in [Han99a] (covering large classes of elliptic second order linear differential equations) normalized μ–harmonic functions are continuous (Theorem 3.5) and lead to a Brelot space (Theorem 4.1) admitting a Green function which is locally (or even globally) comparable with the original one and shares its properties (Theorem 5.1, Theorem 6.1, Theorem 6.3).

This stability result allows us to iterate normalized perturbations (Theorem 5.8). In particular, normalized perturbation by a signed measure μ is really obtained by first perturbing with the positive part μ^+ and then with the negative part $-\mu^-$, given the right density due to the previous normalization (Proposition 5.6).

Moreover, having shown at the beginning that normalized harmonic functions are always Hölder continuous with respect to the intrinsic local quasimetric $\rho : (x,y) \mapsto (G(x,y)^{-1} + G(y,x)^{-1})/2$ (Theorem 2.5) we immediately obtain that normalized μ–harmonic functions are ρ–Hölder continuous as well (Corollary 5.9).

Mostly we shall use only fundamental properties of Green functions and of solutions to the Dirichlet problem. Therefore the results as well as most of the proofs should be accessible also for a reader who is not an expert in (abstract) potential theory.

2 Intrinsic Hölder continuity of harmonic functions

Throughout this paper we shall deal with Brelot spaces $(X, \mathcal{H})$. These spaces can be viewed as a unifying concept which allows us to study simultaneously large classes of elliptic second order linear partial differential operators on open subsets of $\mathbb{R}^d$ (or, more generally, on smooth manifolds). These operators may e.g. be of the following type:

(2.1)
$$\mathcal{L} = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c$$

(a_{ij}, b_i, c (Hölder) continuous, $\xi \mapsto \sum a_{ij}(x)\xi_i\xi_j$ positive definite) or

(2.2)
$$\mathcal{L} = \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{d} a_{ij} \frac{\partial}{\partial x_i} + d_i\right) + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c$$

(a_{ij} measurable, bounded, $(a_{ij}(x))$ uniformly elliptic, mild assumptions on d_i, b_i, c) or

(2.3) $$\mathcal{L} = \sum_{j=1}^{r} X_j^2 + Y$$

($X_1, \ldots, X_r, Y$ smooth vector fields satisfying Hörmander's condition for hypoellipticity of $\sum_{j=1}^{r} X_j^2$). The reader may consult Section 3 of [Han99a] for further details.

Let us fix once and for all a non-empty locally compact and connected topological space X with countable base. For every Borel A of X, let $\mathcal{B}(A)$ ($\mathcal{C}(A)$ resp.) denote the set of all Borel measurable numerical functions (continuous real functions resp.) on A. A subscript "b" (a superscript "+" resp.) will indicate the subset of all bounded functions (positive functions resp.)

Given a numerical function G on $X \times X$, let ${}^*G$ denote the adjoint function, i.e.,

$${}^*G(x, y) := G(y, x) \qquad (x, y \in X).$$

Of course, we shall say that G is symmetric or self-adjoint if $G = {}^*G$.

Let $\mathcal{G}(X)$ denote the set of all measurable functions $G : X \times X \to [0, \infty]$ having the following two properties:

1. G and ${}^*G$ are Green functions for Brelot spaces $(X, \mathcal{H}_G)$ and $(X, \mathcal{H}_{{}^*G})$, respectively, such that the constant 1 is superharmonic.

2. X can be covered by open sets U where G and ${}^*G$ are comparable and $\rho : (x, y) \mapsto (G(x, y)^{-1} + {}^*G(x, y)^{-1})/2$ is a quasimetric, i.e., such that there exists $c \in \mathbb{R}^+$ (which may depend on U) such that, for all $x, y, z \in U$,

   (2.4) $${}^*G(x, y) \le c\, G(x, y), \qquad \rho(x, y) \le c(\rho(x, z) + \rho(z, y))$$

   (and $\rho(x, y) = \rho(y, x)$, $\rho(x, y) = 0$ if and only if $x = y$).

In general, notions with respect to $\mathcal{H}_G$, $G \in \mathcal{G}(X)$, will be distinguished by a prefix or subscript G. E.g., we shall speak about G–harmonic functions or the G–fine topology. If we are dealing with a fixed $G \in \mathcal{G}(X)$, however, we shall usually drop it.

Remarks 2.1. 1. The relationship between G and the pair $(\mathcal{H}_G, \mathcal{H}_{{}^*G})$ is almost one-to-one: Given $G \in \mathcal{G}(X)$, we have $aG \in \mathcal{G}(X)$ and $\mathcal{H}_{aG} = \mathcal{H}_G$, $\mathcal{H}_{a{}^*G} = \mathcal{H}_{{}^*G}$ for every real $a > 0$. Conversely, if $\tilde{G} \in \mathcal{G}(X)$ such that $\mathcal{H}_{\tilde{G}} = \mathcal{H}_G$ and $\mathcal{H}_{{}^*\tilde{G}} = \mathcal{H}_{{}^*G}$, then $\tilde{G} = aG$ for some real $a > 0$.

2. $\mathcal{G}(X)$ is stable under normalization by positive continuous G–superharmonic functions and ${}^*G$–superharmonic functions: If s and s^* are continuous functions on X which are G–superharmonic and ${}^*G$–superharmonic, respectively, bounded and bounded away from 0, then $\tilde{G} = G/(s \otimes s^*) \in \mathcal{G}(X)$, $\mathcal{H}_{\tilde{G}} = (1/s)\mathcal{H}_G$, and $\mathcal{H}_{{}^*\tilde{G}} = (1/s^*)\mathcal{H}_{{}^*G}$. Moreover, this possibility of normalization shows that G– and

G–superharmonicity of the constant 1 is an assumption made almost without loss of generality.

3. Let us note that $\rho(x,x) = 0$ is equivalent to $G(x,x) = \infty$. So we exclude examples where the points are non-polar, i.e., mainly examples where the fine topology coincides with the initial topology and all μ–harmonic functions are continuous anyway.

Assuming that G and $^*G$ are locally comparable, the generalized triangle inequality for ρ is equivalent to the *local triangle property* (LT) for G used in [Han99a]. Consequently, Green functions for operators of the form (2.1), (2.2) or (2.3) have the properties required for functions in $\mathcal{G}(X)$ (see Section 9 of [Han99a]).

Indeed, by definition G has the local triangle property provided the space X can be covered by open sets U such that for some $C \in \mathbb{R}^+$

$$\min\bigl(G(x,z), G(z,y)\bigr) \leq C\, G(x,y) \quad \text{for all } x, y, z \in U. \tag{2.5}$$

Multiplication by $\max(G(x,z), G(z,y))$ transforms (2.5) into

$$G(x,z) \cdot G(z,y) \leq C\, G(x,y) \cdot \max\bigl(G(x,z), G(z,y)\bigr)$$

or, equivalently,

$$G(x,y)^{-1} \leq C \max\bigl(G(x,z)^{-1}, G(z,y)^{-1}\bigr) \quad \text{for all } x, y, z \in U.$$

Therefore (and using $\max(a,b) \leq a + b$) inequality (2.5) implies that

$$\rho(x,y) \leq C\, \bigl(\rho(x,z) + \rho(z,y)\bigr) \quad \text{for all } x, y, z \in U.$$

Let $c \in \mathbb{R}^+$ such that $^*G \leq c\,G$ on $U \times U$. Then

$$\frac{1}{G(x,y)} \leq 2\rho(x,y) \leq \frac{c+1}{G(x,y)} \quad \text{for all } x, y \in U. \tag{2.6}$$

So, conversely, an inequality $\rho(x,y) \leq c'(\rho(x,z) + \rho(z,y))$ leads to

$$\frac{1}{G(x,y)} \leq 2\rho(x,y) \leq 2c' \max\bigl(\rho(x,z), \rho(z,y)\bigr) \leq 4c'(c+1) \max\Bigl(\frac{1}{G(x,z)}, \frac{1}{G(z,y)}\Bigr)$$

for $x, y, z \in U$.

The following will now be obvious for the expert:

Lemma 2.2. *Let $G \in \mathcal{G}(X)$ and let A be any compact subset of X. Then there exists a constant $C > 0$ such that the inequalities (2.4) hold for all $x, y, z \in A$.*

For the convenience of the reader who is not so familiar with potential theory we include a short proof.

Proof of Lemma 2.2. There exist finitely many open subsets $U_1, \dots, U_m$ and constants $C_i > 0$, $1 \le i \le m$, such that the diagonal in $A \times A$ is covered by the sets $U_1 \times U_1, \dots, U_m \times U_m$ and

$$G(y,x) \le C_i\, G(x,y) \quad \text{and} \quad \min(G(x,z), G(z,y)) \le C_i\, G(x,y) \tag{2.7}$$

for all $1 \le i \le m$ and $x, y, z \in U_i$. We fix a metric d for the topology on X, choose $\varepsilon > 0$ such that the neighborhood

$$N := \{(x,y) \in A \times A \colon d(x,y) < 2\varepsilon\}$$

of the diagonal in $A \times A$ is covered by the sets $U_i \times U_i$, $1 \le i \le m$, and define

$$R := \{(x,y) \in A \times A \colon d(x,y) \ge \varepsilon\}.$$

Since R is a compact subset of $X \times X$ not intersecting the diagonal, we have $\inf_R G > 0$ and $\sup_R G < \infty$. So there is a real $C \ge \max(C_1, \dots, C_m)$ satisfying

$$\sup_R G \le C \inf_R G. \tag{2.8}$$

Applying the minimum principle to the functions $G(\cdot, y)$, $y \in A$, we obtain that $\inf_R G \le G$ on N whence, by (2.8),

$$\sup_R G \;\le\; C\, G \quad \text{on } N. \tag{2.9}$$

Finally, fix $x, y, z \in A$. If $(x,y) \in N$, then $(x,y) \in U_i \times U_i$ for some $1 \le i \le m$ and therefore

$$G(y,x) \le C\, G(x,y) \quad \text{and} \quad \min(G(x,z), G(z,y)) \le C\, G(x,y) \tag{2.10}$$

by (2.7) and (2.9). If $(x,y) \notin N$, then $(x,z) \in R$ or $(z,y) \in R$ and (2.10) follows from (2.8). □

For the present, let us fix a function $G \in \mathcal{G}(X)$ and take $\mathcal{H} = \mathcal{H}_G$. Clearly, any result with respect to G immediately implies an adjoint result with respect to ${}^*G$, since ${}^{**}G = G$ and therefore ${}^*G \in \mathcal{G}(X)$.

Lemma 2.3. *The G–fine topology and the ${}^*G$–fine topology coincide. In particular, every bounded G–potential on X is a countable sum of continuous G–potentials.*

Proof. Let T be a subset of X which is G–thin at a point $x \in \overline{T} \setminus T$ and let A be a compact neighborhood of x. Then there exists a measure $\rho \ge 0$ on A such that

$$G^\rho 1(x) \le 1 \quad \text{and} \quad \lim_{y \in T, y \to x} G^\rho 1(y) = \infty.$$

Taking $c > 0$ such that ${}^*G \le cG$ on $A \times A$ we obtain that

$${}^*G^\rho 1(x) \le c \quad \text{and} \quad \lim_{y \in T, y \to x} {}^*G^\rho 1(y) = \infty$$

(since of course $G \le c\,{}^*G$ as well). Thus T is ${}^*G$–thin at x.

Replacing G by ${}^*G$ we get the converse implication. Thus the G–fine topology is the ${}^*G$–fine topology. An application of [JNXL75, Proposition 2.2] finishes the proof. □

Let $\mathcal{V}$ denote the family of all relatively compact regular domains in X (by Lemma 2.3 there is no difference between G–regular and ${}^*G$–regular). Let us note that every compact subset A of X is contained in some $V \in \mathcal{V}$. For every $V \in \mathcal{V}$, we have a harmonic kernel H_V (solving the Dirichlet problem for V) and a Green function G_V.

More generally, for every open set U in X, there is a (generalized) harmonic kernel H_U and a Green function G_U on U obtained by

$$G_U(\cdot, y) = G(\cdot, y) - H_U G(\cdot, y) \qquad (y \in U)$$

(note that $H_X = 0$ and $G_X = G$). For every measure $\rho \geq 0$ on X and for every $f \in \mathcal{B}^+(X)$, we define

$$G_U^\rho f := \int_U G_U(\cdot, y) f(y)\, \rho(dy)$$

so that

$$G_U^\rho f + H_U G^\rho f = G^\rho f.$$

Of course, we extend this definition in the usual way to arbitrary signed measures ρ and arbitrary functions $f \in \mathcal{B}(X)$ whenever the integrals make sense.

Remark 2.4. *Let U be an open subset of X. Then the function G_U is locally comparable with the restriction of G on $U \times U$ (cf. e.g. [Han99a, Remark 9.1]) and ${}^*(G_U) = ({}^*G)_U$ (see [Her62, Proposition 31.3]). Therefore $G_U \in \mathcal{G}(U)$, and G_U is symmetric if G is symmetric.*

Let us recall that, given any quasimetric ρ on a set Y, there exists a metric d on Y and $\gamma, C \in \mathbb{R}^+$ such that

$$C^{-1} d^{\,\gamma} \leq \rho \leq C\, d^{\,\gamma}. \tag{2.11}$$

In fact, if $\rho(x, y) \leq C \max(\rho(x, z), \rho(z, y))$ for all $x, y, z \in Y$, then (2.11) holds for any $\gamma \geq 2 \ln_2 C$ and

$$d(x, y) := C^{1/\gamma} \inf\Big\{ \sum_{j=1}^{k} \rho^{1/\gamma}(z_{j-1}, z_j) \colon k \in \mathbb{N}, z_j \in Y, z_0 = x, z_k = y \Big\}$$

(see [Hei01, Proposition 14.5]).

For our quasimetric $\rho : (x, y) \mapsto (G(x, y)^{-1} + G(y, x)^{-1})/2$ let us introduce ρ–balls:

$$B(x_0, R) := \{x \in X : \rho(x, x_0) < R\} \qquad (x_0 \in X,\ R > 0).$$

It is easily seen that, for every $x_0 \in X$, the balls $B(x_0, R)$, $R > 0$, form a fundamental system of open neighborhoods of x_0. We obtain the following result on uniform Hölder continuity:

Theorem 2.5. *1. Quotients of harmonic functions are ρ–Hölder continuous. More precisely, for every $V \in \mathcal{V}$, there exist constants $\eta, \alpha, C \in \mathbb{R}^+$ such that, for every $x_0 \in V$ and for all harmonic functions g, h on a ball $B(x_0, R) \subset V$ such that $h > 0$,*

$$\left| \frac{g(x)}{h(x)} - \frac{g(x_0)}{h(x_0)} \right| \le C \left(\frac{\rho(x, x_0)}{r} \right)^{\eta} \operatorname*{osc}_{B(x_0,R)} \frac{g}{h}$$

for all $x \in B(x_0, R)$.

2. If there exists a metric d on X and $\gamma \in \mathbb{R}^+$ such that, for every $\delta > 0$, the space X can be covered by open sets W with

$$(1 - \delta) d^{-\gamma} \le G \le (1 + \delta) d^{-\gamma} \quad \text{on } W \times W, \tag{2.12}$$

then, for every $\varepsilon > 0$, quotients of harmonic functions are d–Hölder continuous with exponent $\eta = 1 - \varepsilon$.

To prove Theorem 2.5 we first show that harmonic functions satisfy scaling invariant Harnack inequalities and then apply a trick already used in [Mos61, Mos64].

Lemma 2.6. *Let $V \in \mathcal{V}$, h_1 a ${}^*G$–harmonic function on V, d a metric on V, and $c_1, c_2, \gamma > 0$ such that $c_1^{-1} \le h_1 \le c_1$ on V and $c_2^{-1} d^{-\gamma} \le G \le c_2 d^{-\gamma}$ on $V \times V$. Moreover, let $0 < \alpha < \beta$, $3\beta < c_2^{-2/\gamma}$ and define*

$$c := \frac{c_2^2 (\beta - \alpha)^{-\gamma} - c_1^{-2} (1 + \alpha)^{-\gamma}}{c_2^{-2} (\beta + \alpha)^{-\gamma} - (1 - \alpha)^{-\gamma}}.$$

Finally, let $x_0 \in V$, $r_0 > 0$, define

$$U_r := \{x \in V \colon d(x, x_0) < r\} \qquad (0 < r < r_0),$$

and suppose that $\overline{U}_{r_0} \subset V$.

Then, for every $0 < r \le r_0$ and for every harmonic function $h \ge 0$ on U_r,

$$\sup\nolimits_{U_{\alpha r}} h \le c \inf\nolimits_{U_{\alpha r}} h.$$

Proof. Fix $0 < r \le r_0$. If $0 < r_1 < r_2 \le r$, $x \in U_{r_1}$ and $z \in V \setminus U_{r_2}$, then $r_2 - r_1 \le d(x, z) \le r_1 + r_2$ and therefore

$$G(x, z), G(z, x) \in \left[c_2^{-1} (r_1 + r_2)^{-\gamma},\, c_2 (r_2 - r_1)^{-\gamma} \right]. \tag{2.13}$$

Let ${}^*H_{U_r}$ denote the ${}^*G$–harmonic kernel for U_r. Since 1 is ${}^*G$–superharmonic and $1 \ge c_1^{-1} h_1 \ge c_1^{-2}$ on V, we know that $1 \ge {}^*H_{U_r} 1 \ge c_1^{-1} h_1 \ge c_1^{-2}$. Taking $r_1 = \alpha r$ and $r_2 = r$ we infer from (2.13) that, for all $x \in U_{\alpha r}$,

$$c_1^{-2} c_2^{-1} (1 + \alpha)^{-\gamma} r^{-\gamma} \le {}^*H_{U_r} {}^*G(\cdot, x) \le c_2 (1 - \alpha)^{-\gamma} r^{-\gamma} \quad \text{on } U_r. \tag{2.14}$$

Let $S_{\beta r} := \partial U_{\beta r}$. Then $d(z,x) = \beta r$ for all $z \in S_{\beta r}$. Taking again $r_1 = \alpha r$, but $r_2 = \beta r$, we now conclude from (2.13) and (2.14) that, for all $x \in U_{\alpha r}$ and $z \in S_{\beta r}$,

$$\begin{aligned} 0 &< \big(c_2^{-1}(\beta+\alpha)^{-\gamma} - c_2(1-\alpha)^{-\gamma}\big) r^{-\gamma} \le {}^*G_{U_r}(z,x) \\ &= {}^*G(z,x) - ({}^*H_{U_r}{}^*G(\cdot,x))(z) \le \big(c_2(\beta-\alpha)^{-\gamma} - c_1^{-2}c_2^{-1}(1+\alpha)^{-\gamma}\big) r^{-\gamma}. \end{aligned}$$

By definition of c and since ${}^*G_{U_r}(z,x) = G_{U_r}(x,z)$, this proves that

$$G_{U_r}(x,z) \le c\, G_{U_r}(y,z) \qquad \text{for all } x,y \in U_{\alpha r} \text{ and } z \in S_{\beta r}. \tag{2.15}$$

Finally, fix a harmonic function $h \ge 0$ on U_r. Then

$$p := \inf\{s : s \ge 0 \text{ superharmonic on } U_r,\ s \ge h \text{ on } U_{\beta r}\}$$

is a continuous potential on U_r which is harmonic outside $S_{\beta r}$ and coincides with h on $U_{\beta r}$. There exists a measure $\rho \ge 0$ on $S_{\beta r}$ such that

$$p = G^{\rho}_{U_r} 1 = \int G_{U_r}(\cdot, z)\, \rho(dz).$$

Integrating (2.15) with respect to ρ we conclude that $h(x) = p(x) \le cp(y) = ch(y)$ for all $x, y \in U_{\alpha r}$. □

Proof of Theorem 2.5. 1. Fix $V \in \mathcal{V}$ and let $h_1 := {}^*H_V 1$. Then h_1 is ${}^*G$–harmonic on V and there exists $c_1 \ge 1$ such that $c_1^{-1} \le h_1 \le 1$. Since ρ is a quasimetric on V which can be compared with G^{-1}, there exists a metric d on V and $\gamma, c_2 > 0$ such that

$$c_2^{-1} d^{-\gamma} \le G \le c_2 d^{-\gamma} \qquad \text{on } V \times V.$$

Choose $0 < \alpha < \beta$ such that $3\beta < c_2^{-2/\gamma}$ and define $c > 0$ as in Lemma 2.6.

Fix $x_0 \in V$ and $R > 0$ such that $B(x_0, R) \subset V$. Let $r_0 := \big(R/(2c_2)\big)^{1/\gamma}$. Then $\overline{U}_{r_0} \subset B(x_0, R)$, since $c_2^{-1} d^{\gamma} \le \rho \le c_2 d^{\gamma}$. Consequently, for every $0 < r \le r_0$ and every harmonic function $h \ge 0$ on U_r,

$$\sup_{U_{\alpha r}} h \le c \inf_{U_{\alpha r}} h$$

by Lemma 2.6. Define

$$\eta := \frac{\ln(c^2/(c^2-1))}{\gamma \ln(1/\alpha)}, \quad C_0 := \max\Big(\frac{c^4}{(c^2-1)^2}, \alpha^{-\gamma\eta}\Big), \quad C := c_2^{2\eta} C_0,$$

and let $g, h \in \mathcal{H}(B(x_0, R)$, $h > 0$. By Proposition 7.1 and Remark 7.2,

$$\Big|\frac{g}{h}(x) - \frac{g}{h}(x_0)\Big| \le C_0 \left(\frac{d(x_0,x)}{r_0}\right)^{\gamma\eta} \operatorname*{osc}_{B(x_0,R)} \frac{g}{h} \tag{2.16}$$

for all $x \in U_{\alpha r_0}$. If $x \in B(x_0, R) \setminus U_{\alpha r_0}$, then $d(x, x_0) \ge \alpha r_0$ and (2.16) holds since $C_0 \alpha^{\gamma\eta} \ge 1$. To finish the first part of the proof it suffices to note that $(d/r_0)^{\gamma} \le 2c_2^2 \rho/R$ on $V \times V$.

2. Assume next that (2.12) holds. Fix $0 < \varepsilon < 1$, let $0 < \alpha < (1/3)^{1/\varepsilon}$, and define $\beta := \alpha^\varepsilon$. If we replace the constants c_1, c_2 in the definition of c in Lemma 2.6 by 1 we obtain

$$c_0 := \frac{(\beta-\alpha)^{-\gamma} - (1+\alpha)^{-\gamma}}{(\beta+\alpha)^{-\gamma} - (1-\alpha)^{-\gamma}} = \frac{(1-\alpha^{1-\varepsilon})^{-\gamma} - \alpha^{\varepsilon\gamma}(1+\alpha)^{-\gamma}}{(1+\alpha^{1-\varepsilon})^{-\gamma} - \alpha^{\varepsilon\gamma}(1-\alpha)^{-\gamma}}$$

and hence

$$1 - \frac{1}{c_0} = \frac{(1-\alpha^{1-\varepsilon})^{-\gamma} - (1+\alpha^{1-\varepsilon})^{-\gamma} - \alpha^{\varepsilon\gamma}((1+\alpha)^{-\gamma} - (1-\alpha)^{-\gamma})}{(1-\alpha^{1-\varepsilon})^{-\gamma} - \alpha^{\varepsilon\gamma}(1+\alpha)^{-\gamma}}$$

where, letting $\alpha \to 0$, the denominator tends to 1 and the nominator is approximately $2\gamma\alpha^{1-\varepsilon}$ (the term $\alpha^{\varepsilon\gamma}2\gamma\alpha$ being small with respect to $\alpha^{1-\varepsilon}$). Taking α sufficiently small we therefore have

$$\frac{c_0^2}{c_0^2-1} = \left(1 - \frac{1}{c_0}\right)^{-1}\left(1 + \frac{1}{c_0}\right)^{-1} > \left(\frac{1}{\alpha}\right)^{1-2\varepsilon}.$$

By continuity there exists $\delta > 0$ such that $3\beta < (1+\delta)^{-2/\gamma}$, and

$$\frac{c^2}{c^2-1} > \left(\frac{1}{\alpha}\right)^{1-2\varepsilon} \tag{2.17}$$

provided $c_1 = c_2 = 1 + \delta$.

Fix $x_0 \in V$ and let h_1 be a function which is harmonic on a neighborhood V_1 of x_0 in V and satisfies $h_1(x_0) = 1$. There exists $W \in \mathcal{V}$ such that $x_0 \in W \subset V_1$,

$$(1+\delta)^{-1} \le h \le 1+\delta \quad \text{on } W, \quad (1+\delta)^{-1}d^{-\gamma} \le G \le (1+\delta)d^{-\gamma} \quad \text{on } W \times W.$$

An application of (2.16) and (2.17) finally yields that quotients of harmonic functions on W are d–Hölder continuous with exponent $1 - 2\varepsilon$. □

3 Continuity of normalized μ–harmonic functions

Given $G \in \mathcal{G}(X)$, let $\mathcal{M}_b^+(G)$ denote the set of all measures $\mu \ge 0$ on X which are (globally) G–*bounded*, i.e., such that the G–potential

$$G^\mu 1 := \int G(\cdot, z)\,\mu(dz)$$

is bounded, and let $\mathcal{M}_b(G)$ be the set of all signed measures μ on X such that $\mu^\pm \in \mathcal{M}_b^+(G)$.

Let us begin with a simple result (see [Han99a, Lemma 10.2]):

Lemma 3.1. *Let $G \in \mathcal{G}(X)$ and $V \in \mathcal{V}$. Then there exists a constant $c > 0$ such that, for every $\mu \in \mathcal{M}_b^+(G)$,*

$$\big(G^\mu G(\cdot, y)\big)(x) \le c\|G^\mu 1\|_\infty\, G(x,y) \quad \textit{and} \quad G^{1_V \mu} s \le c\|G^{1_V\mu}\|_\infty\, s$$

for all $x, y \in V$ and for every superharmonic function $s \ge 0$ on X.

Proof. Choose $W \in \mathcal{V}$ containing $\overline{V}$ and $c_0 > 0$ such that

$$G(y,x) \le c_0\, G(x,y) \quad \text{and} \quad \min\big(G(x,z), G(z,y)\big) \le c_0\, G(x,y)$$

for all $x, y, z \in W$. Define

$$c_1 := \inf\{G(x,y) \colon x, y \in \overline{V}\}, \qquad c_2 := \sup\{G(z,y) \colon z \in X \setminus W,\, y \in \overline{V}\},$$

and note that $c_1 > 0$ and $c_2 < \infty$. Now fix $\mu \in \mathcal{M}_b^+(G)$ and $x, y \in V$. We have

$$\big(G^\mu G(\cdot, y)\big)(x) = \int G(x,z) G(z,y)\, \mu(dz)$$

where the integral on $X \setminus W$ can be estimated by

$$c_2 \int_{X \setminus W} G(x,z)\, \mu(dz) \le \frac{c_2}{c_1} \|G^\mu 1\|_\infty\, G(x,y)$$

and the integral on W by

$$c_0 G(x,y) \int_W (G(x,z) + G(z,y))\, \mu(dz) \le c_0(1+c_0)\|G^\mu 1\|_\infty\, G(x,y).$$

So the first inequality of our Lemma follows taking $c := (c_2/c_1) + c_0(1+c_0)$.

Finally, let $s \ge 0$ be superharmonic on X. Then there exist measures $\rho_n \ge 0$ on V, $n \in \mathbb{N}$, such that $G^{\rho_n} \uparrow s$ on V as $n \uparrow \infty$. For every $n \in \mathbb{N}$,

$$G^{1_V\mu} G^{\rho_n} = \int_V G^{1_V\mu} G(\cdot, y)\, \rho_n(dy) \le c\|G^{1_V\mu}\|_\infty \int_V G(\cdot, y)\, \rho_n(dy) = c\|G^{1_V\mu}\|_\infty G^{\rho_n}$$

on V and therefore on X. The proof is finished letting n tend to ∞. □

Remark 3.2. *It is easily seen that the result holds as well if $\mu \in \mathcal{M}_b^+({}^*G)$.*

If $\mu \in \mathcal{M}_b^+(G)$, then each operator $I + G_U^\mu$ on $\mathcal{B}_b(U)$, U open in X, is invertible and

$$\exp(-G_U^\mu s/s)\, s \le (I + G_U^\mu)^{-1} s \le s \tag{3.1}$$

for every bounded superharmonic $s \ge 0$ on U (see [Han99a, Section 4]). This implies that, for all $\mu, \nu \in \mathcal{M}_b^+(X)$, $(I + G_U^\mu)^{-1} G_U^\nu$ defines a bounded kernel on U (denoted by $(I + G_U^\mu)^{-1} G_U^\nu$ as well) and that $(I + G_U^\mu)^{-1} = I - (I + G_U^\mu)^{-1} G_U^\mu$ is a difference of two sub-Markov kernels.

We shall say that $\mu \in \mathcal{M}_b(G)$ is G*–admissible*, if the function

$$\sum_{n=0}^{\infty} \big((I+G^{\mu^+})^{-1}G^{\mu^-}\big)^n\, 1 \tag{3.2}$$

is bounded. The set of all admissible $\mu \in \mathcal{M}_b(G)$ will be denoted by $\mathcal{M}_{ab}(G)$. Of course, $\mathcal{M}_b^+(G) \subset \mathcal{M}_{ab}(G)$.

Assume that $\mu \in \mathcal{M}_{ab}(G)$ and let U be an open subset of X. As in [Han99a] let us define a kernel L_U^μ on U by

$$L_U^\mu := (I+G_U^{\mu^+})^{-1}G_U^{\mu^-}. \tag{3.3}$$

Then

$$L_U^\mu \le G_U^{\mu^-} \quad \text{and} \quad L_W^\mu \le L_U^\mu \le L^\mu := (I+G^{\mu^+})^{-1}G^{\mu^-} \tag{3.4}$$

for every open subset W of U (see (3.1) and [Han99a, Proposition 8.3]). In particular, the kernel

$$S_U^\mu := \sum_{n=0}^{\infty} (L_U^\mu)^n$$

is bounded, and therefore the operator $I+G_U^\mu$ is invertible on $\mathcal{B}_b(U)$,

$$(I+G_U^\mu)^{-1} = S_U^\mu (I+G_U^{\mu^+})^{-1} = \sum_{n=0}^{\infty} \big((I+G_U^{\mu^+})^{-1}G_U^{\mu^-}\big)^n\,(I+G_U^{\mu^+})^{-1}. \tag{3.5}$$

Moreover, $(I+G_U^\mu)^{-1}t \ge (I+G_U^{\mu^+})^{-1}t \ge 0$ for every bounded superharmonic function $t \ge 0$ on U, and the function $s := (I+G_U^\mu)^{-1}1$ satisfies

$$0 < \exp(-\|G_U^{\mu^+}1\|_\infty) \;\le\; s \;\le\; \Big\|\sum_{n=0}^{\infty} \big((I+G_U^{\mu^+})^{-1}G_U^{\mu^-}\big)^n 1\Big\|_\infty < \infty. \tag{3.6}$$

Let us mention the following characterization of the set $\mathcal{M}_{ab}(G)$ (see Proposition 7.3 for further equivalences):

Lemma 3.3. *Let $\mu \in \mathcal{M}_b(G)$. Then the following properties are equivalent:*

1. *μ is G–admissible.*
2. *The spectral radius of the operator $L^\mu = (I+G^{\mu^+})^{-1}G^{\mu^-}$ is strictly less than* 1.
3. *The operator $I+G^\mu$ is invertible and $(I+G^\mu)^{-1}1 \ge 0$.*

Proof. By Proposition 7.3, (1) and (2) are equivalent. By (3.6), it remains to show that (3) implies (1) (see [HM90, Lemma 2.1]). So assume that $s := (I+G^\mu)^{-1}1$ is positive and let $L := L^\mu$. Then

$$t := (I+G^{\mu^+})^{-1}1 = s - Ls.$$

By induction, $\sum_{j=0}^{n-1} L^j t = s - L^n s$ for every $n \in \mathbb{N}$ and hence $\sum_{n=0}^{\infty} L^n t \le s$. The proof is finished since $t \ge \exp(-\|G^{\mu^+}1\|_\infty) > 0$. ◻

Given $G \in \mathcal{G}(X)$ and $\mu \in \mathcal{M}_b(G)$, a measurable real function h on an open subset U of X is called *μ–harmonic* (*quasi-μ–harmonic* resp.) on U if h is finely continuous (finely continuous outside a polar set) and if, for every $V \in \mathcal{V}$ with $\overline{V} \subset U$, there exists a harmonic function g on V and a polar subset P of V such that $G_V^{|\mu|}|h| < \infty$ on $V \setminus P$ and

$$h + G_V^{\mu} h = g \quad \text{on } V \setminus P \tag{3.7}$$

(it suffices to know this for sets V covering U). In fact,

$$g = H_V h \tag{3.8}$$

(see [Han99a, Lemma 3.6]). For the connection with solutions of Schrödinger equations

$$\mathcal{L}u - u\mu = 0$$

we refer the reader to [Han99a, Section 3].

To simplify our considerations let us fix $G \in \mathcal{G}(X)$ and assume that

$$\boxed{\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G).}$$

Theorem 3.4. *For every domain U in X the following holds:*

1. *For every quasi-μ–harmonic function h on U, there exists a (unique) μ–harmonic function $\tilde{h}$ such that $\tilde{h} = h$ outside a polar set.*
2. *Every μ–harmonic function on U is locally bounded.*
3. *Positive μ–harmonic functions on U satisfy Harnack inequalities, i. e., for every compact A in U, there exists a constant $c > 0$ such that*

$$\sup_A h \le c \inf_A h$$

   for every μ–harmonic function $h \ge 0$ on U.

Let us recall that μ–harmonic functions may be very discontinuous: Assume that there exists a harmonic function $g \ge 1$ on X and let φ be any lower semicontinuous function on X such that $0 \le \varphi \le 1$. By [Han99b], there exists a potential $p = G^\rho$ on X such that $0 \le p \le 2$ and, for every $x \in X$, $\limsup_{y \to x} p(y) - p(x) = \limsup_{y \to x} \varphi(y) - \varphi(x)$. Define $h := 3g + p$. Then

$$\mu := -\frac{1}{h}\,\rho \in \mathcal{M}_{ab}(X),$$

since $G^{h^{-1}\rho} 1 \le (1/3) G^\rho 1 \le 2/3$. Moreover,

$$h + G^\mu h = h - G^\rho = 3g \in \mathcal{H}(X)$$

and therefore h is μ–harmonic.

Normalized μ–harmonic functions, however, are always continuous:

Theorem 3.5. *Let h_0 be a strictly positive μ–harmonic function on an open set U in X. Then, for every μ–harmonic function h on U, the quotient h/h_0 is continuous.*

Remark 3.6. *The trivial identity*

$$\frac{h}{h_0} - \frac{h(x)}{h_0(x)} = \frac{1}{h_0}\left(h - \frac{h(x)}{h_0(x)}\,h_0\right)$$

shows that continuity of quotients h/h_0 is equivalent to continuity of μ–harmonic functions at points where they are vanishing.

For the present let us fix an open subset U of X and $V \in \mathcal{V}$ such that $\overline{V} \subset U$.

The proof of Theorem 3.5 will be based on [Han99a, Lemma 11.3] which already was the key for the results collected in Theorem 3.4. But before let us state the following improvement of [Han99a, Lemma 9.6]:

Lemma 3.7. *There exist $a \geq 1$ and $Q \in \mathbb{N}$ such that*

$$\min_{1\le j\le n} G_U(z_{j-1}, z_j) \;\le\; a\, n^Q\, G_U(z_0, z_n), \qquad \min_{1\le j\le n} {}^*G_U(z_{j-1}, z_j) \;\le\; a\, n^Q\, {}^*G_U(z_0, z_n)$$

for every $n \in \mathbb{N}$ and all choices of points $z_0, z_1, \ldots, z_{n-1} \in V$, $z_n \in U$.

Proof. Choose $W \in \mathcal{V}$ such that $\overline{V} \subset W$ and $\overline{W} \subset U$. Since $G_U \in \mathcal{G}(U)$ by Remark 2.4, there exists a constant $C > 0$ such that

$$\min(G_U(x,z), G_U(z,y)) \;\le\; C\, G_U(x,y)$$

for all $x, y, z \in W$ and therefore, by [Han99a, Lemma 9.5],

$$\min_{1\le j\le n} G_U(z_{j-1}, z_j) \;\le\; C\, n^{\log_2 C}\, G_U(z_0, z_n)$$

for all $n \in \mathbb{N}$ and $z_0, \ldots, z_n \in W$. Let us observe that we could as well have used the fact that, on the set $W \times W$, the inverse G_U^{-1} is comparable to some power of a metric d (see (2.11)) and that $d(z_{j-1}, z_j) \geq n^{-1} d(z_0, z_n)$ for some $1 \leq j \leq n$.

If $z_0, \ldots, z_{n-1} \in V$ and $z_n \in U \setminus W$, we have a Harnack inequality $G_U(z_{n-1}, z_n) \leq c\, G_U(z_0, z_n)$. The proof is finished replacing G by ${}^*G$. □

Define

$$\alpha_n := \big\| \big((I + G_U^{\mu^+})^{-1} G_U^{\mu^-}\big)^n 1 \big\|_\infty, \qquad \alpha_n^* := \big\| \big((I + {}^*G_U^{\mu^+})^{-1}\,{}^*G_U^{\mu^-}\big)^n 1 \big\|_\infty.$$

Then [Han99a, Lemma 11.1] yields the following (where a, Q are as in Lemma 3.7 and we have to take into account that G is not necessarily symmetric):

Lemma 3.8. *For all $x \in V$, $y \in U$, and $n \in \mathbb{N}$,*

$$\big((L_U^{\mu^+ - 1_V \mu^-})^n G_U(\cdot, y)\big)(x) \le a(n+1)^Q G_U(x,y)\big(\alpha_n^* + \|G_U^{\mu^-} 1\|_\infty\big) \sum_{j=0}^{n-1} \alpha_j^* \alpha_{n-1-j}\big).$$

Remark 3.9. *Our considerations in Section 5 will show that it would be sufficient to know Lemma 3.8 and its consequence Proposition 3.10 for the case* $\mu^+ = 0$ *where* $L_U^{\mu^+ - 1_V \mu^-} = G_U^{1_V \mu^-}$.

Define

$$c_U^\mu(Q) := (1 + \|G_U^{\mu^-} 1\|_\infty) \sum_{n=0}^{\infty} (n+2)^Q \alpha_n \sum_{n=0}^{\infty} (n+2)^Q \alpha_n^*.$$

By Lemma 3.1, there exists $b > 0$ such that,

$$G_U^\nu s \leq b \, \|G_U^\nu\|_\infty s$$

for every superharmonic $s \geq 0$ on U and for every $\nu \in \mathcal{M}_b^+(G_U)$ supported by V. Then Lemma 3.8 leads to the following version of [Han99a, Proposition 11.2]:

Proposition 3.10. *For every superharmonic function* $s \geq 0$ *on* U,

$$\sum_{n=0}^{\infty} (L_U^{\mu^+ - 1_V \mu^-})^n s \leq (1 + abc_U^\mu(Q) \, \|G_U^{1_V \mu^-}\|_\infty) \, s.$$

Proof. Let $s \geq 0$ be superharmonic on U and $M := L_U^{\mu^+ - 1_V \mu^-}$. Then Lemma 3.8 implies that $\sum_{n=0}^{\infty} M^n s \leq ac_U^\mu(Q) \, s$ on V (see [Han99a, Proposition 11.2]) whence

$$\sum_{n=0}^{\infty} M^n s = s + M\big(\sum_{n=0}^{\infty} M^n s\big) \leq s + ac_U^\mu(Q) G^{1_V \mu^-} s \leq (1 + abc_U^\mu \|G^{1_V \mu^-}\|_\infty) s.$$

□

Since $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$, inequality (3.4) shows that the series $\sum \alpha_n$ and $\sum \alpha_n^*$ are convergent. By Proposition 7.3, there exists $c > 0$ and $0 < \gamma < 1$ such that $\alpha_n + \alpha_n^* \leq c\gamma^n$ for every $n \in \mathbb{N}$. Therefore

$$c_U^\mu(Q) < \infty.$$

Let us recall that part of [Han99a, Lemma 11.3] we shall apply:

Lemma 3.11. *Define* $c := 1 + abc_U^\mu(Q) \|G^{1_V \mu^-}\|_\infty$ *(or* $c := ac_U^\mu(Q)$*). Let* h *be a* μ*–harmonic function on* U *and* s *a positive superharmonic function on* U *such that* $|h| \leq s$ *on* ∂V*. Then* $|h| \leq cH_V s$ *on* V.

Now we are ready to prove Theorem 3.5:

Proof. Fix a μ–harmonic function h on U and $x \in U$ such that $h(x) = 0$. By Remark 3.6, it suffices to show that h is continuous at x.

Given $\varepsilon > 0$, we choose $0 < \delta < \varepsilon/(2c)$ and define

$$F := \{y \in U : |h(y)| \geq \delta\}.$$

Since h is finely continuous and $h(x) = 0$, the set F is finely closed and $x \notin F$. Therefore F is thin at x and there exists a superharmonic function $w \geq 0$ on X such that

$$w(x) < \delta \quad \text{and} \quad \lim_{y \in F, y \to x} w(y) = \infty$$

(we may take $w = 0$ if $x \notin \overline{F}$). Since $|h|$ is locally bounded, there is a compact neighborhood A of x in V such that

$$|h| < w \quad \text{on } A \cap F.$$

We define

$$s := \delta + w\,.$$

Clearly, s is a positive superharmonic function on X such that

$$|h| \leq s \quad \text{on } A, \qquad s(x) < 2\delta.$$

Let W be a regular neighborhood of x in A. By Lemma 3.11, we conclude that

$$|h| \leq cH_W s := g \quad \text{on } W$$

(note that a, Q, b are admissible constants for the pair (W, U)). The function g is harmonic on W and

$$g(x) \leq c\, s(x) \leq c\, 2\delta < \varepsilon.$$

So there exists a neighborhood W' of x in W such that $g < \varepsilon$ on W' whence

$$|h| < \varepsilon \quad \text{on } W'.$$

Thus h is continuous at x. □

Remark 3.12. Looking closely at the proof for [Han99a, Proposition 11.5] it can be verified that positive μ–harmonic functions satisfy ρ–scaling invariant Harnack inequalities and that consequently quotients of μ–harmonic are even ρ–Hölder continuous. We shall not discuss it in detail since we shall get this Hölder continuity as an immediate consequence of results on Green functions obtained by normalized perturbations (see Corollary 5.9).

Using Theorem 3.5 we may even characterize those functions f for which f/h_0 is continuous. We begin with a simple lemma.

Lemma 3.13. *Let $V \in \mathcal{V}$, $\varphi \in \mathcal{B}_b(V)$ and $x \in V$ such that $\lim_{y \to x} \varphi(y) = 0$. Then $G_V^\mu \varphi$ is continuous at x.*

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ such that $\delta G_V^{|\mu|} 1 < \varepsilon$. Fix an open neighborhood W of x in V such that $|\varphi| \leq \delta$ on W. Defining

$$\varphi_1 := 1_W \varphi, \qquad \varphi_2 := 1_{V \setminus W} \varphi$$

we have

$$|G_V^\mu \varphi_1| \leq \delta G_V^{|\mu|} 1 \leq \varepsilon.$$

Moreover, the function $G_V^\mu \varphi_2$ is harmonic on W. In particular, it is continuous at x. So there exists a neighborhood W' of x in W such that

$$|G_V^\mu \varphi_2 - G_V^\mu \varphi_2(x)| < \varepsilon \quad \text{on } W'.$$

Since $\varphi = \varphi_1 + \varphi_2$ we obtain that, for every $y \in W'$,

$$|G_V^\mu \varphi(y) - G_V^\mu \varphi(x)| \le |G_V^\mu \varphi_1(y)| + |G_V^\mu \varphi_1(x)| + |G_V^\mu \varphi_2(y) - G_V^\mu \varphi_2(x)| < 3\varepsilon.$$

□

Corollary 3.14. *Let f be a locally bounded Borel function on an open subset U of X and assume that there exists a μ–harmonic function $h_0 > 0$ on U. Then the following properties are equivalent:*

1. *f/h_0 is continuous on U.*
2. *For every $V \in \mathcal{V}$ with $\overline{V} \subset U$, the function $f + G_V^\mu f$ is continuous on V.*
3. *For every $x \in U$, there exists $V \in \mathcal{V}$ such that $x \in V$, $\overline{V} \subset U$, and $f + G_V^\mu f$ is continuous at x.*

Proof. (1) ⇒ (2): Fix $V \in \mathcal{V}$ with $\overline{V} \subset U$, $x \in V$, and define

$$\varphi := f - \frac{f(x)}{h_0(x)} h_0 = h_0\Big(\frac{f}{h_0} - \frac{f(x)}{h_0(x)}\Big).$$

Then φ is continuous at x, $\lim_{y \to x} \varphi(y) = 0$. By Lemma 3.13, the function $G_V^\mu \varphi$ is continuous at x. The function $g_0 := h_0 + G_V^\mu h_0$ is harmonic on V, $g_0 > 0$. Having

$$f + G_V^\mu f = \varphi + G_V^\mu \varphi + \frac{f(x)}{h_0(x)}\, g_0$$

we thus conclude that $f + G_V^\mu f$ is continuous at x.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Fix $x \in U$ and choose $V \in \mathcal{V}$ such that $x \in V$, $\overline{V} \subset U$, and $u := f + G_V^\mu f$ is continuous at x. Let $\varepsilon > 0$ and take $\delta > 0$ such that

$$\delta \, \|(I + G_V^\mu)^{-1}\| \le \varepsilon h_0 \quad \text{on } V. \tag{3.9}$$

Define

$$f_x := f - \frac{u(x)}{g_0(x)}\, h_0.$$

Obviously it suffices to show that f_x/h_0 is continuous at x. The function

$$u_x := f_x + G_V^\mu f_x = u - \frac{u(x)}{g_0(x)}\, g_0$$

is continuous at x and $u_x(x) = 0$. So there exists $W \in \mathcal{V}$ such that $x \in W \subset V$ and

$$|u_x| < \delta \quad \text{on } W. \tag{3.10}$$

Define

$$\varphi := (I + G_V^\mu)^{-1}(1_W u_x) \quad \text{and } \psi := (I + G_V^\mu)^{-1}(1_{V\setminus W} u_x).$$

By (3.9) and (3.10),

$$|\varphi| \le \delta \, \|(I + G_V^\mu)^{-1}\| \le \varepsilon h_0 \quad \text{on } W. \tag{3.11}$$

Moreover,

$$\psi + G_W^\mu \psi = 1_{V\setminus W} u_x - H_W G_V^\mu \psi = -H_W G_V^\mu \psi \quad \text{on } W.$$

Therefore ψ is μ–harmonic on W and, by Theorem 3.5, there exists a neighborhood W' of x in W such that

$$\left| \frac{\psi}{h_0} - \frac{\psi}{h_0}(x) \right| < \varepsilon \quad \text{on } W'. \tag{3.12}$$

Since $f_x = (I + G_V^\mu)^{-1} u_x = \varphi + \psi$, the inequalities (3.11) and (3.12) imply that

$$\left| \frac{f_x}{h_0} - \frac{f_x}{h_0}(x) \right| \le \left| \frac{\varphi}{h_0} \right| + \left| \frac{\varphi}{h_0}(x) \right| + \left| \frac{\psi}{h_0} - \frac{\psi}{h_0}(x) \right| \le 3\varepsilon \quad \text{on } W'$$

finishing the proof. □

4 Brelot spaces of normalized μ-harmonic functions

Again we fix a signed measure $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$. For every real function ψ on X, let M_ψ denote the multiplication operator $f \mapsto f\psi$.

Given an open subset U of X, let $\mathcal{H}^\mu(U)$ be the set of all μ–harmonic functions on U. For every $V \in \mathcal{V}$, we have a kernel

$$L_V^\mu = (I + G_V^{\mu^+})^{-1} G_V^{\mu^-}$$

(see (3.3)) and we may define a kernel H_V^μ by

$$H_V^\mu := (I + G_V^\mu)^{-1} H_V = \sum_{n=0}^{\infty} (L_V^\mu)^n (I + G_V^{\mu^+})^{-1} H_V.$$

It satisfies

$$(H_V^\mu f)|_V \in \mathcal{H}^\mu(V) \quad \text{for every } f \in \mathcal{B}_b(X).$$

In general, the functions $(H_V^\mu f)|_V$ will not be continuous and they will not tend to $f(z)$ if f is continuous at a point z in the boundary ∂V of V. However, normalizing by

$$s := (I + G^\mu)^{-1} 1$$

we shall obtain a Brelot space and regular harmonic kernels:

Theorem 4.1. *The normalized spaces*

$$\tilde{\mathcal{H}}^\mu(U) := \frac{1}{s}\mathcal{H}^\mu(U) := \{\frac{h}{s} : h \ \mu\text{–harmonic on } U\} \qquad (U \text{ open in } X)$$

form a Brelot space $(X, \tilde{\mathcal{H}}^\mu)$. *The corresponding harmonic kernels for sets* $V \in \mathcal{V}$ *are given by*

$$\tilde{H}_V^\mu := M_s^{-1} H_V^\mu M_s = M_s^{-1}(I + G_V^\mu)^{-1} H_V M_s,$$

i. e., for every continuous function f *on* ∂V, *the function* $(1/s)H_V^\mu(fs)$ *is the unique function in* $\tilde{\mathcal{H}}^\mu(V)$ *tending to* f *at* ∂V, *and it is positive if* $f \geq 0$.

To prepare the proof of this result we will show that at least $\lim_{x\to z} H_V^\mu f(x) = 0$ provided $\lim_{x\to z} f(x) = 0$. A first step is the following:

Lemma 4.2. *Let* $V \in \mathcal{V}$, $z \in \partial V$, *and* $f \in \mathcal{B}_b(V)$ *such that* $\lim_{x\to z} f(x) = 0$. *Then* $\lim_{x\to z} G_V^\mu f(x) = 0$ *and* $\lim_{x\to z} L_V^\mu f(x) = 0$.

Proof. Let $\nu = |\mu|$. Given $\varepsilon > 0$, choose $\delta > 0$ such that $\delta\, G_V^\nu 1 < \varepsilon$. Fix an open neighborhood W of z such that $|f| < \delta$ on $V \cap W$. Defining

$$f_1 := 1_{V\cap W}|f|, \qquad f_2 := 1_{V\setminus W}|f|$$

we have

$$G_V^\nu f_1 \leq \delta G_V^\nu 1 < \varepsilon.$$

Moreover,

$$G_V^\mu f_2 = G^\mu f_2 - H_V G^\mu f_2$$

where $G^\mu f_2$ is harmonic on W. In particular, $G^\mu f_2$ is continuous at z whence

$$\lim_{x\in V, x\to z} H_V G^\mu f_2(x) = G^\mu f_2(z).$$

So there exists $W' \in \mathcal{V}$ such that $z \in W' \subset W$ and

$$G_V^\nu f_2 < \varepsilon \quad \text{ on } V \cap W'.$$

Therefore $|G_V^\mu f| \leq G_V^\nu |f| < 2\varepsilon$ on $V \cap W'$. To finish the proof it suffices to note that $0 \leq |L_V^\mu f| \leq G_V^\nu |f|$ by (3.4). □

Proposition 4.3. *Let* $V \in \mathcal{V}$, $z \in \partial V$, $f \in \mathcal{B}_b(\partial V)$ *such that* $\lim_{y\in\partial V, y\to z} f(y) = 0$. *Then* $\lim_{x\in V, x\to z} H_V^\mu f(x) = 0$.

Proof. Define $u := (I + G_V^{\mu^+})^{-1} H_V |f|$ so that

$$H_V^\mu |f| = \sum_{n=0}^{\infty} (L_V^\mu)^n u. \tag{4.1}$$

We know that $\lim_{x\to z} u(x) = 0$, since the positive harmonic function $H_V|f|$ on V tends to zero at z and $0 \le u \le H_V|f|$. Applying Lemma 4.2 we obtain by induction that

$$\lim_{x\to z}(L_V^\mu)^n u(x) = 0 \quad \text{for all } n = 0, 1, 2, \dots \tag{4.2}$$

Fix $\varepsilon > 0$. By Proposition 7.3,

$$\|u\|_\infty \sum_{n=N}^{\infty} (L_V^\mu)^n 1 \le \varepsilon \tag{4.3}$$

for some $N \in \mathbb{N}$. By (4.2), there exists an open neighborhood W of z such that

$$\sum_{n=0}^{N-1} (L_V^\mu)^n u \le \varepsilon \quad \text{on } V \cap W. \tag{4.4}$$

Combining (4.1), (4.3), and (4.4), we finally conclude that $|H_V^\mu f| \le H_V^\mu|f| \le 2\varepsilon$ on $V \cap W$. $\square$

Proof of Theorem 4.1. We know that

$$\tilde{\mathcal{H}}^\mu := \{\tilde{\mathcal{H}}^\mu(U) : U \text{ open in } X\}$$

is a sheaf of real functions and that the spaces $\tilde{\mathcal{H}}^\mu(U)$ are closed under locally uniform convergence.

Let us fix $V, W \in \mathcal{V}$ such that $\overline{V} \subset W$ and define

$$h_0 := (I + G_W^\mu)^{-1} H_W 1.$$

Then h_0 is a strictly positive μ–harmonic function on W. Given a μ–harmonic function h on V, we know from Corollary 3.14 that s/h_0 is continuous on W, h/h_0 is continuous on V, and therefore the function

$$\frac{h}{s} = \frac{h}{h_0} \cdot \frac{h_0}{s}$$

is continuous on V. This shows that $\tilde{\mathcal{H}}^\mu$ is a sheaf of *continuous* real functions. It satisfies the Brelot convergence axiom, since we have Harnack inequalities for positive μ–harmonic functions and the functions s, s^{-1} are bounded.

Let f be a continuous real function on ∂V. Then $H_V^\mu(fs)$ is a bounded μ–harmonic function on V whence

$$\frac{1}{s} H_V^\mu(fs) \in \tilde{\mathcal{H}}^\mu(V) \cap \mathcal{B}_b(V).$$

Let $z \in \partial V$ and define

$$u_z := \big(f - f(z)\,\frac{s(z)}{h_0(z)}\,\frac{h_0}{s}\big)s.$$

Obviously, $\lim_{x\in\partial V, x\to z} u_z(x) = 0$ since

$$\lim_{x\to z} \frac{s(z)}{h_0(z)} \frac{h_0}{s}(x) = 1.$$

So, by Proposition 4.3,

$$\lim_{x\to z} H_V^\mu u_z(x) = 0.$$

Since $H_V^\mu h_0 = h_0$ by (3.7) and (3.8), we have

$$\frac{1}{s} H_V^\mu u_z = \frac{1}{s} H_V^\mu (fs) - f(z) \frac{s(z)}{h_0(z)} \frac{h_0}{s}.$$

Thus

$$\lim_{x\to z} \frac{1}{s(x)} H_V^\mu (fs)(x) = f(z),$$

i.e., the function $(H_V^\mu(fs))/s$ solves the Dirichlet problem for V and f with respect to $\tilde{\mathcal{H}}^\mu$. Moreover, obviously $(1/s)H_V^\mu(fs) \geq 0$ if $f \geq 0$.

To prove uniqueness of these solutions consider a μ–harmonic function h on V such that h/s tends to zero at the boundary of V. Then h is bounded on V and vanishes at ∂V. Therefore $g := h + G_V^\mu h$ is harmonic on V and $\lim_{x\to z} g(x) = 0$ for every $z \in \partial V$ since, by Lemma 4.2, also $\lim_{x\to z} G_V^\mu h(x) = 0$ for every $z \in \partial V$. Thus $g = 0$ and, finally, $h = (I + G_V^\mu)^{-1} g = 0$ on V. □

For every open subset U of X, let $\mathcal{S}(U)$ ($\mathcal{P}(U)$ resp.) denote the set of all functions on U which are superharmonic (potentials resp.) with respect to $\mathcal{H}$. Using a corresponding notation with respect to $\tilde{\mathcal{H}}^\mu$ we obtain the following supplement to Theorem 4.1:

Theorem 4.4. *$(X, \tilde{\mathcal{H}}^\mu)$ is even a $\mathcal{P}$–harmonic Brelot space (i.e., there exists a strictly positive $\tilde{p} \in \tilde{\mathcal{P}}^\mu(X)$) and $1 \in \tilde{\mathcal{S}}^\mu(X)$.*

For every open subset U of X, the bijection $M_s^{-1}(I + G_U^\mu)^{-1} : \mathcal{C}_b(U) \to \mathcal{C}_b(U)$ maps $\mathcal{S}(U) \cap \mathcal{C}_b(U)$ onto $\tilde{\mathcal{S}}^\mu(U) \cap \mathcal{C}_b(U)$, $\mathcal{S}(U) \cap \mathcal{C}_b^+(U)$ onto $\tilde{\mathcal{S}}^\mu(U) \cap \mathcal{C}_b^+(U)$, and $\mathcal{P}(U) \cap \mathcal{C}_b(U)$ onto $\tilde{\mathcal{P}}^\mu(U) \cap \mathcal{C}_b(U)$.

Remark 4.5. *In fact, the statements of Theorem 4.4 still hold if we replace $\mathcal{C}_b(U)$ by $\mathcal{B}_b(U)$ (see Corollary 5.7).*

Proof of Theorem 4.4. 1. Let U be an open subset of X, $t \in \mathcal{S}(U) \cap \mathcal{C}_b(U)$, and

$$\tilde{t} := \frac{1}{s}(I + G_U^\mu)^{-1} t.$$

If $t \geq 0$, then $\tilde{t} \geq 0$ by (3.1) and (3.5). For every $V \in \mathcal{V}$ such that $\overline{V} \subset U$,

$$s\tilde{t} + G_V^\mu(s\tilde{t}) - H_V(s\tilde{t}) = s\tilde{t} + G_U^\mu(s\tilde{t}) - H_V(s\tilde{t} + G_U^\mu(s\tilde{t})) = t - H_V t$$

and therefore

$$\tilde{t} - \tilde{H}_V^\mu \tilde{t} = \frac{1}{s}(I + G_V^\mu)^{-1}(t - H_V t). \tag{4.5}$$

Then, for every $V \in \mathcal{V}$ with $\overline{V} \subset U$, $t - H_V t \in \mathcal{S}^+(V)$ and therefore $\tilde{t} - \tilde{H}_V^\mu \tilde{t} \geq 0$ by (4.5) and (3.1). So $\tilde{t} \in \tilde{\mathcal{S}}^\mu(U)$. In particular,

$$1 = M_s^{-1}(I + G^\mu)^{-1} 1 \in \tilde{\mathcal{S}}^\mu(U).$$

Suppose next that $t \in \mathcal{P}(U)$. In particular, $t \geq 0$ whence $\tilde{t} \geq 0$. Let $\tilde{h} \in \tilde{\mathcal{H}}^\mu(U)$ such that $0 \leq \tilde{h} \leq \tilde{t}$. Then $h := s\tilde{h} + G_U^\mu(s\tilde{h}) \in \mathcal{H}(U)$, $t - h \in \mathcal{S}(U)$. Since

$$t - h = s\tilde{t} - s\tilde{h} + G_U^\mu(s\tilde{t} - s\tilde{h}) \geq G_U^\mu(s\tilde{t} - s\tilde{h})$$

where $G_U^\mu(s\tilde{t} - s\tilde{h}) \in \mathcal{P}(U)$ we conclude that $t - h \geq 0$. Since $t \in \mathcal{P}(U)$, this implies that $-h \geq 0$ whence $-\tilde{h} = (1/s)(I + G_U^\mu)^{-1}(-h) \geq 0$. Thus $\tilde{h} = 0$ and $\tilde{t} \in \tilde{\mathcal{P}}^\mu(U)$.

In particular, taking a strictly positive continuous bounded $p \in \mathcal{P}(X)$ we have $\tilde{p} := (1/s)(I + G^\mu)^{-1} p \in \tilde{\mathcal{P}}^\mu(X)$ and $\tilde{p} \neq 0$. Since X is connected, $\tilde{p} > 0$.

2. Let U be an open subset of X, $\tilde{t} \in \tilde{\mathcal{S}}^\mu(U)$, and $t := (I + G_U^\mu)(s\tilde{t})$. If $\tilde{t} \geq 0$, then obviously $t \geq 0$ provided $\mu \geq 0$. In the general case we shall use the trivial identity

$$M_s^{-1}(I + G_U^\mu) M_s = \big(I - M_s^{-1}(I + G_U^\mu)^{-1} G_U^\mu M_s\big)^{-1}.$$

We note first that, for every positive $f \in \mathcal{B}_b(U)$, the functions $G_U^{\mu^\pm}(sf)$ are countable sums of continuous functions in $\mathcal{P}(U)$ and therefore the functions $(1/s)(I + G_U^\mu)^{-1} G_U^{\mu^\pm}(sf)$ are countable sums of continuous functions in $\tilde{\mathcal{P}}^\mu(U)$ by the first part of our proof. Moreover, outside the support $S(f)$ of f the functions $G_U^{\mu^\pm}(sf)$ are harmonic with respect to $\mathcal{H}$ and therefore the functions $(1/s)(I + G_U^\mu)^{-1} G_U^{\mu^\pm}(sf)$ are harmonic with respect to $\tilde{\mathcal{H}}^\mu$ on $U \setminus S(f)$. Finally,

$$\big(I - M_s^{-1}(I + G_U^\mu)^{-1} G_U^\mu M_s\big)^{-1} 1 = M_s^{-1}(I + G_U^\mu) s$$

where

$$(I + G_U^\mu) s = (I + G^\mu - H_U G^\mu) s = 1 - H_U(1 - s) = (1 - H_U 1) + H_U s \geq 0.$$

By a version of the implication (3) $\Rightarrow$ (1) in Lemma 3.3 and the considerations before which only uses the abstract notion of potentials on a harmonic space (not necessarily having a Green function, see [HM90, Lemma 2.1])[1] we conclude that

$$M_s^{-1}(I + G_U^\mu) M_s \tilde{w} = (I - M_s^{-1}(I + G_U^\mu)^{-1} G_U^\mu M_s)^{-1} \tilde{w} \geq 0 \tag{4.6}$$

for every positive $\tilde{w} \in \tilde{\mathcal{S}}^\mu(U)$.

Therefore $t \geq 0$ if $\tilde{t} \geq 0$. Now fix $V \in \mathcal{V}$ such that $\overline{V} \subset U$. Then $\tilde{t} - \tilde{H}_V^\mu \tilde{t}$ is a positive function in $\tilde{\mathcal{S}}^\mu(V)$ and

$$t - H_V t = (I + G_V^\mu) M_s (\tilde{t} - \tilde{H}_V^\mu \tilde{t}) \tag{4.7}$$

[1] In fact, Theorem 5.1 will show that $\tilde{\mathcal{H}}^\mu = \mathcal{H}_{\tilde{G}}$ for some $\tilde{G} \in \mathcal{G}(X)$. Its proof, however, will use Theorem 4.4.

(see (4.5)). Applying (4.6) to V in place of U we conclude that $t - H_V t \geq 0$ and therefore $t \in \mathcal{S}(U)$.

Finally, assume that $\tilde{t} \in \tilde{\mathcal{P}}^\mu(U) \cap \mathcal{C}(U)$. Since $\tilde{\mathcal{P}}^\mu(U) \subset \tilde{\mathcal{S}}^\mu(U)$, clearly $t \in \mathcal{S}(U)$, $t \geq 0$. Let $h \in \mathcal{H}(U)$ such that $0 \leq h \leq t$. Then $\tilde{h} := (1/s)(I + G_U^\mu)^{-1} h \in \tilde{\mathcal{H}}^\mu(U)$, $\tilde{h} \geq 0$. Since $w := t - h \in \mathcal{S}(U) \cap \mathcal{C}_b^+(U)$, we know that $\tilde{t} - \tilde{h} = (1/s)(I + G^\mu)^{-1} w$ is positive. Thus $\tilde{h} = 0$, $h = 0$, and $t \in \mathcal{P}(U)$. □

5 Normalized perturbation within a class of Green functions

In this section we intend to show that our family of Brelot sheaves $\mathcal{H}_G$, $G \in \mathcal{G}(X)$, is stable under normalized perturbation (Theorem 5.1, Proposition 5.6). An interesting consequence is the possibility of iterating such perturbations (Theorem 5.8). In particular, perturbation by a signed measure can be truly decomposed into a positive perturbation followed by a negative perturbation.

Theorem 5.1. *Let $G \in \mathcal{G}(X)$, $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$, and define*

$$\tilde{G}(x,y) := \frac{\sum_{n=0}^{\infty} \big(((I + G^{\mu^+})^{-1} G^{\mu^-})^n (I + G^{\mu^+})^{-1} G(\cdot, y)\big)(x)}{(I + G^\mu)^{-1} 1(x) \cdot (I + {}^*G^\mu)^{-1} 1(y)} \qquad (x, y \in X).^2$$

Then $\tilde{G} \in \mathcal{G}(X)$, $\mathcal{H}_{\tilde{G}} = \tilde{\mathcal{H}}_G^\mu$, and $\tilde{G}$ is locally comparable with G, i.e., for every compact subset A of X, there exists a constant $C \geq 1$ such that

$$C^{-1} G \leq \tilde{G} \leq CG \quad \text{on } A \times A. \tag{5.1}$$

In the following we suppose that $G \in \mathcal{G}(X)$, $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$ and denote

$$s := (I + G^\mu)^{-1} 1 \quad \text{and} \quad s^* := (I + {}^*G^\mu)^{-1} 1.$$

We shall use without further comment that the functions $s, s^*, 1/s, 1/s^*$ are bounded (see (3.1)).

To show that a function $\tilde{G}$ is a Green function for $\tilde{\mathcal{H}}_G^\mu$ the following result will be useful:

Lemma 5.2. *Let $\tilde{G} : X \times X \to [0, \infty]$ be a measurable function such that, for every $\rho \in \mathcal{M}_b(G)$,*

$$\tilde{G}^\rho = M_s^{-1} (I + G^\mu)^{-1} G^\rho M_{s^*}^{-1}.$$

Then, for every $\tilde{p} \in \tilde{\mathcal{P}}^\mu(X) \cap \mathcal{C}(X)$ which is $\tilde{\mathcal{H}}_G^\mu$-harmonic outside a compact set, there exists a measure $\rho \geq 0$ on X such that $\tilde{p} = \tilde{G}^\rho$.

[2] The use of $(I + G^\mu)^{-1} G(\cdot, y)$ will be justified in Proposition 5.3.

Proof. Let $\tilde{p} \in \tilde{\mathcal{P}}^\mu(X) \cap \mathcal{C}(X)$ and S compact such that $\tilde{p}|_{X\setminus S} \in \tilde{\mathcal{H}}^\mu_G(X \setminus S)$. Then $\tilde{p}$ is bounded since $1 \in \tilde{\mathcal{S}}^\mu(X)$. By Theorem 4.4,

$$p := (I + G^\mu)(\tilde{p}s) \in \mathcal{P}(X) \cap \mathcal{C}_b(X),$$

and $p|_{X\setminus S} \in \mathcal{H}_G(X \setminus S)$. So there exists a measure $\rho_0 \geq 0$ on X (supported by S) such that $p = G^{\rho_0}1$ and therefore

$$\tilde{p} = M_s^{-1}(I + G^\mu)^{-1}G^{\rho_0}1 = \tilde{G}^{s^*\rho_0}1.$$

□

To prove Theorem 5.1 we first consider the two special cases $\mu \geq 0$ and $\mu \leq 0$.

Proposition 5.3. *Assume that $\mu \geq 0$. Then there exist unique finely continuous functions $g(\cdot, y) \geq 0$ on X such that*

$$g(\cdot, y) + G^\mu g(\cdot, y) = G(\cdot, y) \qquad (y \in X) \tag{5.2}$$

(justifying to write $(I + G^\mu)^{-1}G(\cdot, y)$ instead of $g(\cdot, y)$). Define

$$T^\mu(G)(x, y) := \tilde{G}(x, y) := \frac{g(x, y)}{s(x)\, s^*(y)} \qquad (x, y \in X).$$

Then $\tilde{G} \in \mathcal{G}(X)$, $\tilde{G}$ is locally comparable with G,

$$\mathcal{H}_{\tilde{G}} = \tilde{\mathcal{H}}^\mu_G \quad \text{and} \quad {}^*\tilde{G} = T^\mu({}^*G).$$

Moreover, for every $\rho \in \mathcal{M}_b(G)$,

$$\tilde{G}^\rho = M_s^{-1}(I + G^\mu)^{-1}G^\rho M_{s^*}^{-1}. \tag{5.3}$$

Proof. By Lemma 3.1, for every $V \in \mathcal{V}$, there exists a constant $c_V > 0$ such that, for every $y \in V$,

$$G^\mu\, G(\cdot, y) \leq c_V\, G(\cdot, y) \quad \text{on } V. \tag{5.4}$$

Let us fix $y \in X$ and define

$$s_m := \min\big(G(\cdot, y), m\big) \qquad (m \in \mathbb{N}).$$

By (3.1), for every $m \in \mathbb{N}$,

$$s_m - (I + G^\mu)^{-1}G^\mu s_m = (I + G^\mu)^{-1}s_m, \quad \exp(-\frac{G^\mu s_m}{s_m})s_m \leq (I + G^\mu)^{-1}s_m \leq s_m.$$

Letting m tend to ∞ and recalling that $(I + G^\mu)^{-1}G^\mu$ is a kernel, we obtain that, for every $x \in X \setminus \{y\}$,

$$g(x, y) := G(x, y) - \big((I + G^\mu)^{-1}G^\mu G(\cdot, y)\big)(x) \geq 0$$

and

$$\exp\Big(-\frac{(G^\mu G(\cdot,y))(x)}{G(x,y)}\Big)\, G(x,y) \le g(x,y) \le G(x,y).$$

We define $g(y,y) := \infty$. Using (5.4) we see that

$$\exp(-c_V)\, G(x,y) \le g(x,y) \le c_V\, G(x,y) \qquad \text{if } x,y \in V \in \mathcal{V}.$$

So

$$\lim_{x\to y} g(x,y) = \infty \tag{5.5}$$

and the measurable function g on $X\times X$ is locally comparable with G. Moreover, the definition of $g(\cdot,y)$ implies that $(I+G^\mu)g(\cdot,y) + G^\mu G(\cdot,y) = (I+G^\mu)G(\cdot,y)$ and therefore

$$(I+G^\mu)g(\cdot,y) = G(\cdot,y), \tag{5.6}$$

since $G^\mu G(\cdot,y) < \infty$ on $X\setminus\{y\}$ by (5.4). It follows immediately from (5.6) and (5.5) that $g(\cdot,y)$ is finely continuous.

If s is any positive finely continuous function such that $G(\cdot,y) = s + G^\mu s$ then an application of $(I+G^\mu)^{-1}G^\mu$ yields that $(I+G^\mu)^{-1}G^\mu G(\cdot,y) = G^\mu s$ and hence $s = g(\cdot,y)$ on $X\setminus\{x\}$. By fine continuity, $s(y) = \text{f-lim}_{x\neq y, x\to y}\, g(x,y) = \infty$. Thus $s = g(\cdot,y)$.

We now claim that $\tilde G(\cdot,y) \in \tilde{\mathcal{P}}^\mu_G(X)$. Indeed, the identity (5.6) implies that $\tilde G(\cdot,y) = s^*(y)^{-1} g(\cdot,y)/s$ is $\tilde{\mathcal{H}}^\mu_G$-harmonic on $X\setminus\{y\}$ and therefore $\tilde G(\cdot,y) \in \tilde{\mathcal{S}}^\mu_G(X)$. Next, let $\tilde h \in \tilde{\mathcal{H}}^\mu_G(X)$ such that $0 \le \tilde h \le \tilde G(\cdot,y)$. If $V\in\mathcal{V}$ contains the point y, then $\tilde h$ is bounded on $X\setminus V$. Moreover, $\tilde h = \tilde H^\mu_V \tilde h \le \sup_{\partial V} \tilde h$. Therefore $\tilde h$ is bounded on X and $h := \tilde h s + G^\mu(\tilde h s) \in \mathcal{H}_G(X)$, $0 \le h \le s^*(y)^{-1}G(\cdot,y)$. This implies that $h=0$ and $\tilde h = 0$. Thus $\tilde G(\cdot,y) \in \tilde{\mathcal{P}}^\mu_G(X)$.

If $\rho \in \mathcal{M}_b(G)$, then $\tilde G^\rho = M_s^{-1} g^\rho M_{s^*}^{-1}$ by definition of $\tilde G$ where

$$g^\rho + G^\mu g^\rho = G^\rho$$

by (5.6) and hence

$$\tilde G^\rho = M_s^{-1}(I+G^\mu)^{-1}G^\rho M_{s^*}^{-1}.$$

Applying Lemma 5.2 we obtain that $\tilde G$ is a Green function for $\tilde{\mathcal{H}}^\mu_G$. By [Han99a, Lemma 15.1], ${}^*\tilde G = T^\mu({}^*G)$. So ${}^*\tilde G$ is a Green function for $\tilde{\mathcal{H}}^\mu_{{}^*G}$.

Moreover, G is locally comparable with g and therefore with $\tilde G$. In particular, $(x,y) \mapsto (\tilde G(x,y)^{-1} + \tilde G(y,x)^{-1})/2$ is locally a quasimetric on X. Therefore $\tilde G \in \mathcal{G}(X)$. □

Proposition 5.4. *Assume that $\mu \le 0$ and define*

$$(T^\mu(G))(x,y) := \tilde G(x,y) := \frac{\sum_{n=0}^{\infty}\big((G^{-\mu})^n G(\cdot,y)\big)(x)}{s(x)\, s^*(y)} \qquad (x,y\in X).$$

Then $\tilde G \in \mathcal{G}(X)$, $\tilde G$ is locally comparable with G,

$$\mathcal{H}_{\tilde G} = \tilde{\mathcal{H}}_G^\mu \quad \text{and} \quad {}^*\tilde G = T^\mu({}^*G).$$

Moreover, for every $\rho \in \mathcal{M}_b(G)$,

$$\tilde G^\rho = M_s^{-1}(I+G^\mu)^{-1}G^\rho M_{s^*}^{-1}.$$

Proof. Define $\nu := \mu^- = -\mu$. By Proposition 3.10, for every $V \in \mathcal{V}$, there exists a constant $c_V > 0$ such that, for every $y \in V$,

$$\sum_{n=0}^\infty \left(G^{1_V\nu}\right)^n G(\cdot, y) \le c_V\, G(\cdot, y)$$

and therefore

$$G_V(\cdot,y) \le p_V^y := \sum_{n=0}^\infty \left(G_V^\nu\right)^n G_V(\cdot,y) \le c_V\, G(\cdot,y). \tag{5.7}$$

In particular, $p_V^y \in \mathcal{P}_G(V)$. We claim that

$$\tilde p_V^y := \frac{p_V^y}{s} \in \tilde{\mathcal{P}}_G^\mu(V) \cap \tilde{\mathcal{H}}_G^\mu(V\setminus\{y\}) \quad \text{and} \quad \tilde p_V^y \uparrow \frac{g(\cdot,y)}{s} \quad \text{as } V \uparrow X.$$

The second statement follows immediately from the definitions. To prove the first one we define

$$s_m := \min\bigl(G_V(\cdot,y), m\bigr) \qquad (m \in \mathbb{N}).$$

By Theorem 4.4, $\tilde s_m := (1/s)(I+G_V^\mu)^{-1}s_m \in \tilde{\mathcal{S}}_G^\mu(V)\cap\tilde{\mathcal{H}}_G^\mu(\{G_V(\cdot,y)<m\})$. Since

$$\tilde s_m = \frac{1}{s}\sum_{n=0}^\infty (G_V^\nu)^n s_m \uparrow \tilde p_V^y \qquad \text{as } m \uparrow \infty$$

and $\tilde p_V^y \le c_V G(\cdot,y) < \infty$ on $V\setminus\{y\}$, we infer that $\tilde p_V^y \in \tilde{\mathcal{S}}_G^\mu(V)\cap\tilde{\mathcal{H}}_G^\mu(V\setminus\{y\})$.

Next let $\tilde h \in \tilde{\mathcal{H}}_G^\mu(V)$ such that $0 \le \tilde h \le \tilde p_V^y$. Choose $W \in \mathcal{V}$ such that $y \in W$ and $\overline{W} \subset V$. Then $\tilde h$ is obviously bounded on $V \setminus W$ (where $G(\cdot,y)$ is bounded). Since $\tilde h = \tilde H_W^\mu \tilde h \le \sup_{\partial W} \tilde h$ on W, we obtain that in fact $\tilde h$ is bounded on V. So $h := \tilde h s - G_V^\nu(\tilde h s) \in \mathcal{B}_b(V)$. Knowing that $h \in \mathcal{H}_G(V)$ and $-G_V^\nu(\tilde h s) \le h \le p_V^y$ where $G_V^\nu(\tilde h s), p_V^y \in \mathcal{P}_G(V)$, we conclude that $h = 0$. The injectivity of $I - G_V^\nu = I + G_V^\mu$ on $B_b(V)$ finally implies that $\tilde h s = 0$, i.e., $\tilde h = 0$.

Suppose now that $y \in V \subset W$ where $V, W \in \mathcal{V}$. We claim that $\tilde p_W^y$ and $\tilde p_V^y$ differ on V by an $\tilde{\mathcal{H}}_G^\mu$-harmonic function. Indeed, on $V \setminus \{y\}$

$$\begin{aligned}(p_W^y - G_V^\nu p_W^y) - (p_V^y - G_V^\nu p_V^y) &= (G_W(\cdot,y) + H_V G_W^\nu p_W^y) - G_V(\cdot,y)\\ &= H_V(G_W^\nu p_W^y + G_W(\cdot,y))\end{aligned}$$

where $h := H_V(G_W^\nu p_W^y + G_W(\cdot,y)) \in \mathcal{H}_G^+(V)$ and h is bounded, since $G_W^\nu p_W^y \le p_W^y \le c_W G(\cdot,y)$. Note that

$$\sum_{n=0}^{\infty}(G_V^\nu)^n p_V^y \le \sum_{n=0}^{\infty}(G_V^\nu)^n p_W^y \le c_W \sum_{n=0}^{\infty}(G^{1_V\nu})^n G(\cdot,y) \le c_V c_W G(\cdot,y).$$

Applying $\sum_{n=0}^{\infty}(G_V^\nu)^n$ to the equation $h + p_V^y + G_V p_W^y = p_W^y + G_V^\nu p_V^y$ we therefore obtain that

$$(I - G_V^\nu)^{-1}h + p_V^y = p_W^y \quad \text{on } V. \tag{5.8}$$

Thus $\tilde h := (1/s)(I - G_V^\nu)^{-1}h \in \tilde{\mathcal{H}}_G^\mu(V)$, $\tilde h \ge 0$ and $\tilde h + \tilde p_V^y = \tilde p_W^y$ on V.

By [Bau66, Satz 5.3.6] there exists a unique $\tilde p^y \in \tilde{\mathcal{P}}_G^\mu(X)$ which is $\tilde{\mathcal{H}}_G^\mu$-harmonic on $X \setminus \{y\}$ and such that, for every $V \in \mathcal{V}$ containing y, $\tilde p^y|_V = \tilde p_V^y + \tilde h$ for some (positive) $\tilde h \in \tilde{\mathcal{H}}_G^\mu(V)$. Therefore

$$\frac{g(\cdot,y)}{s} = \sup_{V\uparrow X} \tilde p_V^y \le \tilde p^y$$

and it follows immediately that $g(\cdot,y)/s$ is contained in $\tilde{\mathcal{P}}_G^\mu(X)$ and differs from each function $\tilde p_V^y$, $V \in \mathcal{V}$, by a function in $\tilde{\mathcal{H}}_G^\mu(V)$. In other words, $g(\cdot,y)/s$ has the properties characterizing $\tilde p^y$ and therefore $g(\cdot,y)/s = \tilde p^y$.

By definition of $g(\cdot,y)$,

$$g(\cdot,y) = G(\cdot,y) + G^\nu g(\cdot,y) \quad \text{and} \quad \tilde G(x,y) = \frac{g(x,y)}{s(x)s^*(y)}.$$

Moreover, for every $\rho \in \mathcal{M}_b(G)$, $g^\rho = G^\rho + G^\nu g^\rho = G^\rho - G^\mu g^\rho$ and therefore

$$\tilde G^\rho = M_s^{-1}(I + G^\mu)^{-1} G^\rho M_{s^*}^{-1}. \tag{5.9}$$

Hence $\tilde G$ is a Green function for $\tilde{\mathcal{H}}_G^\mu$ by Lemma 5.3. Furthermore, it is easily verified that $\big(({}^*G^\nu)^n {}^*G(\cdot,y)\big)(x) = \big((G^\nu)^n G(\cdot,x)\big)(y)$ for every $n \in \mathbb{N}$ and therefore ${}^*\tilde G = T^\mu({}^*G)$. So ${}^*\tilde G$ is a Green function for $\tilde{\mathcal{H}}_{{}^*G}^\mu$.

Given a compact subset A of a set $V \in \mathcal{V}$, there exists a constant $c' > 0$ with

$$\tilde G \le c' \tilde G_V \quad \text{on } A \times A. \tag{5.10}$$

It follows readily that, for every $y \in V$,

$$\tilde G_V(\cdot,y) = \frac{p_V^y}{s^*(y)s}. \tag{5.11}$$

Indeed, both functions are contained in $\tilde{\mathcal{P}}_G^\mu(V)$ and differ from $\tilde G(\cdot,y)$ by a function in $\tilde{\mathcal{H}}_G^\mu(V)$.

Combining (5.7), (5.10) and (5.11) we obtain that $\tilde G$ and G are comparable on $A \times A$. Thus finally $\tilde G \in \mathcal{G}(X)$ finishing the proof. □

To iterate perturbations the following result will be useful:

Lemma 5.5. *Let $G \in \mathcal{G}(X)$, $\mu_1, \mu_2 \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$, $s_1 := (I + G^{\mu_1})^{-1}1$, $s_1^* := (I + {}^*G^{\mu_1})^{-1}1$, $\varphi := s_1 s_1^*$, $s_2 := (I + G^{\mu_2})^{-1}1$. Suppose that $G_1 \in \mathcal{G}(X)$ and, for every $\rho \in \mathcal{M}_b(G) \cap \mathcal{M}_b({}^*G)$,*

$$G_1^\rho = M_{s_1}^{-1}(I + G^{\mu_1})^{-1} G^\rho M_{s_1^*}^{-1} \quad \text{and} \quad {}^*G_1^\rho = M_{s_1^*}^{-1}(I + {}^*G^{\mu_1})^{-1} {}^*G^\rho M_{s_1}^{-1}.$$

Then $\varphi(\mu_2 - \mu_1) \in \mathcal{M}_{ab}(G_1) \cap \mathcal{M}_{ab}({}^*G_1)$, $s_{21} := (I + G_1^{\varphi(\mu_2 - \mu_1)})^{-1}1 = s_2/s_1$, and

$$M_{s_2}^{-1}(I + G^{\mu_2})^{-1} = M_{s_{21}}^{-1}(I + G_1^{\varphi(\mu_2-\mu_1)})^{-1} M_{s_1}^{-1}(I + G^{\mu_1})^{-1}. \tag{5.12}$$

In particular, $\tilde{\mathcal{H}}_G^{\mu_2} = \tilde{\mathcal{H}}_{G_1}^{\varphi(\mu_2-\mu_1)}$.

Proof. By our assumption $G_1^{\varphi(\mu_2-\mu_1)^\pm} = M_{s_1}^{-1}(I + G^{\mu_1})^{-1} G^{(\mu_2-\mu_1)^\pm} M_{s_1}$. Therefore

$$\begin{aligned} & (I + G^{\mu_1}) M_{s_1} (I + G_1^{\varphi(\mu_2-\mu_1)}) M_{s_1}^{-1} \\ = \; & (I + G^{\mu_1})(I + (I + G^{\mu_1})^{-1} G^{\mu_2 - \mu_1}) = (I + G^{\mu_2}). \end{aligned}$$

This shows that $I + G_1^{\varphi(\mu_2-\mu_1)}$ is invertible and

$$(I + G^{\mu_2})^{-1} = M_{s_1}(I + G_1^{\varphi(\mu_2-\mu_1)})^{-1} M_{s_1}^{-1}(I + G^{\mu_1})^{-1}. \tag{5.13}$$

In particular,

$$s_2 = (I + G^{\mu_2})^{-1}1 = M_{s_1}(I + G_1^{\varphi(\mu_2-\mu_1)})^{-1} M_{s_1}^{-1}(I + G^{\mu_1})^{-1}1 = s_1 s_{21}. \tag{5.14}$$

So $s_{21} \geq 0$ and therefore $\varphi(\mu_2 - \mu_1) \in \mathcal{M}_{ab}(G_1)$ by Proposition 3.3. Similarly, our adjoint assumption implies that $\varphi(\mu_2 - \mu_1) \in \mathcal{M}_{ab}({}^*G_1)$.

In addition, (5.13) and (5.14) show that (5.12) holds. By Theorem 4.4, this implies that the harmonic sheaves $\tilde{\mathcal{H}}_G^{\mu_2}$ and $\tilde{\mathcal{H}}_{G_1}^{\varphi(\mu_2-\mu_1)}$ have the same set of positive continuous bounded superharmonic functions and hence are identical. Alternatively, we might show that (5.12) holds as well for the corresponding kernels on sets $V \in \mathcal{V}$, and then it follows immediately that the two harmonic sheaves in question have the same harmonic kernels.) □

We now combine perturbation by a positive measure (Proposition 5.3) and perturbation by a negative measure (Proposition 5.4) to obtain perturbation by a signed measure. This will prove Theorem 5.1.

Proposition 5.6. *Let $G \in \mathcal{G}(X)$ and $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$. Define*

$$\tilde{G}(x,y) := \frac{\Big(\sum_{n=0}^{\infty} \big((I + G^{\mu^+})^{-1} G^{\mu^-}\big)^n (I + G^{\mu^+})^{-1} G(\cdot, y)\Big)(x)}{s(x) s^*(y)} \qquad (x, y \in X).$$

Then $T^\mu(G) := \tilde{G} \in \mathcal{G}(X)$, $\tilde{G}$ is locally comparable with G, $T^\mu({}^*G) = {}^*(T^\mu(G))$, $\mathcal{H}_{\tilde{G}} = \tilde{\mathcal{H}}_G^\mu$ and, for every $\rho \in \mathcal{M}_b(G)$,

$$\tilde{G}^\rho = M_s^{-1}(I + G^\mu)^{-1} G^\rho M_{s^*}^{-1}. \tag{5.15}$$

Moreover, defining $\varphi := (I + G^{\mu^+})^{-1}1 \cdot (I + {}^*G^{\mu^+})^{-1}1$ *the signed measure* $-\varphi\mu^-$ *is contained in* $\mathcal{M}_{ab}(T^{\mu^+}(G)) \cap \mathcal{M}_{ab}({}^*T^{\mu^+}(G))$ *and*

$$\tilde{\mathcal{H}}^{\mu}_G = \tilde{\mathcal{H}}^{-\varphi\mu^-}_{T^{\mu^+}(G)}, \qquad T^{\mu}(G) = T^{-\varphi\mu^-}\big(T^{\mu^+}(G)\big).$$

Proof. We shall apply Lemma 5.5 with $\mu_1 := \mu^+$ and $\mu_2 := \mu$ so that $\mu_2 - \mu_1 = -\mu^-$. Let $G_1 := T^{\mu^+}(G)$, define s_1, s_2, s_{21} as in Lemma 5.5 and s_1^*, s_2^*, s_{21}^* in the same way replacing G by ${}^*G$. By Proposition 5.3, ${}^*G_1 = T^{\mu^+}({}^*G)$. Therefore (5.3) implies that

$$G_1^{\rho} = M_{s_1}^{-1}(I + G^{\mu^+})^{-1}G^{\rho}M_{s_1^*}^{-1} \quad \text{and} \quad {}^*G_1^{\rho} = M_{s_1^*}^{-1}(I + {}^*G^{\mu^+})^{-1}\,{}^*G^{\rho}M_{s_1}^{-1}$$

for every $\rho \in \mathcal{M}_b(G) \cap \mathcal{M}_b({}^*G)$. By Lemma 5.5, $-\varphi\mu^- \in \mathcal{M}_{ab}(G_1) \cap \mathcal{M}_{ab}({}^*G_1)$ and

$$\tilde{\mathcal{H}}^{\mu}_G = \tilde{\mathcal{H}}^{-\varphi\mu^-}_{G_1}.$$

By (5.3),

$$G_1^{\varphi\mu^-} = M_{s_1}^{-1}(I + G^{\mu^+})^{-1}G^{\varphi\mu^-}M_{s_1^*}^{-1} = M_{s_1}^{-1}(I + G^{\mu^+})^{-1}G^{\mu^-}M_{s_1} \tag{5.16}$$

and therefore

$$\sum_{n=0}^{\infty}(G_1^{\varphi\mu^-})^n = M_{s_1}^{-1}\sum_{n=0}^{\infty}\big((I + G^{\mu^+})^{-1}G^{\mu^-}\big)^n M_{s_1}. \tag{5.17}$$

By definition in Proposition 5.3,

$$\begin{aligned}\big(T^{-\varphi\mu^-}(G_1)\big)(x,y) &= \frac{\sum_{n=0}^{\infty}\big((G_1^{-\varphi\mu^-})^n G_1(\cdot,y)\big)(x)}{s_{21}(x)s_{21}^*(y)} \\ &= \frac{\sum_{n=0}^{\infty}\big[\big((I + G^{\mu^+})^{-1}G^{\mu^-}\big)^n M_{s_1}G_1(\cdot,y)\big](x)}{s_1(x)s_{21}(x)s_{21}^*(y)}\end{aligned}$$

where

$$M_{s_1}G_1(\cdot,y) = \frac{1}{s_1^*(y)}(I + G^{\mu^+})^{-1}G(\cdot,y).$$

By Lemma 5.5, $s_1 s_{21} = s_2$, $s_1^* s_{21}^* = s_2^*$, and we obtain that

$$\tilde{G} = T^{-\varphi\mu^-}(G_1) = T^{-\varphi\mu^-}(T^{\mu^+}(G)) \in \mathcal{G}(X)$$

and

$$\mathcal{H}_{\tilde{G}} = \mathcal{H}_{T^{-\varphi\mu^-}(G_1)} = \tilde{\mathcal{H}}^{-\varphi\mu^-}_{G_1} = \tilde{\mathcal{H}}^{\mu}_G.$$

Moreover, by Proposition 5.3 and Proposition 5.4,

$$T^{\mu}({}^*G) = T^{-\varphi\mu^-}(T^{\mu^+}({}^*G)) = T^{-\varphi\mu^-}({}^*(T^{\mu^+}(G))) = {}^*(T^{-\varphi\mu^-}(T^{\mu^+}(G))) = {}^*\tilde{G}.$$

Finally, fix $\rho \in \mathcal{M}_b(G)$, $\rho \geq 0$. The definition of $\tilde{G}$ implies that

$$\tilde{G}^{\rho} = M_s^{-1}\sum_{n=0}^{\infty}\big((I + G^{\mu^+})^{-1}G^{\mu^-}\big)^n(I + G^{\mu^+})^{-1}G^{\rho}M_{s^*}^{-1} = M_s^{-1}(I + G^{\mu})^{-1}G^{\rho}M_{s^*}^{-1}.$$

□

Corollary 5.7. *Let $G \in \mathcal{G}(X)$, $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$ and $\tilde{G} := T^\mu(G)$. Then, for every open subset U of X, the bijection $M_s^{-1}(I + G_U^\mu)^{-1}$ on $\mathcal{B}_b(U)$ maps $\mathcal{P}_G(U) \cap \mathcal{B}_b(U)$ on $\mathcal{P}_{\tilde{G}}(U) \cap \mathcal{B}_b(U)$ and $\mathcal{S}_G(U) \cap \mathcal{B}_b^+(U)$ on $\mathcal{S}_{\tilde{G}}(U) \cap \mathcal{B}_b^+(U)$.*

Proof. Since $G, \tilde{G} \in \mathcal{G}(X)$, bounded functions in $\mathcal{P}_G(U)$ ($\mathcal{P}_{\tilde{G}}(U)$ resp.) are countable sums of continuous functions in $\mathcal{P}_G(U)$ ($\mathcal{P}_{\tilde{G}}(U)$ resp.). So the first statement follows from Theorem 4.4. Since $M_s^{-1}(I + G_U^\mu)^{-1}$ maps $\mathcal{H}_G(U)) \cap \mathcal{B}_b^+(U)$ on $\mathcal{H}_{\tilde{G}}(U) \cap B_b^+(U)$ and each positive superharmonic function is a sum of a positive harmonic function and a potential, the second statement is an immediate consequence. □

Having Lemma 5.5 and Theorem 5.6, we may easily iterate perturbation by arbitrary signed measures μ, ν in $\mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$. In particular, perturbing $T^\mu(G)$ and $\tilde{\mathcal{H}}_G^\mu$ we can get back to G and $\mathcal{H}_G$:

Theorem 5.8. *Let $G \in \mathcal{G}(X)$, $\mu, \nu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$, and*

$$\varphi := (I + G^\mu)^{-1}1 \cdot (I + {}^*G^\mu)^{-1}1.$$

Then $\varphi(\nu - \mu) \in \mathcal{M}_{ab}(T^\mu(G)) \cap \mathcal{M}_{ab}({}^*T^\mu(G))$,

$$\tilde{\mathcal{H}}_G^\nu = \tilde{\mathcal{H}}_{T^\mu(G)}^{\varphi(\nu-\mu)} \quad \text{and} \quad T^\nu(G) = T^{\varphi(\nu-\mu)}\big(T^\mu(G)\big).$$

In particular,

$$\mathcal{H}_G = \tilde{\mathcal{H}}_{T^\mu(G)}^{-\varphi\mu} \quad \text{and} \quad G = T^{-\varphi\mu}\big(T^\mu(G)\big).$$

Proof. Let $\mu_1 := \mu$, $\mu_2 := \nu$, and $G_1 := T^\mu(G)$. Define $s_1, s_2, s_{21}, s_1^*, s_2^*, s_{21}^*$ as before. Applying Proposition 5.6 to G and ${}^*G$, we obtain that, for every $\rho \in \mathcal{M}_b(G) \cap \mathcal{M}_b({}^*G)$,

$$\tilde{G}^\rho = M_s^{-1}(I + G^\mu)^{-1} G^\rho M_{s^*}^{-1}, \qquad {}^*\tilde{G}^\rho = M_{s^*}^{-1}(I + {}^*G^\mu)^{-1}\,{}^*G^\rho M_s^{-1}.$$

By Lemma 5.5, $s_1^* s_{21}^* = s_2^*$, $\varphi(\nu - \mu) \in \mathcal{M}_{ab}(G_1) \cap \mathcal{M}_{ab}({}^*G_1)$, $\tilde{\mathcal{H}}_G^\nu = \tilde{\mathcal{H}}_{T^\mu(G)}^{\varphi(\nu-\mu)}$ and

$$M_{s_2}^{-1}(I + G^\nu)^{-1} = M_{s_{21}}^{-1}(I + G_1^{\varphi(\nu-\mu)})^{-1} M_{s_1}^{-1}(I + G^\mu)^{-1}.$$

Using (5.15) three times we infer that, for every $\rho \in \mathcal{M}_b(G)$,

$$\begin{aligned}
(T^\nu(G))^\rho &= M_{s_2}^{-1}(I + G^\nu)^{-1} G^\rho M_{s_2^*}^{-1} \\
&= M_{s_{21}}^{-1}(I + G_1^{\varphi(\nu-\mu)})^{-1} M_{s_1}^{-1}(I + G^\mu)^{-1} G^\rho M_{s_2^*}^{-1} \\
&= M_{s_{21}}^{-1}(I + G_1^{\varphi(\nu-\mu)})^{-1} G_1^\rho M_{s_1^*} M_{s_2^*}^{-1} \\
&= \big(T^{\varphi(\nu-\mu)}(G_1)\big)^\rho.
\end{aligned}$$

Thus $T^\nu(G) = T^{\varphi(\nu-\mu)}(G_1)$. □

Corollary 5.9. *Let $G \in \mathcal{G}(X)$, $\rho = (G^{-1} + {}^*G^{-1})/2$, and $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$. Then quotients of μ–harmonic functions (quotients of $\tilde{\mathcal{H}}_G^\mu$–harmonic functions) are ρ–Hölder continuous.*

Proof. By Theorem 5.1, $\tilde{G} = T^\mu(G)$ is locally comparable with G. Therefore $\tilde{\rho} := (\tilde{G}^{-1} + {}^*\tilde{G}^{-1})/2$ is locally comparable with ρ. By Theorem 2.5, quotients of $\tilde{\mathcal{H}}_G^\mu$–harmonic functions are $\tilde{\rho}$–Hölder continuous and therefore ρ–Hölder continuous. □

6 Global comparison of Green functions

As in the previous section suppose that $G \in \mathcal{G}(X)$ and $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$. Then we know by Theorem 5.1 that the perturbed Green function $T^\mu(G)$ and G are locally comparable. In this section we shall see that $T^\mu(G)$ and G are *globally* comparable provided μ^+ and μ^- are $\mathcal{S}^+(X)$-bounded, $\mathcal{S}^+(X)$-contractive resp. at infinity.

Theorem 6.1. *Assume that $\mu \in \mathcal{M}_b^+(G)$. Then the following properties are equivalent:*

1. *$T^\mu(G)$ and G are globally comparable.*
2. *μ^+ is $\mathcal{S}^+(X)$-bounded, i.e., there exists a constant $\alpha \geq 0$ such that, for every $s \in \mathcal{S}^+(X)$,*
$$G^\mu s \leq \alpha\, s.$$
3. *There exists a compact subset A of X and a constant $\alpha \geq 0$ such that, for all $x, y \in A^c$,*

$$\int_{A^c} G(x,z)G(z,y)\,\mu(dz) \leq \alpha\, G(x,y). \tag{6.1}$$

Proof. (1) $\Rightarrow$ (2): Property (1) implies that for some $\alpha > 0$ and all $y \in X$, $G(\cdot, y) \leq \alpha(I + G^\mu)^{-1}G(\cdot, y)$ and therefore

$$G^\mu G(\cdot, y) \leq (I + G^\mu)G(\cdot, y) \leq \alpha G(\cdot, y).$$

Integrating with respect to measures $\rho \geq 0$ on X and taking increasing limits we obtain (2).

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): By Lemma 3.1, there exists a constant $\beta \geq 0$ such that

$$G^{1_A\mu} s \leq \beta s$$

for every $s \in \mathcal{S}^+(X)$ (it suffices to choose $V \in \mathcal{V}$ containing A). The proof is finished by the decomposition $G^\mu = G^{1_A\mu} + G^{1_{A^c}\mu}$ and the following simple lemma. □

Lemma 6.2. *Let U be an open subset of X, $\nu \geq 0$ a measure on X, and $\alpha > 0$ such that $\int_U G(x,z)G(z,y)\,\nu(dz) \leq \alpha G(x,y)$ for all $x, y \in U$. Then $G^{1_U\nu} s \leq \alpha s$ for all $s \in \mathcal{S}^+(X)$.*

Proof. Fix $s \in \mathcal{S}^+(X)$. There exist measures ρ_n with compact support in U such that $G^{\rho_n}1 \uparrow s$ on U. By integration, our assumption implies that $G^\nu G^{\rho_n}1 \leq \alpha G^{\rho_n}1 \leq \alpha s$ on U whence on X by domination principle. The proof is finished letting n tend to infinity. □

Theorem 6.3. *Suppose that $\mu \in \mathcal{M}_{ab}(G) \cap \mathcal{M}_{ab}({}^*G)$ and that for some compact subset A of X, constants $\alpha_+ \geq 0$, $0 \leq \alpha_- < 1$ and all $x, y \in A^c$,*

$$\int_{A^c} G(x,z)G(z,y)\,\mu^+(dz) \leq \alpha_+\, G(x,y) \tag{6.2}$$

and

$$\int_{A^c} G(x,z)G(z,y)\,\mu^-(dz) \leq \alpha_-\, G(x,y) \tag{6.3}$$

(or $\int_{A^c} g(x,z)g(z,y)\,\mu^-(dz) \leq \alpha_- g(x,y)$ where $g(\cdot,z) := (I+G^{\mu^+})^{-1}G(\cdot,z)$). Then $T^\mu(G)$ and G are globally comparable.

Before proving Theorem 6.3 let us recall that

$$(T^\mu(G))(\cdot,y) = \frac{\sum_{n=0}^{\infty}(L^\mu)^n g(\cdot,y)}{s\,s^*(y)}$$

where $L^\mu = (I+G^{\mu^+})^{-1}G^{\mu^-}$, $s = (I+G^{\mu^+})^{-1}1$, $s^* = (I+{}^*G^{\mu^+})^{-1}1$, and the functions $s, s*, s^{-1}, (s^*)^{-1}$ are bounded. Assume that (6.2) and (6.3) hold. Then g and G are globally comparable by Theorem 6.1. So we know from Theorem 5.1 that there exists a constant $c \geq 1$ such that

$$\left(\sum_{n=0}^{\infty}(L^\mu)^n G(\cdot,y)\right)(x) \leq c\,G(x,y) \tag{6.4}$$

for all $x, y \in A$, and we have to show that (6.4) holds for all $x, y \in X$ if we replace c by a suitable constant C. We recall that $L^\mu \leq G^{\mu^-}$ and that there exists a constant $c_- \geq 0$ such that

$$G^{\mu^-} s \leq c_-\, s \tag{6.5}$$

for all $s \in \mathcal{S}^+(X)$ (apply the equivalence (2) $\Leftrightarrow$ (3) in Theorem 6.1 to μ^-). The following lemma will allow us to obtain the desired extension of (6.4) in two steps.

Lemma 6.4. *Suppose that $y \in X$ and $\gamma \geq 1$ such that*

$$\sum_{n=0}^{\infty}(L^\mu)^n G(\cdot,y) \leq \gamma\, G(\cdot,y) \quad \text{on } A.$$

Then

$$\sum_{n=0}^{\infty}(L^\mu)^n G(\cdot,y) \leq \gamma\frac{c_- + 1}{1-\alpha_-}\, G(\cdot,y) \quad \text{on } X.$$

Proof. Fix $n \in \mathbb{N}$ and let

$$f := \sum_{k=0}^{n} (L^\mu)^k G(\cdot, y), \qquad s := G^{\mu^-} f.$$

Since $L^\mu \le G^{\mu^-}$, we know by (6.5) that f and s are bounded by a multiple of $G(\cdot, y)$ and therefore finite on $X \setminus \{y\}$. Clearly

$$f = G(\cdot, y) + L^\mu \big(\sum_{k=0}^{n-1} (L^\mu)^k G(\cdot, y) \big) \le G(\cdot, y) + s. \tag{6.6}$$

Moreover,

$$s = G^{1_A \mu^-} f + G^{1_{A^c} \mu^-} f \in \mathcal{S}^+(X)$$

where, by assumption and (6.5),

$$G^{1_A \mu^-} f \le \gamma G^{1_A \mu^-} G(\cdot, y) \le \gamma c_- \, G(\cdot, y)$$

and, by Lemma 6.2,

$$G^{1_{A^c} \mu^-} f \le G^{1_{A^c} \mu^-} (G(\cdot, y) + s) \le \alpha_- (G(\cdot, y) + s).$$

Consequently, $s \le (\gamma c_- + \alpha_-) G(\cdot, y) + \alpha_- s$ whence, using (6.6),

$$f \le \big(1 + \frac{\gamma c_- + \alpha_-}{1 - \alpha}\big) G(\cdot, y) \le \gamma \, \frac{c_- + 1}{1 - \alpha_-} \, G(x, y).$$

□

Proof of Theorem 6.3. a) Assume that (6.2) and (6.3) hold. By (6.4) and Lemma 6.4,

$$\left(\sum_{n=0}^{\infty} (L^\mu)^n G(\cdot, y) \right)(x) \le c \, \frac{c_- + 1}{1 - \alpha_-} \, G(x, y) \tag{6.7}$$

for all $y \in A$ and $x \in X$. Replacing G by ${}^*G$ we obtain that (6.7) holds for all $y \in X$ and $x \in A$. Applying Lemma 6.4 again we conclude that, for all $x, y \in X$,

$$\left(\sum_{n=0}^{\infty} (L^\mu)^n G(\cdot, y) \right)(x) \le c \left(\frac{c_- + 1}{1 - \alpha_-} \right)^2 G(x, y).$$

b) Suppose now that we have $\int_{A^c} g(x, z) g(z, y) \, \mu^-(dz) \le \alpha_- g(x, y)$ instead of (6.3). Let

$$s := (I + G^{\mu^+})^{-1} 1, \quad s^* := (I + {}^*G^{\mu^+})^{-1} 1, \quad G_1(x, y) := \frac{g(x, y)}{s(x) s^*(y)}.$$

Then $-ss^*\mu^- \in \mathcal{M}_{ab}(G_1) \cap \mathcal{M}_{ab}(G_1)$ and $T^\mu(G) = T^{-ss^*\mu^-}(G_1)$ by Proposition 5.6. Moreover, for all $x, y \in A^c$,

$$\begin{aligned}
& \int_{A^c} G_1(x,z)G_1(z,y)\,(ss^*\mu^-)(dz) \\
= \; & \frac{1}{s(x)s^*(y)} \int_{A^c} g(x,z)g(z,y)\,\mu^-(dz) \\
\le \; & \frac{\alpha_-}{s(x)s^*(y)}\, g(x,y) = \alpha_-\, G_1(x,y).
\end{aligned}$$

Thus the proof can be finished applying the first part of the proof to G_1 and $-ss^*\mu^-$. □

7 Appendix

The following proposition presents Moser's trick (see [Mos61, Mos64]) for getting Hölder continuity from scaling invariant Harnack inequalities under general assumptions. In fact, there is an equivalence:

Proposition 7.1. *Let ρ be a positive real function on a set E vanishing at a point $x_0 \in E$. For every $r > 0$, let $\mathcal{F}_r$ be a linear space of bounded real functions on*

$$B_r := \{x \in E \colon \rho(x) < r\}$$

such that $1 \in \mathcal{F}_r$ and $\mathcal{F}_r|_{B_s} \subset \mathcal{F}_s$ whenever $0 < s < r$.

Let $s, r \in]0,\infty[$, $\alpha, \gamma \in]0,1[$, $c, C \in]1,\infty[$, and consider the following properties.

(Ha) Scaling invariant Harnack inequalities: *For all $0 < r' \le r$, positive $f \in \mathcal{F}_{r'}$ and $x, y \in B_{\alpha r'}$,*

$$f(x) \le c\, f(y).$$

(Hö) Hölder continuity: *For all $f \in \mathcal{F}_r$ and $x \in B_s$,*

$$|f(x) - f(x_0)| \le C\big(f(x_0) - \inf f(B_r)\big)\, \Big(\frac{\rho(x)}{r}\Big)^\eta.$$

Then (Ha) *implies* (Hö) *if $s \le \alpha r$, $C \ge c^2/(c-1)$ and $\eta \le \ln(c/(c-1))/\ln(1/\alpha)$.*
Conversely, (Hö) *implies* (Ha) *if $\alpha r \le s$, $C\alpha^\eta < 1$, $c \ge (1 + C\alpha^\eta)/(1 - C\alpha^\eta)$.*

Proof. 1. Suppose that (Ha) holds. Let $0 < r' \le r$, $A := B_{\alpha r'}$ and $B := B_{r'}$. Given $g \in \mathcal{F}_r'$, $g \ge 0$, we then have $\sup_A g \le c \inf_A g$, i. e.,

$$\operatorname*{osc}_A g := \sup_A g - \inf_A g \le (c-1) \inf_A g. \tag{7.1}$$

Now fix $f \in \mathcal{F}_r$ and define

$$a := \sup_B f, \qquad g := a - f|_B.$$

Of course, $g \geq 0$ and, by our general assumption, $g \in \mathcal{F}_{r'}$. Since $\operatorname{osc}_A g = \operatorname{osc}_A f$ and $\inf_B f \leq \inf_A f$, the inequality (7.1) implies that

$$\inf_A g = \sup_B f - \sup_A f \leq \operatorname*{osc}_B f - \operatorname*{osc}_A f.$$

So we conclude that $\operatorname{osc}_A f \leq (c-1)(\operatorname{osc}_B f - \operatorname{osc}_A f)$, i. e.,

$$\operatorname*{osc}_A f \leq \frac{c-1}{c} \operatorname*{osc}_B f.$$

Proceeding by induction we thus obtain that, for every $n \in \mathbb{N}$,

$$\sup_{B_{\alpha^n r}} |f - f(x_0)| \leq \operatorname*{osc}_{B_{\alpha^n r}} f \leq \left(\frac{c-1}{c}\right)^{n-1} \operatorname*{osc}_{B_{\alpha r}} f. \tag{7.2}$$

Applying (7.1) to the positive function $f|_{B_r} - \inf_{B_r} f \in \mathcal{F}_r$ we get

$$\operatorname*{osc}_{B_{\alpha r}} f \leq (c-1) \inf_{B_{\alpha r}} \bigl(f|_{B_r} - \inf_{B_r} f\bigr) \leq (c-1)\bigl(f(x_0) - \inf_{B_r} f\bigr). \tag{7.3}$$

Finally, fix $x \in B_{\alpha r}$. If $\rho(x) = 0$, then $x \in B_{\alpha^n r}$ for every $n \in \mathbb{N}$ and (7.2) implies that $f(x) - f(x_0) = 0$. So suppose that $\rho(x) > 0$. Then there exists $n \in \mathbb{N}$ such that $\alpha^{n+1} r \leq \rho(x) \leq \alpha^n r$ and we conclude from (7.2) and (7.3) that

$$|f(x) - f(x_0)| \leq \frac{c^2}{c-1} \bigl(1 - c^{-1}\bigr)^{n+1} \bigl(f(x_0) - \inf f(B_r)\bigr)$$

where

$$\bigl(1 - c^{-1}\bigr)^{n+1} \leq \left(\frac{\rho(x)}{r}\right)^{\dfrac{\ln(1-c^{-1})}{\ln \alpha}}.$$

2. Assume now conversely that (Hö) holds, let $\alpha r \leq s$ and $C\alpha^\eta < 1$. Fix $0 < r' \leq r$, $f \in \mathcal{F}_{r'}$, $f \geq 0$, and $x_1, x_2 \in B_{\alpha r'}$. Then $x_j \in B_s$ and $\rho(x_j)/r \leq \alpha$, hence

$$|f(x_j) - f(x_0)| \leq C\alpha^\eta f(x_0),$$

i. e., $f(x_0)(1 - C\alpha^\eta) \leq f(x_j) \leq f(x_0)(1 + C\alpha^\eta)$ for $j = 1, 2$. This implies that

$$f(x_1) \leq \frac{1 + C\alpha^\eta}{1 - C\alpha^\eta} f(x_2).$$

□

Remark 7.2. *If the assumption* $1 \in \mathcal{F}_r$ *is replaced by* $f_0|_{B_r} \in \mathcal{F}_r$ *for some bounded function* $f_0 \geq 1$ *on* E, *property* (Ha) *leads to* (Hö) *for quotients* f/f_0, $f \in \mathcal{F}_r$, *provided we replace* c *by* c^2 *(this follows immediately from the fact that* $f(x) \leq cf(y)$ *and* $f_0(y) \leq cf_0(x)$ *imply that* $f(x)/f(x_0) \leq c^2 f(y)/f_0(y)$*).*

For the next two results let $(E, \mathcal{E})$ be a measurable space and let $\mathcal{E}_b$, $\mathcal{E}^+$ denote the set of all $\mathcal{E}$–measurable numerical functions on E which are bounded, positive resp.

The following proposition is partly known (see [HH88, Lemma 1.3], [HM90, Lemma 2.1] where the equivalence of (1), (2) and (3) is shown), implication (1) $\Rightarrow$ (5) seems to be new.

Proposition 7.3. *For every bounded kernel L on $(E, \mathcal{E})$ the following properties are equivalent:*

1. *The operator $I - L$ on $\mathcal{E}_b$ is invertible and the inverse of $I - L$ is positive.*
2. *The operator $I - L$ on $\mathcal{E}_b$ is invertible and $(I - L)^{-1} = \sum_{n=0}^{\infty} L^n$.*
3. *The function $\sum_{n=0}^{\infty} L^n 1$ is bounded.*
4. *$\sum_{n=0}^{\infty} \|L^n 1\|_\infty < \infty$.*
5. *There exist $c \in \mathbb{R}^+$ and $0 \le \alpha < 1$ such that $L^n 1 \le c\alpha^n$ for every $n \in \mathbb{N}$.*
6. *There exists $n \in \mathbb{N}$ and $a \in \mathbb{R}^+$ such that $L^n 1 \le a < 1$.*
7. *The spectral radius of the bounded operator L on $\mathcal{E}_b$ is strictly less than 1.*

Proof. (1) $\Rightarrow$ (5): Taking $f := (I - L)^{-1} 1 \in \mathcal{E}_b^+$ and $c \in \mathbb{R}^+$ with $f \le c$ we have $0 \le Lf = f - 1 \le \big(1 - (1/c)\big) f$. Trivial induction yields that, for every $n \in \mathbb{N}$,

$$L^n 1 \le L^n f \le (1 - \frac{1}{c})^n f \le c\,(1 - \frac{1}{c})^n.$$

(5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1): Trivial.

(5) $\Rightarrow$ (7) $\Rightarrow$ (6): It suffices to note that $\|L^n\| = \|L^n 1\|_\infty$.

(6) $\Rightarrow$ (5): Fix $N \in \mathbb{N}$ and $0 \le a < 1$ such that $L^N 1 \le a$ and choose a real b such that $b \ge L^r 1$ for all $r = 0, 1, \dots, N-1$. Given $n \in \mathbb{N} \cup \{0\}$, we have $n = kN + r$ with $k, r \in \mathbb{N} \cup \{0\}$, $0 \le r \le N - 1$, and then

$$L^n 1 = (L^N)^k (L^r 1) \le b a^k \le (b/a)\,(a^{1/N})^n.$$

□

It may be of independent interest to note that (considering $f := \sum_{n=0}^{\infty} L^n s$) the first part of the preceding proof actually shows the following:

Proposition 7.4. *If L is a kernel on $(E, \mathcal{E})$, $c > 0$ real, and $s \in \mathcal{E}^+$ such that $\sum_{n=0}^{\infty} L^n s \le cs$, then*

$$L^n s \le c\big(1 - \frac{1}{c}\big)^n s \quad \textit{for every } n \in \mathbb{N}. \tag{7.4}$$

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