# LARGE NOISE ASYMPTOTICS FOR ONE-DIMENSIONAL DIFFUSIONS

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Abstract. We establish a law of large number and a central limit theorem for a class of additive functionals related to the solution of a one-dimensional stochastic differential equation perturbed by a large noise.

## 0. Introduction

There is a vast literature on small noise perturbation of dynamical systems. This includes topics such as the Freidlin–Wentzell large deviation estimates and the Varadhan estimates on law densities with connections to the Malliavin calculus. On the other hand dynamical systems with large noise have been, to our knowledge, much less considered. It is worth noting that the influence of small noise on solutions to stochastic differential equations can be closely approached by the small time asymptotics behaviour. However the influence of large noise generally cannot be reduced to the large time behaviour.

We consider the following one dimensional stochastic differential

$$\mathrm{d}X_t = (b(X_t) + vX_t)\,\mathrm{d}t + \eta\,\mathrm{d}B_t, \qquad X_0 = 0, \tag{0.1}$$

where $b$ is a real Borel bounded function, $\eta$ is a large real constant, and $B$ is a 1-dimensional Brownian motion. This equation has a unique strong solution, see e.g. [LG], [N], [P], or [RY, Th. 3.8, Chap. IX] for $v = 0$, and [FR] for $v \neq 0$.

One can ask under which conditions on $b$ the solution $X = X^{v,\eta}$ does not see at macroscopic level the non-linear part $b(X)$. In other words denoting by $\eta B^v$ the Ornstein–Uhlenbeck process being the solution to (0.1) with $b = 0$ one would like to know whether

$$\sup_{t\in[0,T]} |X_t^{v,\eta} - \eta B_t^v| \overset{(\mathbb{P})}{\to} 0. \tag{0.2}$$

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If (0.2) takes place, then one says that the *triviality phenomenon* occurs.

In examining the convergence of high dimensional white noise driven stochastic PDEs, Oberguggenberger and Russo [OR] introduced the concept of a massless at zero Schwartz tempered distribution. Namely, a Borel real function $b$ has a Fourier transform massless at zero, or simply is *Fourier massless at zero* if

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} b\left(\frac{y}{\varepsilon}\right) e^{-\frac{y^2}{2}}\,\mathrm{d}y = 0. \tag{0.3}$$

Notice that $p$-integrable functions with an arbitrary $p\in[1,\infty)$ and bounded measurable functions wanishing at $\infty$ are Fourier massless at zero. Notice that if $|b|$ is Fourier massless at zero, then the triviality phenomenon occurs. In fact we have

$$\mathbb{E}\left(\int_0^T |b(\eta B^v_s)|\,\mathrm{d}s\right)^p \to 0, \qquad p\geq 1,$$

which yields (0.2) by means of the Girsanov transformation.

By analogy with the case of high dimensional stochastic PDEs, one might expect that the triviality phenomenon takes place for any Fourier massless at zero function. This however does not happen in the present framework of large noise analysis. Namely, taking for simplicity $v=0$ consider a Lipschitz bounded function satisfying

$$\lim_{x\to+\infty} b(x) = \ell = -\lim_{x\to-\infty} b(x),$$

with a certain $\ell>0$. Then (0.3) is fulfilled but the triviality phenomenon does not occur since

$$|X^{0,\eta}_t - \eta B_t| \to l\left|\int_0^t \operatorname{sgn}(B_s)\,\mathrm{d}s\right|,$$

which is not even a Gaussian process.

One objective of this paper is to study precise asymptotics for a class of functions for which triviality takes place. This will include some kind of laws of large numbers and central limit theorems. We focus on the class $\mathcal{C}$ of functions $b$ having a bounded primitive. Integrating by parts, it is easy to see that $b\in\mathcal{C}$ fulfills (0.3), and so $b$ is Fourier massless at zero. Moreover, as illustrated by Proposition 1.1, the triviality phenomenon occurs by means of the inverse Itô formula applied to

$$X^{v,\eta}_t - \eta B^v_t = \int_0^t b(X^{v,\eta}_s)\,\mathrm{d}s. \tag{0.4}$$

The class $\mathcal{C}$ includes

(a) the class of integrable functions,
(b) trigonometric polynomials.

Clearly, if $b \in L^1$ then $|b|$ is Fourier massless at zero. This however generally fails for a trigonometric polynomial $b$. For take $b(x) = \cos(x)$, $x \in \mathbb{R}$.

In fact we consider (0.1) for any bounded measurable function $b$ and we study precise asymptotics of additive functionals

$$A_t^{v,\eta}(\rho) = \int_0^t \rho(\eta B_s^v)\,\mathrm{d}s \quad \text{and} \quad \mathcal{A}_t^{v,\eta}(\rho) = \int_0^t \rho(X_s^{v,\eta})\,\mathrm{d}s \tag{0.5}$$

for any Schwartz distribution $\rho$ belonging to the union of

(a') the class of finite signed measures,

(b') the class of the Schwartz distribution $\rho = H'$ where $H \in W^{1,\infty}$ and $H^2 - c_\rho$ has a bounded primitive for a certain $c_\rho$.

Note that the first class is a generalization of the space $L^1$, whereas the second class contains the space of trigonometric polynomials.

The triviality phenomenon (0.2) occurs also for $b \in L^2$, and for additive functionals (0.5) driven by $\rho \in L^2$. This will be however an object of our future studies. We note that in the case $v = 0$, that is for the Brownian motion, Yamada [Y1,Y2] proved that uniformly in $t$ on bounded intervals

$$\frac{1}{\lambda}\int_0^{\lambda t} \rho(B_s)\,\mathrm{d}s \overset{(\mathbb{P})}{\to} \frac{1}{\pi}\int_{\mathbb{R}} (\mathcal{H}^{-1}\rho)(x)\,\mathrm{d}x\, C_t^0 \qquad \text{as } \lambda \to +\infty, \tag{0.6}$$

where $C^0$ is a given process depending on the Brownian motion and $\mathcal{H}^{-1}\rho$ is the inverse Hilbert transform of $\rho \in L^2$. Clearly, through an obvious change of time variable $\eta = \sqrt{\lambda}$, (0.6) can be formulated in terms of asymptotic behaviour of additive functionals, for which we are interested to get a non-zero limit, see Proposition 1.2 for pathwise convergence when $\rho$ is a finite measure and Theorem 1.2 for convergence in law when $\rho$ belongs to the class (b').

The paper is organized as follows. In Section 1, we state the basic limit results for $\eta A^{v,\eta}(\rho)$ and $\eta \mathcal{A}^{v,\eta}(\rho)$. Proposition 1.2 states a law of large numbers (pathwise convergence) when $\rho$ is a finite measure. If moreover, $\rho$ has finite first moment, then Theorem 1.1 gives a central limit theorem. The class (b') is treated in Theorem 1.2. Sections 2, 3, 4 are devoted to the proof of the limit results.

## 1. Formulation of the results

Let $(L^\infty, \|\cdot\|_\infty)$ be the space of classes (with respect to Lebesgue measure) of bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the essential supremum norm. Let $W^{-1,\infty}$ be the space of all distributions $F'$ where $F \in L^\infty$. We equipped $W^{-1,\infty}$ with the norm $\|\rho\|_{W^{-1,\infty}} = \|F\|_\infty$, where $F \in L^\infty$ is such that $F' = \rho$ and $F(0) = 0$. Let $C_0(\mathbb{R})$ be the class of all continuous functions with a compact support. Clearly, $C_0(\mathbb{R})$ is dense in $W^{-1,\infty}$.

In this paper, all the initial conditions are assumed to be equal to zero and all the equations will be taken on a compact interval $[0,T]$.

Let $X^{v,\eta}$ be the solution to the stochastic differential equation (0.1), where $B$ is a 1-dimensional Brownian motion defined on a probability space $\mathfrak{A} = (\Omega, \mathfrak{F}, \mathbb{P})$, $b : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function, and the parameter $v \in \mathbb{R}$. Let $B_t^v := \int_0^t \mathrm{e}^{v(t-s)}\, \mathrm{d}B_s$ be the Ornstein–Uhlenbeck process being the solution to the equation obtained from (0.1) by putting $b = 0$.

Given $\rho \in C_0(\mathbb{R})$ and $\eta > 0$ we define additive functionals $A^{v,\eta}(\rho)$ and $\mathcal{A}^{v,\eta}(\rho)$ by (0.5). The lemma below enables us to extend the functionals $A^{v,\eta}$ and $\mathcal{A}^{v,\eta}$ from the space $C_0(\mathbb{R})$ to the space $W^{-1,\infty}$. In the paper we use $\|\cdot\|_T$ to denote the supremum norm on $C([0,T];\mathbb{R})$.

**Lemma 1.1.** *Let $T \in (0,\infty)$ and $p \in [1,\infty)$. Then:*
*(i) The linear operators $\rho \to A^{v,\eta}(\rho)$ and $\rho \to \mathcal{A}^{v,\eta}(\rho)$ are continuous from the space $(C_0(\mathbb{R}), \|\cdot\|_{W^{-1,\infty}})$ into $\mathcal{L}_T^p := L^p(\Omega, \mathfrak{F}, \mathbb{P}; C([0,T];\mathbb{R}))$, and hence $A^{v,\eta}$ and $\mathcal{A}^{v,\eta}$ can be uniquely extended to the continuous linear operators (denoted also by $A^{v,\eta}$ and $\mathcal{A}^{v,\eta}$) acting from $W^{-1,\infty}$ to $\mathcal{L}_T^p$.*
*(ii) For any $\rho \in W^{-1,\infty}$ the following inverse Itô formulae hold true*

$$A_t^{v,\eta}(\rho) = \frac{2H(\eta B_t^v) - 2H(0)}{\eta^2} - \frac{2}{\eta}\int_0^t F\left(\eta B_s^v\right)\, \mathrm{d}B_s^v \tag{1.1}$$

*and*

$$\mathcal{A}_t^{v,\eta}(\rho) = \frac{2H(X_t^{v,\eta}) - 2H(0)}{\eta^2} - \frac{2}{\eta^2}\int_0^t F\left(X_s^{v,\eta}\right)\, \mathrm{d}X_s^{v,\eta}, \tag{1.2}$$

*where $\rho = F'$, $F \in L^\infty$ and $H$ is a primitive of $F$.*

*Proof.* By Itô's formula (1.1) and (1.2) hold true for $\rho \in C_0(\mathbb{R})$. Since any primitive $H$ of a bounded function $F$ has a linear growth one can get $(i)$. Then by standard approximation arguments one obtains $(ii)$. □

As a consequence of Lemma 1.1$(ii)$ and Girsanov's theorem, we have the following asymptotic result on $A^{v,\eta}$ and $\mathcal{A}^{v,\eta}$.

**Proposition 1.1.** *Let $0 < r < 1$. For all $\rho \in W^{-1,\infty}$, $T \in (0,\infty)$ and $p \in [1,\infty)$ one has*

$$\lim_{\eta\to\infty} \mathbb{E}\,\|\eta^r A^{v,\eta}(\rho)\|_T^p = 0 = \lim_{\eta\to\infty} \mathbb{E}\,\|\eta^r \mathcal{A}^{v,\eta}(\rho)\|_T^p .$$

Let us denote by $\mathcal{M}_{\mathrm{fin}}(\mathbb{R})$ the collection of all finite Radon signed measures on $\mathbb{R}$. We denote by $\|\rho\|_{\mathrm{Var}}$ the total variation of a $\rho \in \mathcal{M}_{\mathrm{fin}}(\mathbb{R})$. Let $\mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$ be the subspace of $\mathcal{M}_{\mathrm{fin}}(\mathbb{R})$ consisting of all measures $\rho$ with finite first moment, that is

$$\int_{\mathbb{R}} |x|\, \|\rho\|_{\mathrm{Var}}(\,\mathrm{d}x) < \infty.$$

Clearly, $\mathcal{M}_{1,\text{fin}}(\mathbb{R}) \subset \mathcal{M}_{\text{fin}}(\mathbb{R}) \subset W^{-1,\infty}$.

Let $L^B$ be the local time of $B$, let $f \in L^1(\mathbb{R})$, and let

$$M_t^\eta(f) := \eta A_t^{0,\eta}(f) - \int_{\mathbb{R}} f(x)\,\mathrm{d}x\, L_t^B(0).$$

Then, see [RY, Prop. 2.1, Chap. XIII], $\mathbb{E}\|M^\eta(f)\|_T^p \to 0$ for all $p \in [1,\infty)$ and $T < \infty$. The theorem below provides similar results on the asymptotic behavior of

$$M^{v,\eta}(\rho) := \eta A^{v,\eta}(\rho) - \rho(\mathbb{R})\, L^{B^v}(0) \quad \text{and} \quad \mathcal{M}^{v,\eta}(\rho) := \eta \mathcal{A}^{v,\eta}(\rho) - \rho(\mathbb{R})\, L^{B^v}(0),$$

where $\rho \in \mathcal{M}_{\text{fin}}(\mathbb{R})$ and $L^{B^v}$ is the local time of $B^v$.

**Proposition 1.2.** *For all $\rho \in \mathcal{M}_{\text{fin}}(\mathbb{R})$, $T \in (0,\infty)$, and $p \in [1,\infty)$,*

$$\lim_{\eta\to\infty} \mathbb{E}\,\|M^{v,\eta}(\rho)\|_T^p = 0 = \lim_{\eta\to\infty} \mathbb{E}\,\|\mathcal{M}^{v,\eta}(\rho)\|_T^p\,.$$

*Remark* 1.1. If $\rho \in L^1(\mathbb{R})$, then using Girsanov's transformation one can easily derive from [RY, Prop. 2.1, Chap. III], that for any finite $T$, $\|M^{v,\eta}(\rho)\|_T \to 0$, $\mathbb{P}$-a.s..

Our next result provides a weak convergence of $\sqrt{\eta}M^{v,\eta}(\rho)$ and $\sqrt{\eta}\mathcal{M}^{v,\eta}(\rho)$ under the assumption $\rho \in \mathcal{M}_{1,\text{fin}}(\mathbb{R})$. Let $F_\rho(x) = \rho((-\infty,x))$, and let

$$\alpha_\rho = \sqrt{\int_{\mathbb{R}} \left(F_\rho - \rho(\mathbb{R})\,\chi_{(0,+\infty)}\right)^2(x)\,\mathrm{d}x}.$$

Clearly $F_\rho \in L^\infty$ and $\rho = F'_\rho$.

*Remark* 1.2. Note that if $\rho \in \mathcal{M}_{1,\text{fin}}(\mathbb{R})$, then $\alpha_\rho < \infty$. For

$$\begin{aligned}
\alpha_\rho^2 &= \int_{-\infty}^0 \left(\rho((-\infty,x))\right)^2\,\mathrm{d}x + \int_0^\infty \left(\rho([x,\infty))\right)^2\,\mathrm{d}x \\
&\le \|\rho\|_{\text{Var}}(\mathbb{R}) \left(\int_{-\infty}^0 \|\rho\|_{\text{Var}}((-\infty,x])\,\mathrm{d}x + \int_0^\infty \|\rho\|_{\text{Var}}([x,\infty))\,\mathrm{d}x\right) \\
&\le \|\rho\|_{\text{Var}}(\mathbb{R}) \int_{\mathbb{R}} |x|\,\|\rho\|_{\text{Var}}(\,\mathrm{d}x).
\end{aligned}$$

*Remark* 1.3. Note that $\alpha_\rho = 0$ iff $\rho = c\delta_0$ for a certain $c \in \mathbb{R}$. Then obviously, $M^{v,\eta}(\rho) = 0$. This shows that next result is somehow optimal and it is not possible to get a non-trivial limit, renormalizing further by a bigger power of $\eta$, when $\rho \in \mathcal{M}_{1,\text{fin}}(\mathbb{R})$.

**Theorem 1.1.** *Assume that $\rho \in \mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$. Then uniformly in $t$ on compact intervals*

$$(B_t, \sqrt{\eta}\, M_t^{v,\eta}(\rho)) \implies \left(\beta_t, 2\alpha_\rho \gamma_{L_t^{\beta^v}(0)}\right) \quad \text{as } \eta \to \infty \tag{1.3}$$

*and*

$$(B_t, \sqrt{\eta}\, \mathcal{M}_t^{v,\eta}(\rho)) \implies \left(\beta_t, 2\alpha_\rho \gamma_{L_t^{\beta^v}(0)}\right) \quad \text{as } \eta \to \infty, \tag{1.4}$$

*where $\beta$ and $\gamma$ are independent standard Brownian motions, and $L^{\beta^v}$ is the local time of the Ornstein–Uhlenbeck process $\beta^v$.*

Let $W^{1,\infty}(\mathbb{R})$ be the space of all bounded absolutely continuous functions $H : \mathbb{R} \to \mathbb{R}$ such that $H' \in L^\infty$, and let $D_b^{-1}(\mathbb{R})$ be the space of all Schwartz distributions $\rho$ such that $\rho = H''$ for a certain $H \in W^{1,\infty}(\mathbb{R})$. Clearly $D_b^{-1}(\mathbb{R}) \subset W^{-1,\infty}$.

Note that for any $\rho \in D_b^{-1}(\mathbb{R})$ there is a unique $H_\rho \in W^{1,\infty}(\mathbb{R})$ such that $\rho = H_\rho''$. Let $\mathcal{D}$ be the class of all $\rho \in D_b^{-1}(\mathbb{R})$ for which there is a (unique) constant $c_\rho$ such that $(H_\rho')^2 - c_\rho \in W^{-1,\infty}$. Note that $c_\rho \geq 0$.

The last result of this section provides a generalized central limit theorem for additive functionals $A^{v,\eta}(\rho)$ and $\mathcal{A}^{v,\eta}(\rho)$, where $\rho \in \mathcal{M}_{\mathrm{fin}} + \mathcal{D}$. We note that the class $\mathcal{D}$ contains trigonometrical polynomials, see Example 1.2.

**Theorem 1.2.** *Assume that $\rho = \rho_\mathcal{M} + \rho_\mathcal{D}$, where $\rho_\mathcal{M} \in \mathcal{M}_{\mathrm{fin}}$ and $\rho_\mathcal{D} \in \mathcal{D}$. Then uniformly in $t$ on compact intervals*

$$(B_t, \eta A_t^{v,\eta}(\rho)) \implies \left(\beta_t, \rho_\mathcal{M}(\mathbb{R}) L_t^{\beta^v}(0) + 2\sqrt{c_{\rho_\mathcal{D}}}\, \gamma_t\right)$$

*and*

$$(B_t, \eta \mathcal{A}_t^{v,\eta}(\rho)) \implies \left(\beta_t, \rho_\mathcal{M}(\mathbb{R}) L_t^{\beta^v}(0) + 2\sqrt{b_{\rho_\mathcal{D}}}\, \gamma_t\right),$$

*where $\beta$ and $\gamma$ are independent standard Brownian motions.*

**Example** 1.1. Suppose that $\rho = F'$ where $F(x) = \int_\mathbb{R} \mathrm{e}^{ixy} \mu(\mathrm{d}y)$ is the Fourier transform of a complex measure $\mu$ on $\mathbb{R}$. Clearly, $\rho \in D_b^{-1}(\mathbb{R})$ if $\mu$ and $x^{-1}\mu(\mathrm{d}x)$ are finite. Assuming this we obtain:

(*i*) If $\rho \in \mathcal{D}$, then $c_\rho = \mu * \overline{\mu}(\{0\}) = \int_\mathbb{R} \overline{\mu}(\{x\})\, \mu(\mathrm{d}x)$. Thus in particular $c_\rho = 0$ if $\mu$ is atomless.

(*ii*) A sufficient condition for $\rho \in \mathcal{D}$ is

$$\int\int_{\{|y-x|\neq 0\}} \frac{\|\mu\|_{\mathrm{Var}}(\mathrm{d}y)\, \|\mu\|_{\mathrm{Var}}(\mathrm{d}x)}{|y-x|} < \infty.$$

For $F^2 = F\overline{F}$ is the Fourier transform of $\mu * \overline{\mu}$, which again is a measure. Then we derive (*i*) from the fact that any constant function $C$ is the Fourier transform of $C\delta_0$, and the following observation : if a Fourier transform of a measure $\nu$ has a bounded primitive say $h$, then $\nu(\{0\})=0$, and $h(x)=-i\int_{\mathbb{R}} \mathrm{e}^{ixy}y^{-1}\nu(\,\mathrm{d}y)$, $x\in\mathbb{R}$. Set

$$\xi(\,\mathrm{d}y)=-iy^{-1}\left(\mu*\overline{\mu}-\mu*\overline{\mu}(\{0\})\delta_0\right)(\,\mathrm{d}y).$$

We infer that if there is a bounded primitive $H$ of $F^2-\mu*\overline{\mu}(\{0\})$, then $H(x)=\int_{\mathbb{R}} \mathrm{e}^{ixy}\xi(\,\mathrm{d}y)$. Finally $H$ given by the formula above is bounded if $\xi$ is a finite measure, which is equivalent to $(ii)$.

Taking in the example above any purely atomic spectral measure $\mu$ with a finite number of atoms we obtain the following.

**Example** 1.2. Any trigonometrical polynomial $\rho(x)=\sum_{j=-m}^{m} a_j \mathrm{e}^{ib_jx}$, where $0<m<\infty$, $b_j\neq 0$, and $a_j=\overline{a_{-j}}$ belongs to $\mathcal{D}$ with $c_\rho=\sum_{j=-m}^{m}|a_j|^2|b_j|^{-2}$.

We can now formulate some result concerning triviality phenomenon for $X^{v,\eta}$. To do this note that $X^{v,\eta}$ is the solution to

$$X^{v,\eta}_t=\int_0^t \mathrm{e}^{v(t-s)}b\left(X^{v,\eta}_s\right)\,\mathrm{d}t+\eta B^v_t.$$

Thus

$$\begin{aligned}\mathcal{D}^{v,\eta}_t(b):=&X^{v,\eta}_t-\eta B^v_t=\int_0^t \mathrm{e}^{v(t-s)}b\left(X^{v,\eta}_s\right)\,\mathrm{d}t=\int_0^t \mathrm{e}^{v(t-s)}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}^{v,\eta}_s(b)\,\mathrm{d}s\\ =&\mathcal{A}^{v,\eta}_t(b)+v\int_0^t \mathrm{e}^{v(t-s)}\mathcal{A}^{v,\eta}_s(b)\,\mathrm{d}s=\mathcal{J}^v\left(\mathcal{A}^{v,\eta}(b)\right)_t,\end{aligned}\tag{1.5}$$

where $\mathcal{J}^v$ is a bounded linear operator on $C([0,\infty);\mathbb{R})$ given by

$$\mathcal{J}^v(\psi)_t=\psi_t+v\int_0^t \mathrm{e}^{v(t-s)}\psi_s\,\mathrm{d}s,\qquad \psi\in C([0,\infty);\mathbb{R}).\tag{1.6}$$

Consequently we have the following corollary to Propositions 1.1 and 1.2, and Theorems 1.1 and 1.2. Recall that in the present paper $b$ is a bounded measurable function. If $b\in L^1$ then we set

$$\mathfrak{M}^{v,\eta}_t:=\eta\mathcal{D}^{v,\eta}_t(b)-\int_{\mathbb{R}} b(x)\,\mathrm{d}x\,\mathcal{J}^v\left(L^{B^v}(0)\right)_t.$$

**Corollary** 1.1. *(i) If $b\in W^{-1,\infty}$ then for all $T\in[0,\infty)$, $r\in(0,1)$ and $p\in[1,\infty)$ one has $\mathbb{E}\|\eta^r\mathcal{D}^{v,\eta}(b)\|^p_T\to 0$ as $\eta\to\infty$.*
*(ii) If $b\in L^1$ then for all $T\in[0,\infty)$ and $p\in[1,\infty)$ one has $\mathbb{E}\|\mathfrak{M}^{v,\eta}\|^p_T\to 0$ as $\eta\to\infty$.*
*(iii) If $b$ and $x\to xb(x)$ belong to $L^1$, then uniformly in $t$ on compact intervals*

$$\left(B_t,\sqrt{\eta}\,\mathfrak{M}^{v,\eta}_t(b)\right)\implies\left(\beta_t,2\alpha_b\,\mathcal{J}^v\left(\gamma_{L^{\beta^v}(0)}\right)_t\right)\quad \text{as } \eta\to\infty,$$

*where $\beta$ and $\gamma$ are independent standard Brownian motions, and $L^{\beta^v}$ is the local time of the Ornstein–Uhlenbeck process $\beta^v$.*
*(iv) If $b = b_{\mathcal{M}} + b_{\mathcal{D}}$, where $b_{\mathcal{M}} \in L^1$ and $b_{\mathcal{D}} \in \mathcal{D}$, then uniformly in $t$ on compact intervals*

$$(B_t, \eta\mathcal{D}_t^{v,\eta}(b)) \implies \left(\beta_t, \int_{\mathbb{R}} b(x)\,\mathrm{d}x\,\mathcal{J}^v\left(L^{\beta^v}(0)\right)_t + 2\sqrt{c_{b_{\mathcal{D}}}}\,\mathcal{J}^v(\gamma)_t\right),$$

*where $\beta$ and $\gamma$ are independent standard Brownian motions.*

## 2. Proof of Proposition 1.2

Let us fix $T$, $p$, and $\rho \in \mathcal{M}_{\mathrm{fin}}(\mathbb{R})$. Let $t \in [0,T]$. By occupation density times formula, see e.g. [RY, Corollary 1.6, Chap. VI, p. 224] for any $f \in L^1(\mathbb{R})$ we have

$$\eta A_t^{v,\eta}(f) = \eta\int_{\mathbb{R}} f(\eta x)\, L_t^{B^v}(x)\,\mathrm{d}x = \int_{\mathbb{R}} L_t^{B^v}\left(\eta^{-1}x\right) f(x)\,\mathrm{d}x.$$

Thus by a standard approximation argument we get

$$\eta A_t^{v,\eta}(\rho) = \int_{\mathbb{R}} L_t^{B^v}\left(\eta^{-1}x\right)\rho(\mathrm{d}x). \tag{2.1}$$

Consequently

$$M_t^{v,\eta}(\rho) = \int_{\mathbb{R}}\left\{L_t^{B^v}\left(\eta^{-1}x\right) - L_t^{B^v}(0)\right\}\rho(\mathrm{d}x)$$

and by Jensen's inequality,

$$\mathbb{E}\,\|M^{v,\eta}(\rho)\|_T^p \le (\|\rho\|_{\mathrm{Var}}(\mathbb{R}))^{p-1}\int_{\mathbb{R}}\mathbb{E}\left\|L^{B^v}\left(\eta^{-1}x\right) - L^{B^v}(0)\right\|_T^p \|\rho\|_{\mathrm{Var}}(\mathrm{d}x).$$

Next, by Tanaka's formula, see e.g. [RY, Th. 1.5, Chap. VI, p. 222], we have

$$L_t^{B^v}\left(\eta^{-1}x\right) - L_t^{B^v}(0) = I_t^\eta(x) + J_t^\eta(x),$$

where

$$I_t^\eta(x) := \left|B_t^v - \eta^{-1}x\right| - \left|\eta^{-1}x\right| - \left|B_t^v\right|,$$

and

$$J_t^\eta(x) := -\int_0^t\left\{\mathrm{sgn}\left(B_s^v - \eta^{-1}x\right) - \mathrm{sgn}\left(B_s^v\right)\right\}\mathrm{d}B_s^v.$$

Note that $|I_t^\eta(x)| \le 2|B_t^v|$. Hence it is easy to see that

$$\sup_{x\in\mathbb{R}}\mathbb{E}\left\|L^{B^v}\left(\eta^{-1}x\right) - L^{B^v}(0)\right\|_T^p < \infty.$$

Moreover, since $\mathbb{P}\left(B_s^v = 0\right) = 0$, $s > 0$, for any $x$ one has

$$\lim_{\eta\to\infty} \mathbb{E}\left\| L^{B^v}\left(\eta^{-1}x\right) - L^{B^v}(0)\right\|_T^p = 0.$$

Therefore one obtains $\mathbb{E}\| M^{v,\eta}(\rho)\|_T^p \to 0$ by means of Lebesgue's dominated convergence theorem. Clearly, to prove $\mathbb{E}\|\mathcal{M}^{v,\eta}(\rho)\|_T^p \to 0$ it is enough to show that $\mathbb{E}\|\eta\left(A^{v,\eta}(\rho) - \mathcal{A}^{v,\eta}(\rho)\right)\|_T^p \to 0$. To do this, note that $Y^{v,\eta} := \eta^{-1}X^{v,\eta}$ is a continuous semi-martingale and $Y^{v,\eta} = B^v + \eta^{-1}\mathcal{B}^{v,\eta}$, where

$$\mathcal{B}_t^{v,\eta} = \int_0^t \mathrm{e}^{v(t-s)} b\left(X_s^{v,\eta}\right) \mathrm{d}s.$$

Let us denote by $L^\eta$ the local time of $Y^{v,\eta}$. Then using the arguments contained in the proof of (2.1) we obtain

$$\eta\left(\mathcal{A}_t^{v,\eta}(\rho) - A_t^{v,\eta}(\rho)\right) = \int_{\mathbb{R}} \left\{L_t^\eta\left(\eta^{-1}x\right) - L_t^{B^v}\left(\eta^{-1}x\right)\right\} \rho(\mathrm{d}x).$$

Applying again Tanaka's formula we obtain

$$L_t^\eta\left(\eta^{-1}x\right) - L_t^{B^v}\left(\eta^{-1}x\right) = I_t^\eta(x) + J_t^\eta(x),$$

where

$$I_t^\eta(x) := \left|Y_t^{v,\eta} - \eta^{-1}x\right| - \left|B_t^v - \eta^{-1}x\right| - \eta^{-1}\int_0^t \operatorname{sgn}\left(Y_s^{v,\eta} - \eta^{-1}x\right) \mathrm{d}\mathcal{B}_s^{v,\eta},$$

and

$$J_t^\eta(x) := \int_0^t \left\{\operatorname{sgn}\left(B_s^v - \eta^{-1}x\right) - \operatorname{sgn}\left(Y_s^{v,\eta} - \eta^{-1}x\right)\right\} \mathrm{d}B_s^v.$$

Since $b$ is bounded it is easy to see that there is a constant $c$ depending on $v$ and $T$ such that

$$\|Y^{v,\eta} - B^v\|_T + \|I^\eta(x)\|_T \le c\eta^{-1}, \qquad \eta > 0,\ x\in\mathbb{R}. \tag{2.2}$$

Thus the proof will be completed as soon as we show that

$$\sup_{x\in\mathbb{R}} \mathbb{E}\|J^\eta(x)\|_T^p < \infty \quad \forall\eta \qquad \text{and} \qquad \lim_{\eta\to\infty} \mathbb{E}\|J^\eta(x)\|_T^p = 0 \quad \forall x\in\mathbb{R}. \tag{2.3}$$

Since

$$|B_t^u| = \left|B_t + u\int_0^t \mathrm{e}^{u(t-s)} B_s \,\mathrm{d}s\right| \le \left(1 + |u|\mathrm{e}^{|u|t}\right)\|B\|_t,$$

we have $|J_t^\eta(x)| \le C_1\left(J^{\eta,1}(x) + J_t^{\eta,2}(x)\right)$, where

$$J^{\eta,1}(x) := \|B\|_T \int_0^T \left|\operatorname{sgn}\left(B_t^v - \eta^{-1}x\right) - \operatorname{sgn}\left(Y_t^{v,\eta} - \eta^{-1}x\right)\right| \mathrm{d}t$$

and

$$J_t^{\eta,2}(x) := \left|\int_0^t \left\{\operatorname{sgn}\left(B_s^v - \eta^{-1}x\right) - \operatorname{sgn}\left(Y_s^{v,\eta} - \eta^{-1}x\right)\right\} \mathrm{d}B_s\right|.$$

Then (2.3) follows easily from (2.2) and Burkholder's inequality. □

## 3. Proof of Theorem 1.1

Let us fix $T < \infty$ and $\rho \in \mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$. Recall that $\rho = F'_\rho$, where $F_\rho(x) = \rho((-\infty, x))$. Let $g = F_\rho - \rho(\mathbb{R})\chi_{(0,\infty)}$. Then $g$ is bounded and, see Remark 1.2, it is square integrable. Clearly $\|g\|_{L^2(\mathbb{R})} = \alpha_\rho$. Define

$$G^\eta_t := \int_0^t g\left(\eta B^v_s\right) \mathrm{d}B^v_s \qquad \text{and} \qquad \mathcal{G}^\eta_t := \eta^{-1} \int_0^t g\left(X^{v,\eta}_s\right) \mathrm{d}X^{v,\eta}_s.$$

We have

$$G^\eta_t = \int_0^t F_\rho\left(\eta B^v_s\right) \mathrm{d}B^v_s - \rho(\mathbb{R}) \int_0^t \chi_{(0,\infty)}\left(\eta B^v_s\right) \mathrm{d}B^v_s.$$

By Tanaka's formula, see e.g. [RY, p. 222],

$$\int_0^t \chi_{(0,\infty)}\left(\eta B^v_s\right) \mathrm{d}B^v_s = (B^v_t)^+ - \frac{1}{2} L^{B^v}_t(0).$$

Thus

$$-2\int_0^t F_\rho\left(\eta B^v_s\right) \mathrm{d}B^v_s = -2G^\eta_t - 2\rho(\mathbb{R})(B^v_t)^+ + \rho(\mathbb{R}) L^{B^v}_t(0).$$

Let $H$ be a primitive of $F$. Then using the inverse Itô formula (1.1) we obtain

$$M^{v,\eta}(\rho) = -2G^\eta + I^\eta, \tag{3.1}$$

where

$$I^\eta_t = \frac{2}{\eta}\left(H\left(\eta B^v_t\right) - H(0)\right) - 2\rho(\mathbb{R})(B^v_t)^+.$$

First we will show that uniformly in $t \in [0, T]$,

$$\left(B_t, \sqrt{\eta} G^\eta_t\right) \implies \left(\beta_t, \alpha_\rho \gamma_{L^{\beta^v}_t(0)}\right), \tag{3.2}$$

where $\beta$ and $\gamma$ are independent Brownian motions. To do this note that $G^\eta = vJ^{\eta,1} + J^{\eta,2}$, where

$$J^{\eta,1}_t := \int_0^t g\left(\eta B^v_s\right) B^v_s \,\mathrm{d}s \qquad \text{and} \qquad J^{\eta,2}_t := \int_0^t g\left(\eta B^v_s\right) \mathrm{d}B_s.$$

Note that the function $x \to xg(x)$ is integrable, therefore it has a bounded primitive. Thus we have

$$\begin{aligned}\int_{\mathbb{R}} |g(x)x| \,\mathrm{d}x &\le \int_{-\infty}^0 |x| \, \|\rho\|_{\mathrm{Var}}((-\infty, x]) \,\mathrm{d}x + \int_0^\infty |x| \, \|\rho\|_{\mathrm{Var}}([x, \infty)) \,\mathrm{d}x \\ &\le \int_{\mathbb{R}} |x| \, \|\rho\|_{\mathrm{Var}}(\mathrm{d}x) < \infty.\end{aligned}$$

Thus Proposition 1.1 yields $\eta^{p/2}\mathbb{E}\left\| J^{\eta,1}\right\|_T^p \to 0$, and it is left to show that uniformly in $t \in [0,T]$,

$$\left(B_t, \sqrt{\eta}J_t^{\eta,2}\right) \Longrightarrow \left(\beta_t, \alpha_\rho \gamma_{L_t^{\beta^v}}\right). \tag{3.3}$$

Let $W^\eta$ be DDS Brownian motion of $\sqrt{\eta}J^{\eta,2}$, see e.g. [RY, Chap. V.1, p. 181]. Then $\sqrt{\eta}J_t^{\eta,2} = W^\eta_{\psi^\eta(t)}$, where $\psi^\eta(t) = \eta\int_0^t g^2\left(\eta B_s^v\right) \mathrm{d}s$. By Proposition 1.1,

$$\left\langle B, \sqrt{\eta}J^{\eta,2}\right\rangle_t = \sqrt{\eta}\int_0^t g\left(\eta B_s^v\right) \mathrm{d}s \to 0$$

and by Proposition 1.2, $\left\langle \sqrt{\eta}J^{\eta,2}, \sqrt{\eta}J^{\eta,2}\right\rangle_t = \psi_t^\eta \to \alpha_\rho^2 L_t^{B^v}(0)$. Thus (3.2) follows from [RY, Th. 2.3, Chap. XIII, p. 524], see also the proof of Th. 2.6, Chap. XIII, p. 526 from [RY].

Having shown (3.1) and (3.2), the proof of (1.3) will be completed as soon as we show that

$$\sqrt{\eta}I_t^\eta \to 0, \qquad \mathbb{P}\text{-a.s. uniformly in } t \in [0.T]. \tag{3.4}$$

To see this, set

$$h_\eta(x) = \sqrt{\eta}\left(\eta^{-1}H(\eta x) - \rho(\mathbb{R})(x)^+\right). \tag{3.5}$$

Then for $x \le 0$,

$$\begin{aligned}
|h_\eta(x)| &\le \eta^{-1/2}\left|H(\eta x)\right| \le \eta^{-1/2}\left(|H(0)| + \int_{\eta x}^0 |F(z)|\,\mathrm{d}z\right) \\
&\le \eta^{-1/2}\left(|H(0)| + \int_{-\infty}^0 \|\rho\|_{\mathrm{Var}}((-\infty,y])\,\mathrm{d}y\right) \\
&\le \eta^{-1/2}\left(|H(0)| + \int_{-\infty}^0 |y|\,\|\rho\|_{\mathrm{Var}}(\mathrm{d}y)\right).
\end{aligned}$$

Note that, by Fubini's

$$\int_0^{\eta x} F(z)\,\mathrm{d}z = \eta x\rho((-\infty,0]) + \int_0^{\eta x}\rho((0,z))\,\mathrm{d}z = \eta x\rho(-\infty,\eta x) - \int_{(0,\eta x)} y\rho(\mathrm{d}y).$$

Hence for $x > 0$,

$$\begin{aligned}
|h_\eta(x)| &= \sqrt{\eta}\left|\eta^{-1}\left(H(0) + \int_0^{\eta x} F(z)\,\mathrm{d}z\right) - x\rho(\mathbb{R})\right| \\
&= \sqrt{\eta}\left|\eta^{-1}H(0) - \eta^{-1}\int_{(0,\eta x)} y\rho(\mathrm{d}y) - x\rho([\eta x,\infty))\right| \\
&\le \eta^{-1/2}|H(0)| + 2\eta^{-1/2}\int_{\mathbb{R}} |y|\,\|\rho\|_{\mathrm{Var}}(\mathrm{d}y),
\end{aligned}$$

as

$$|\eta x\rho([\eta x,\infty))| \le \int_{\mathbb{R}} |y|\, \|\rho\|_{\mathrm{Var}}(\,\mathrm{d}y).$$

Thus there is a constant $C<\infty$ such that

$$|h_\eta(x)| \le C\eta^{-1/2}\left(1+\int_{\mathbb{R}}|y|\,\|\rho\|_{\mathrm{Var}}(\,\mathrm{d}y)\right) \qquad \text{for all } \eta>0,\ x\in\mathbb{R} \tag{3.6}$$

and (3.4) follows from the identity $\sqrt{\eta}I^\eta_t = 2h_\eta\left(B^v_t\right) - 2\left(\sqrt{\eta}\right)^{-1}H(0)$.

We proceed to the proof of (1.4). Let $L^\eta$ be the local time of $Y^{v,\eta} := \eta^{-1}X^{v,\eta}$. Since

$$\int_0^t \chi_{(0,\infty)}\left(X^{v,\eta}_s\right)\mathrm{d}Y^{v,\eta}_s = \left(Y^{v,\eta}_t\right)^+ - \frac{1}{2}L^\eta_t(0),$$

the inverse Itô formula (1.2) yields $\mathcal{M}^{v,\eta}(\rho) = -2\mathcal{G}^\eta + \mathcal{I}^\eta$, where

$$\mathcal{I}^\eta_t = \frac{2}{\eta}\left(H\left(X^{v,\eta}_t\right) - H(0)\right) - 2\rho(\mathbb{R})\left(Y^{v,\eta}_t\right)^+.$$

Thus the proof will be completed as soon as we show that

$$\left(B_t, \sqrt{\eta}\mathcal{G}^\eta_t\right) \implies \left(\beta_t, \alpha_\rho \gamma_{L^{\beta^v}_t(0)}\right) \tag{3.7}$$

and

$$\sqrt{\eta}\mathcal{I}^\eta_t \to 0,\ \mathbb{P}\text{-a.s. uniformly in } t\in[0.T]. \tag{3.8}$$

To show (3.7) we will use the ideas from the proof of (3.2). Namely, first we note that $\mathcal{G}^\eta = \mathcal{J}^{\eta,1} + \mathcal{J}^{\eta,2}$, where

$$\mathcal{J}^{\eta,1}_t := \eta^{-1}\int_0^t g\left(X^{v,\eta}_s\right)\left(b\left(X^{v,\eta}_s\right) + vX^{v,\eta}\right)\mathrm{d}s \quad \text{and} \quad \mathcal{J}^{\eta,2}_t := \int_0^t g\left(X^{v,\eta}_s\right)\mathrm{d}B_s.$$

Recall, see the proof of (3.2) that the function $x\to xg(x)$ is integrable. Hence, as $b$ is bounded the function $x\to g(x)(b(x)+vx)$ is integrable, and Proposition 1.1 yields $\eta^{p/2}\mathbb{E}\left\|\mathcal{J}^{\eta,1}\right\|^p_T \to 0$. To show that uniformly in $t\in[0,T]$,

$$\left(B_t, \sqrt{\eta}\mathcal{J}^{\eta,2}_t\right) \implies \left(\beta_t, \alpha_f\gamma_{L^{\beta^v}_t}\right)$$

we note that $\sqrt{\eta}\mathcal{J}^{\eta,2}_t = W^\eta_{\psi^\eta(t)}$ where $\psi^\eta(t) = \eta\int_0^t g^2\left(X^{v,\eta}_s\right)\mathrm{d}s$ and $W$ is DDS Brownian motion of $\sqrt{\eta}\mathcal{J}^{\eta,2}$. By Proposition 1.1,

$$\left\langle B, \sqrt{\eta}\mathcal{J}^{\eta,2}\right\rangle_t = \sqrt{\eta}\int_0^t g\left(X^{v,\eta}_s\right)\mathrm{d}s \to 0$$

and by Proposition 1.2,

$$\left\langle \sqrt{\eta}\mathcal{J}^{\eta,2}, \sqrt{\eta}\mathcal{J}^{\eta,2}\right\rangle_t = \psi^\eta_t \to \alpha^2_\rho L^{B^v}_t(0).$$

Thus (3.7) follows from [RY, Th. 2.3, Chap. XIII, p. 524]. To see (3.8) note that $\sqrt{\eta}\mathcal{I}^\eta_t = 2h_\eta\left(Y^{v,\eta}_t\right) - 2\left(\sqrt{\eta}\right)^{-1}H(0)$, where $h_\eta$ is given by (3.5), and consequently (3.8) follows from (3.6). □

## 4. Proof of Theorem 1.2

We will need the following lemma.

**Lemma 4.1.** *Let $(B^\eta)$ be a family of two sided Brownian motions, let $\psi : [0,\infty) \to \mathbb{R}$ be a deterministic function, and let $(\psi^\eta(t), t \geq 0)$ be a family of continuous processes such that for any $0 \leq T < \infty$, $\|\psi^\eta - \psi\|_T \to 0$ in probability, as $\eta \to \infty$. Then there is a Brownian motion $\beta$ such that $B^\eta_{\psi^\eta(t)} \implies \beta_{\psi(t)}$ uniformly in $t$ on compact intervals.*

*Proof.* Clearly it is enough to show that for a fixed $T < \infty$, $\gamma^\eta_t := B^\eta_{\psi^\eta(t)} - B^\eta_{\psi(t)}$, converges in probability to 0, uniformly with respect to $t \in [0,T]$. This follows from the convergence of $\psi^\eta$ to $\psi$, the uniform continuity of Brownian motion, and the fact that for all $\eta$, and $\varepsilon, \delta > 0$, one has

$$\begin{aligned}\mathbb{P}\left\{\|\gamma^\eta\|_T > \varepsilon\right\} &\leq \mathbb{P}\left\{\|\gamma^\eta\|_T > \varepsilon;\ \|\psi^\eta - \psi\|_T \leq \delta\right\} + \mathbb{P}\left\{\|\psi^\eta - \psi\|_T > \delta\right\} \\ &\leq \mathbb{P}\left\{\sup_{t,s:\,0\leq t,s\leq T,\,|t-s|\leq\delta} |B^\eta_t - B^\eta_s| > \varepsilon\right\} + \mathbb{P}\left\{\|\psi^\eta - \psi\|_T > \delta\right\}.\end{aligned}$$

□

*Proof of Theorem* 1.2. Let $L^\eta$ be the local time of $Y^\eta := \eta^{-1}X^{v,\eta}$. Write

$$\begin{aligned}V^\eta_t &:= \int_0^t \left(L^{B^v}_t\left(\eta^{-1}x\right) - L^{B^v}_t(0)\right)\rho_{\mathcal{M}}(\mathrm{d}x), \\ \mathcal{V}^\eta_t &:= \int_0^t \left(L^\eta_t\left(\eta^{-1}x\right) - L^\eta_t(0)\right)\rho_{\mathcal{M}}(\mathrm{d}x).\end{aligned}$$

Then, see the proof of Proposition 1.2, we have

$$\eta A^{v,\eta}(\rho) = V^\eta_t + \rho_{\mathcal{M}}(\mathbb{R})L^{B^v}_t(0) + \eta A^{v,\eta}\left(\rho_{\mathcal{D}}\right)$$

and

$$\eta \mathcal{A}^{v,\eta}(\rho) = V^\eta_t + \rho_{\mathcal{M}}(\mathbb{R})L^\eta_t(0) + \eta \mathcal{A}^{v,\eta}\left(\rho_{\mathcal{D}}\right).$$

Moreover, see again the proof of Proposition 1.2, we have $\mathbb{E}\left(\|V^\eta\|_T + \|\mathcal{V}^\eta\|_T\right) \to 0$.

Let $H \in W^{1,\infty}(\mathbb{R})$ be such that $\rho_{\mathcal{D}} = H''$. Let $F = H'$, and let

$$R^\eta_t = \int_0^t F\left(\eta B^v_s\right)\mathrm{d}B_s \qquad \text{and} \qquad \mathcal{R}^\eta_t = \int_0^t F\left(X^{v,\eta}_s\right)\mathrm{d}B_s. \tag{4.1}$$

Let

$$N^{v,\eta}_t := \int_0^t F\left(\eta B^v_s\right)B^v_s\,\mathrm{d}s \qquad \text{and} \qquad \mathcal{N}^{v,\eta}_t := \eta^{-1}\int_0^t F\left(X^{v,\eta}_s\right)X^{v,\eta}_s\,\mathrm{d}s.$$

Finally let $I^\eta_t = 2\eta^{-1}\left(H(\eta B^v_t) - H(0)\right)$, and let

$$\mathcal{I}^\eta = \frac{2}{\eta}\left(H(X^{v,\eta}_t) - H(0)\right) - \frac{2}{\eta}\int_0^t F\left(X^{v,\eta}_s\right) b\left(X^{v,\eta}_s\right) \mathrm{d}s.$$

By (1.1) and (1.2),

$$\eta A^{v,\eta}_t(\rho_{\mathcal{D}}) = I^\eta_t - 2vN^{v,\eta}_t - 2R^\eta_t$$

and

$$\eta \mathcal{A}^{v,\eta}_t(\rho_{\mathcal{D}}) = \mathcal{I}^\eta_t - 2v\mathcal{N}^{v,\eta}_t - 2\mathcal{R}^\eta_t.$$

Obviously, as $H$, $F$, and $b$ are bounded, we have $\mathbb{E}\left(\|I^\eta\|_T + \|\mathcal{I}^\eta\|_T\right) \to 0$ for any $T < \infty$. Thus, as

$$L^{B^v}_t(0) = |B^v_t| - \int_0^t \operatorname{sgn} B^v_s \,\mathrm{d}B^v_s$$

and

$$L^\eta_t(0) = |Y^\eta| - \int_0^t \operatorname{sgn} Y^\eta_s \,\mathrm{d}Y^\eta_s,$$

the proof will be completed as soon as we show that

$$\mathbb{E}\left(\|N^{v,\eta}\|_T + \|\mathcal{N}^{v,\eta}\|_T\right) \to 0 \qquad \text{for any } T < \infty \tag{4.2}$$

and that uniformly with respect to $t$ on compact intervals,

$$\left(B^v_t, \int_0^t \operatorname{sgn} B^v_s \,\mathrm{d}B^v_s, R^\eta_t\right) \Longrightarrow \left(\beta^v_t, \int_0^t \operatorname{sgn} \beta^v_s \,\mathrm{d}\beta^v_s, \sqrt{c_{\rho_{\mathcal{D}}}}\gamma_t\right) \tag{4.3}$$

and

$$\left(Y^\eta_t, \int_0^t \operatorname{sgn} Y^\eta_s \,\mathrm{d}Y^\eta_s, \mathcal{R}^\eta_t\right) \Longrightarrow \left(\beta^v_t, \int_0^t \operatorname{sgn} \beta^v_s \,\mathrm{d}\beta^v_s, \sqrt{c_{\rho_{\mathcal{D}}}}\gamma_t\right), \tag{4.4}$$

where $\beta$ and $\gamma$ are independent Brownian motions.

To see (4.2) note that

$$N^{v,\eta}_t = A^{v,\eta}_t(\rho_{\mathcal{D}})B^v_t - \int_0^t A^{v,\eta}_s(\rho_{\mathcal{D}}) \,\mathrm{d}B^v_s$$

and

$$\mathcal{N}^{v,\eta}_t = \eta^{-1}\mathcal{A}^{v,\eta}_t(\rho_{\mathcal{D}})X^{v,\eta}_t - \eta^{-1}\int_0^t \mathcal{A}^{v,\eta}_s(\rho_{\mathcal{D}}) \,\mathrm{d}X^{v,\eta}_s.$$

Thus (4.2) follows easily from Proposition 1.1.

It remains to check (4.3) and (4.4). Let $W^\eta$, and $\mathcal{W}^\eta$ be DDS Brownian motions of $R^\eta$, and $\mathcal{R}^\eta$, respectively. Then $R^\eta_t = W^\eta_{\psi^\eta(t)}$ and $\mathcal{R}^\eta_t = \mathcal{W}^\eta_{\varphi^\eta(t)}$, where

$$\psi^\eta(t) = \int_0^t F^2\left(\eta B^v_s\right) \mathrm{d}s \qquad \text{and} \qquad \varphi^\eta(t) = \int_0^t F^2\left(X^{v,\eta}_s\right) \mathrm{d}s.$$

Proposition 1.1 yields $\psi^\eta_t = \langle R^\eta\rangle_t \to c_{\rho_\mathcal{D}} t$ and $\varphi^\eta = \langle \mathcal{R}^\eta\rangle_t \to c_{\rho_\mathcal{D}} t$ uniformly in $t$ on bounded intervals. Thus by Lemma 4.1, $R^\eta_t \implies \sqrt{c_{\rho_\mathcal{D}}}\,\gamma_t$ and $\mathcal{R}^\eta_t \implies \sqrt{c_{\rho_\mathcal{D}}}\,\gamma_t$ uniformly in $t \in [0,T]$, where $\gamma$ is a Brownian motion. Finally, by Proposition 1.1 we have

$$\langle B^v, R^\eta\rangle_t = \int_0^t F\left(\eta B^v_s\right) \mathrm{d}s \to 0 \quad \text{and} \quad \langle Y^\eta, \mathcal{R}^\eta\rangle_t = \int_0^t F\left(\eta X^{v,\eta}_s\right) \mathrm{d}s \to 0,$$

and, again by Proposition 1.1, for $Z_t = \int_0^t \operatorname{sgn} B^v_s \, \mathrm{d}B^v_s$, and $\mathcal{Z}^\eta_t = \int_0^t \operatorname{sgn} Y^\eta_s \, \mathrm{d}Y^\eta_s$,

$$\langle Z, R^\eta\rangle_t = \int_0^t F\left(\eta B^v_s\right) \operatorname{sgn} \eta B^v_s \, \mathrm{d}s \to 0$$

and

$$\langle \mathcal{Z}^\eta, \mathcal{R}^\eta\rangle_t = \int_0^t F\left(\eta X^{v,\eta}_s\right) \operatorname{sgn} \eta X^{v,\eta}_s \, \mathrm{d}s \to 0,$$

because the function $\tilde{F}(x) = \operatorname{sgn} x\, F(x)$, $x \in \mathbb{R}$ has a bounded primitive. Thus the desired conclusion follows from [RY, Th. 2.3, Chap. XIII, p. 524] and [RY, Th. 2.6, Chap. XIII, p. 526]. □

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