The Lévy -Ito decomposition theorem on separable Banach spaces
(revised)

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Abstract

A direct definition of stochastic integrals for deterministic Banach valued functions on separable Banach spaces with respect to compensated Poisson random measures is given. For the case of Banach spaces of type 2 this definition yields a direct proof of the Lévy -Ito decomposition of a càdlàg process with stationary, independent increments into a jump and Brownian component.

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1 Introduction

In the first part of this article we give a direct definition of stochastic integrals for deterministic Banach valued functions $f : E \to F$ with respect to compensated Poisson random measure of Lévy processes $(X_t)_{t\geq 0}$ (Definition 2.23) on separable Banach spaces E . (In [3] the results of this article will be extended for the case of compensated Poisson random measure of additive processes $(X_t)_{t\geq 0}$. In [42] the whole approach is extended to the case of random functions.) We introduce different kinds of integrability conditions, strong p -integrability (in Definition 3.11) and simple p -integrability (in Definition 3.18), $p \geq 1$.
The simple p -integrability condition is satisfied when the "natural" integral of f on a set $\Lambda_n := \{\delta_n < \|x\| \leq 1\}$ (definition 3.3), which is the sum of the compensated jumps in Λ_n of $(X_t)_{t\geq 0}$ evaluated on the function f, converges, when $\delta_n > 0$ goes to zero, in $L^p(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is the probability space where the Lévy process $(X_t)_{t\geq 0}$ is defined. (We recall that the compensated Poisson random measure $N_t - t\nu$ of $(X_t)_{t\geq 0}$ might not be finite on $\{0 < \|x\| \leq 1\}$).
The strong p -integral is defined by approximation in L^p of the "natural" integrals of simple functions (Definition 3.4). This concept generalizes the known

definition of stochastic integration of real valued functions with respect to martingales measures [17], [49]), to Banach space valued functions, for the case where the martingale measures are given by compensated Poisson random measures. These results extend also the concept of stochastic integration of real valued functions with respect to compensated Poisson random measures associated to real valued Lévy processes $(X_t)_{t\geq 0}$ with stable law introduced in [45].
We prove that functions which are Bochner integrable w.r.t. the Lévy measure ν of $(X_t)_{t\geq 0}$ are strong 1 -integrable, and functions with values on separable Banach spaces F of type 2 (Definition 1.1 below), which satisfy the condition $\int \|f(x)\|^2\nu(dx) < \infty$ are strong 2 -integrable, the strong p -integrability being equivalent to the simple p -integrability, under the above conditions. Moreover we introduce the notion of simple integral (Definition 3.22) and prove that under the above conditions a function is simply integrable and the simple integral coincides with the simple or strong p -integrals, $p = 1, 2$.
In the second part of the paper we use these results to extend the Lévy -Ito decomposition theorem, known for Lévy -processes $(X_t)_{t\geq 0}$ with state space $\mathbb{R}^d$ [27],[28], [18], (see also [20], [10], [43]), to the case where the state is a separable Banach space of type 2 (see Theorem 4.1, last Section). To the best of our knowledge the only generalizations of this decomposition of càdlàg processes with stationary, independent increments (Lévy processes) into a jump and a Brownian part on infinite dimensional state spaces are given in [19], [47], [48], where the decomposition is proven for the case where the state space is a (co -) nuclear space, and in [12], for the case of Banach spaces (the latter with a rather sketchy proof). In these works however no direct expression of the Lévy part in terms of stochastic integrals is given (see also Remark 4.7 below).
For our proof of the decomposition theorem for the Lévy -processes $(X_t)_{t\geq 0}$ on separable Banach spaces $(E, \mathcal{B}(E))$, we need that the corresponding Lévy measure ν (see Definition 2.20) satisfies the condition

$$\int_{E\setminus\{0\}} min(1, \|x\|^2)\nu(dx) < \infty \qquad (1)$$

It is well known that on $(\mathbb{R}\setminus 0, \mathcal{B}(\mathbb{R}\setminus 0))$ the condition (1) is satisfied by any Lévy measure. Viceversa, any σ -finite measure on $(\mathbb{R}\setminus\{0\}, \mathcal{B}(\mathbb{R}\setminus\{0\}))$, satisfying (1), is a Lévy measure. The same correspondence between Lévy measures and the above condition holds in any separable Hilbert space ([35]), while in general separable Banach spaces $(E, \mathcal{B}(E))$ (1) is neither necessary nor sufficient for a σ -finite measure on $(E\setminus 0, \mathcal{B}(E\setminus 0))$ to be a Lévy measure. However separable Banach spaces of cotype 2 are characterized by the condition that any Lévy measure satisfies condition (1), while separable Banach spaces of type 2 are characterized by the condition that any σ -finite measure on $(E\setminus\{0\}, \mathcal{B}(E\setminus\{0\}))$, which satisfies (1), is a Lévy measure (see e.g. [4], [5], [15], [31]). We recall the definition of type -p, resp. cotype -p Banach spaces (see e.g. [4]).

Definition 1.1 *A separable Banach space B is of type p, $1 \leq p \leq 2$, if there is a constant K_p, such that if $\{X_i\}_{i=1}^n$ is any finite set of centered independent B-valued random variables, such that $E[||X_i||^p] < \infty$, then*

$$E[||\sum_{i=1}^n X_i||^p] \leq K_p \sum_{i=1}^n E[||X_i||^p] \tag{2}$$

Definition 1.2 *A separable Banach space B is of cotype p, $p \geq 2$, if there is a constant C_p, such that if $\{X_i\}_{i=1}^n$ is any finite set of centered independent B-valued random variables, such that $E[||X_i||^p] < \infty$, then*

$$E[||\sum_{i=1}^n X_i||^p] \geq C_p \sum_{i=1}^n E[||X_i||^p] \tag{3}$$

A Banach space is of type 2 as well as of cotype 2 if and only if it is isomorphic to a Hilbert space [25]. Typical examples of separable Banach spaces of type 2 (resp. cotype 2) are the spaces $L_p(\Omega, P)$, $p \in [2, \infty)$ (resp. $p \in [1, 2]$), where (Ω, P) is any measure space.

It is well known that on separable Banach spaces there is a one to one correspondence between infinitely divisible laws μ and weakly continuous convolution semigroups $(\mu_t)_{t \geq 0}$, the correspondence being such that $\mu_1(\cdot) = \mu(\cdot)$, $\hat{\mu}_t(\cdot) = (\hat{\mu}(\cdot))^t$ (where in general with $\hat{P}$ we denote the Fourier transform of a probability measure P). For the (canonical) process $(X_t)_{t \geq 0}$ with stationary independent increments associated with μ_t we have $P(X_t \in A) = \mu_t(A)$, $A \in \mathcal{B}(E)$. The Lévy -Khinchine decomposition theorem for infinitly divisible laws on separable Banach spaces (see e.g. Theorem 5.7.3 [29]), states that any infinitely divisible law μ is the convolution of a centered normal distribution, a delta distribution and a Poisson type distribution. For the proof of the Lévy -Khinchine formula on separable Banach spaces condition (1) is not needed. The decomposition theorem of the Lévy processes $(X_t)_{t \geq 0}$ (cfr. Theorems 4.1, 4.6) is however a path decomposition and our proof needs the stronger condition (1) to define in Section 3 the (deterministic, stochastic) integral $\int_{E \setminus \{0\}} f(x) \, (N_t(dx) - t\nu(dx))$ with respect to the compensated Poisson random measure $N_t - t\nu$. Using the scalar product in $\mathbb{R}$ (and in a similar way this could be proven on any separable Hilbert space H), the integral $\int_{E \setminus \{0\}} f(x) \, (N_t(dx) - t\nu(dx))$ can be defined by proving an isometry between $M_\nu^2 := \{f : E \to E \quad \mathcal{B}(E \setminus \{0\})/\mathcal{B}(E) \quad -measurable, \quad \int ||f(x)||^2 \, \nu(dx) < \infty\}$ and a subset of the space of square integrable random variables $L_2(\Omega, \mathcal{F}, P)$ ($(\Omega, \mathcal{F}, P)$ being the probability space over which the process is defined). In the case of separable Banach spaces of type 2 this isometry is replaced by an inequality ((71), below), which holds for any $f \in M_\nu^2$.

We expect our results to be applicable to the study of processes with jumps and the corresponding stochastic differential equations (SDEs) (with non Gaussian Lévy white noise) on separable Banach spaces, as e.g. [2], where we study processes (and corresponding SDE) obtained by subordination of Ornstein Uhlenbeck processes, or e.g. [32], where we prove existence and uniqueness of non linear SDEs with Lévy noise on Hilbert spaces, which arise when analyzing the dynamics of an infinite interacting particle system in statistical mechanics. For applications of stochastic processes with jumps on finite and infinite dimensional spaces see, e.g., [9].

The structure of the paper is as follows. In Sect.2 we define Lévy measures and processes on separable Banach spaces, as well as the associated Poisson random measures. In Section 3 we define and discuss deterministic stochastic integrals on separable Banach spaces. Section 4 gives our Lévy -Ito decomposition theorem (Theorems 4.1 and 4.6).

2 Poisson and Lévy measures of Lévy processes on separable Banach spaces

We assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq +\infty}, P)$, satisfying the "usual hypothesis", is given:
i) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$
ii) $\mathcal{F}_t = \mathcal{F}_t^+$, where $\mathcal{F}_t^+ = \cap_{u>t}\mathcal{F}_t$ for all t such that $0 \leq t \leq +\infty$, i.e. the filtration is right continuous

We shall define and study Lévy processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq +\infty}, P)$ with values in $(E, \mathcal{B}(E))$, where in the whole paper we assume that E is a separable Banach space with norm $\|\cdot\|$ and $\mathcal{B}(E)$ is the corresponding σ -algebra.

Definition 2.1 "Lévy process" *A process $(X_t)_{t\geq 0}$ with state space $(E, \mathcal{B}(E))$, is an $\mathcal{F}_t$ - Lévy process on $(\Omega, \mathcal{F}, P)$ if*
i) $(X_t)_{t\geq 0}$ is adapted (to $(\mathcal{F}_t)_{t\geq 0}$)
ii) $X_0 = 0$ a.s.
iii) $(X_t)_{t\geq 0}$ has increments independent of the past, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ if $0 \leq s < t$
iv) $(X_t)_{t\geq 0}$ has stationary increments, that is $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t$
v) $(X_t)_{t\geq 0}$ is stochastically continuous
vi) $(X_t)_{t\geq 0}$ is càdlàg.

Remark 2.2 *The class satisfying i) -v) is given canonically once an infinitely divisible probability measure E is fixed (as easily seen e.g. from [29] and [8]). That any such process has a càdlàg version follows from its being, after compensation, a martingale (see e.g. [13], [22],[36]).*

Definition 2.3 "Lévy process in law" *A process $(X_t)_{t\geq 0}$ with state space $(E, \mathcal{B}(E))$, is an $\mathcal{F}_t$ - Lévy process in law on $(\Omega, \mathcal{F}, P)$ if it satisfies the conditions i) to v) in definition 2.1.*

Definition 2.4 Counting process *Let $(T_n)_{n\geq 0}$ be a strictly increasing sequence of random variables with values in $I\!R_+$, such that $T_0 = 0$. Let*

$$\mathbf{1}_{t\geq T_n}(\omega) \quad = 1 \, if \, t \geq T_n(\omega) \tag{4}$$

$$0 \, if \, t < T_n(\omega) \tag{5}$$

$(N_t)_{t\geq 0}$ with $N_t = \sum_{n\geq 1} \mathbf{1}_{t\geq T_n}$ is called the counting process associated to $(T_n)_{n\geq 0}$.
$T = sup_n T_n$ is the explosion time of $(N_t)_{t\geq 0}$. If $T = +\infty$ a.s. then $(N_t)_{t\geq 0}$ is a counting process without explosion.

Remark 2.5 *The counting process $(N_t)_{t\geq 0}$ is adapted iff $(T_n)_{n\geq 0}$ are stopping times (see e.g. Theorem 22, Chapt I [38]).*

Definition 2.6 (Poisson process) *An adapted counting process without explosion is a Poisson process if it satisfies properties iii) and iv) in the definition 2.1 of Lévy process with state space E replaced by $I\!N \cup \{0\}$.*

We recall (see for example [38] Th. 2.3) that if $(N_t)_{t\geq 0}$ is a Poisson process, then there is $\lambda \geq 0$, called the "intensity of the Poisson process", such that

$$P(N_t = n) = \exp(-\lambda t)(\lambda t)^n/n! \quad n \in I\!N \cup \{0\} \tag{6}$$

$$E[N_t] = t\lambda \tag{7}$$

$$Var[N_t] = t\lambda \tag{8}$$

where $E[\cdot]$ (resp. $Var[\cdot]$) denotes the expectation (resp. variance) on $(\Omega, \mathcal{F}, P)$.

Now let $(X_t)_{t\geq 0}$ be a Lévy process on $(E, \mathcal{B}(E))$ (in the sense of Definition 2.1). Set $X_{s^-} := lim_{s\uparrow t} X_s$ and $\Delta X_s := X_s - X_s^-$

Theorem 2.7 *Let* $\Lambda \in \mathcal{B}(E)$, $0 \in (\overline{\Lambda})^c$ *(where as usual* $\overline{\Lambda}$ *denotes the closure of the set* Λ *and with* N^c *we denote the complement of the set* N*), and let*

$$N_t^\Lambda := \sum_{0<s\leq t} \mathbf{1}_\Lambda(\Delta X_s) = \sum_{n\geq 1} \mathbf{1}_{t\geq T_n^\Lambda} \tag{9}$$

where

$$T_1^\Lambda := inf\{s > 0 : \Delta X_s \in \Lambda\} \tag{10}$$

$$T_{n+1}^\Lambda := inf\{s > T_\Lambda^n : \Delta X_s \in \Lambda\}, \quad n \in I\!N. \tag{11}$$

N_t^Λ *is an adapted counting process without explosion. Moreover it is a Poisson process*

In order to prove Theorem 2.7 we need first to state and prove two further Propositions:

Proposition 2.8 *Let*

$$f: \quad I\!R_+ \to E \tag{12}$$

$$t \to f(t) \tag{13}$$

be a càdlàg function, then $\forall T > 0$, $\forall \Lambda \in \mathcal{B}(E)$, *such that* $0 \in \overline{\Lambda}^c$, f *has at most a finite number of jumps with values in* Λ *in the time* $[0, T]$*, i.e.*

$$N_T^\Lambda := number\{t \in [0, T] : \Delta f(t) \in \Lambda\} < +\infty \tag{14}$$

where

$$\Delta f(t) := f(t) - f^-(t) \tag{15}$$

$$f^-(t) := \lim_{s\uparrow t} f(t) \tag{16}$$

Proof of Proposition 2.8:
Suppose $\Lambda \subset E \setminus \mathcal{B}(0, 2\epsilon)$, $\epsilon > 0$, and ad absurdum that there is an infinite sequence $\{t_n\}_{n\in I\!N} \subset [0, T]$, such that $\Delta f(t_n) \in \Lambda$. There exists then a sub -sequence, which for simplicity we still denote by $\{t_n\}_{n\in I\!N}$, converging for $n \to +\infty$ to a point $t_0 \notin \{t_n\}_{n\in I\!N}$. We can suppose , taking if necessary a further subsequence, that $t_n \uparrow t_0$, or $t_n \downarrow t_0$.
Suppose $t_n \downarrow t_0$, then, as f is càdlàg, there exists $n \in I\!N$, s.th. $\forall t \in (t_0, t_n)$ $\|f(t) - f(t_0)\| < \epsilon$. This implies that $\forall j > n$ $\|f(t_j) - f(t_0)\| < \epsilon$, $\|f^-(t_j) - f(t_0)\| < \epsilon$, which implies $\|f(t_j) - f^-(t_j)\| < 2\epsilon$, against the ad absurdum hypothesis.
If on the other hand $t_n \uparrow t_0$, then, as f is càdlàg, there exists $n \in I\!N$, s.th. $\forall t \in (t_n, t_0)$ $\|f(t) - f^-(t_0)\| < \epsilon$. This implies that $\forall j > n$ $\|f(t_j) - f^-(t_0)\| < \epsilon$, $\|f^-(t_j) - f^-(t_0)\| < \epsilon$, which implies $\|f(t_j) - f^-(t_j)\| < 2\epsilon$, once more against the absurdum hypothesis. ■

Proposition 2.9 *Let $(X_t)_{t\geq 0}$ be a Lévy process with state space $(E, \mathcal{B}(E))$. Let $\Lambda \in \mathcal{B}(E)$, $0 \notin \overline{\Lambda}$, and let*

$$T_\Lambda^1 := inf\{s > 0 : \Delta X_s \in \Lambda\} \tag{17}$$

$$... \tag{18}$$

$$T_\Lambda^{n+1} := inf\{s > T_\Lambda^n : \Delta X_s \in \Lambda\} \tag{19}$$

$\{T_\Lambda^n\}_{n\in\mathbb{N}}$ is a strictly increasing sequence of stopping times.

Proof of Proposition 2.9:
That $\{T_\Lambda^n\}_{n\in\mathbb{N}}$ is strictly increasing, follows from the fact that $(X_t)_{t\geq 0}$ is càdlàg and Proposition 2.8 (which in particular tells us that the sequence $\{T_\Lambda^n\}_{n\in\mathbb{N}}$ can not have an accumulation point).
Let us prove that T_Λ^1 is a stopping time. As by hypothesis the filtration $(\mathcal{F}_t)_{0\leq 0\leq\infty}$ is right continuous, it is sufficient to prove that $\{T_\Lambda^1 < t\} \in \mathcal{F}_t$. Let $\Lambda_m := \{x \in E : d(x, \Lambda) < \frac{1}{m}\}$. We prove that

$$\{T_\Lambda^1 < t\} = \cap_{m\in\mathbb{N}} \cap_{n\in\mathbb{N}} \cup_{p,q\in\mathbb{Q}:|p-q|<\frac{1}{n},0<p<q<t}\{(X_p - X_q) \in \Lambda_m\} \tag{20}$$

(where $\mathbb{Q}$ denotes the rational numbers) and hence $\{T_\Lambda^1 < t\} \in \mathcal{F}_t$. We have:

$$\{T_\Lambda^1 < t\} = \cup_{a\in(0,t)}\Delta X_a \in \Lambda \tag{21}$$

$$\subset \cup_{a\in(0,t)}\{\Delta X_a \in \Lambda \cap \cap_{m\in\mathbb{N}} \cap_{n\in\mathbb{N}} \cup_{p,q\in\mathbb{Q}:|p-q|<\frac{1}{n},0<p<a<q<t} \tag{22}$$

$$\{\|X_q - X_a\| < \frac{1}{2m} \cup \|X_p - X_a^-\| < \frac{1}{2m}\}\} \tag{23}$$

$$\subset \cup_{a\in(0,t)}\{\Delta X_a \in \Lambda \cap \cap_{m\in\mathbb{N}} \cap_{n\in\mathbb{N}} \cup_{p,q\in\mathbb{Q}:|p-q|<\frac{1}{n},0<p<q<t}\{(X_p - X_q) \in \Lambda_m\} \tag{24}$$

$$\subset \cap_{m\in\mathbb{N}} \cap_{n\in\mathbb{N}} \cup_{p,q\in\mathbb{Q}:|p-q|<\frac{1}{n},0<p<q<t}\{(X_p - X_q) \in \Lambda_m\} \tag{25}$$

$$\cap_{m\in\mathbb{N}} \cap_{n\in\mathbb{N}} \cup_{p,q\in\mathbb{Q}:|p-q|<\frac{1}{n},0<p<q<t}\{(X_p - X_q) \in \Lambda_m\} \tag{26}$$

$$\subset \cap_{m\in\mathbb{N}} \cup_{0<a_m<t} \{X_{a_m} - X_{a_m^-} \in \Lambda_m\} \tag{27}$$

$$\subset \cup_{0<a<t}\{X_a - X_{a^-} \in \Lambda\} = \{T_\Lambda^1 < t\} \tag{28}$$

where the latter inclusion follows from Propostion 2.8.

In the same way we see that

$$\{T_\Lambda^k < t\} = \cap_{m\in\mathbb{N}} \cap_{n\in\mathbb{N}} \{\cup_{p^1,q^1\in\mathbb{Q}:|p^1-q^1|<\frac{1}{n},0<p^1<q^1<t}\{(X_{p^1} - X_{q^1}) \in \Lambda_m\} \tag{29}$$

$$... \cup_{p^k,q^k\in\mathbb{Q}:|p^k-q^k|<\frac{1}{n},0<p^k<q^k<t} \{(X_{p^k} - X_{q^k}) \in \Lambda_m\} \tag{30}$$

so that $\{T_\Lambda^k < t\} \in \mathcal{F}_t \; \forall k \in \mathbb{N}$ ■

Proof of Theorem 2.7:

$$N_t^\Lambda := \sum_{0<s\leq t} \mathbf{1}_\Lambda(\Delta X_s) = \sum_{n=1}^{\infty} \mathbf{1}_{T_\Lambda^n \leq t} \tag{31}$$

moreover, $T_\Lambda^1 > 0$ and $\lim_{n\to\infty} T_\Lambda^n = +\infty$, so that N_t^Λ is a counting process without an explosion.
Moreover N_t^Λ has independent identically distributed (i.i.d.) increments, i.e. it satisfies property iii) and iv) in the definition of Lévy processes, as can easily be verified by the fact that if $0 \leq s < t$, $\sigma\{N_t^\Lambda - N_s^\Lambda\} = \sigma\{X_u - X_v; s \leq v < u < t\}$ is independent of $\mathcal{F}_s$. By the stationarity of the distribution of $(X_t)_{t\geq 0}$ we also have that $N_t^\Lambda - N_s^\Lambda$ is identically distributed as N_{t-s}^Λ. ■

The following definitions coincide with those, e.g., in [8].

Definition 2.10 *A ring $\mathcal{F}$ on a set S (we write $(\mathcal{F}, S)$) is a collection of subsets, such that the following properties hold:*
i) $\phi \in \mathcal{F}$
ii) $A, B \in \mathcal{F}$ *implies* $A \setminus B \in \mathcal{F}$
iii) $A, B \in \mathcal{F}$ *implies* $A \cup B \in \mathcal{F}$

Definition 2.11 *A pre- measure μ on $(\mathcal{F}, S)$ is a set function*

$$\mu: \quad \mathcal{F} \to I\!R_+ \tag{32}$$

which satisfies
i) $\mu(\phi) = 0$
ii) $A_i \in \mathcal{F}$, $i \in I\!N$, $A_i \cap A_j = \phi$ *if* $i \neq j$, *and* $\cup_{i=1}^\infty A_i \in \mathcal{F}$ *implies* $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$

Definition 2.12 *A pre- measure μ on $(\mathcal{F}, S)$ is a σ -finite pre- measure if there is a sequence $\{\Lambda_n\}_{n\in I\!N}$, such that $\Lambda_n \in \mathcal{F}$, $\mu(\Lambda_n) < \infty$ $\forall n \in I\!N$, $\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_n \subset \ldots \subset \cup_1^\infty \Lambda_n = S$.*

Theorem 2.13 *Let $\mathcal{B}(E \setminus \{0\})$ be the trace σ -algebra on $E \setminus \{0\}$ of the Borel σ -algebra $\mathcal{B}(E)$ on E , and $\mathcal{F}((E \setminus \{0\}) := \{\Lambda \in \mathcal{B}(E \setminus \{0\}) : 0 \in (\overline{\Lambda})^c\}$, then $\mathcal{F}((E \setminus \{0\})$ is a ring and for all $\omega \in \Omega$ the set function*

$$N_t^\cdot := N_t(\omega, \cdot): \quad \mathcal{F}(E \setminus \{0\}) \to I\!R_+ \tag{33}$$

$$\Lambda \to N_t^\Lambda(\omega) \tag{34}$$

is a σ -finite pre- measure

We first assume Theorem 2.13 and prove the following Corollary 2.14. Then we prove Theorem 2.13.

Corollary 2.14 *For any $\omega \in \Omega$ there is a unique σ -finite measure on $\mathcal{B}(E\setminus\{0\})$*

$$N_t(\omega,\cdot): \quad \mathcal{B}(E\setminus\{0\}) \to I\!R_+ \tag{35}$$
$$A \to N_t^A(\omega) \tag{36}$$

which is the continuation of the σ -finite pre- measure on $\mathcal{F}(E\setminus\{0\})$ given by Theorem 2.13.

Proof of Corollary 2.14:
We prove that $\mathcal{B}(E\setminus\{0\})$ is the σ -algebra generated by $\mathcal{F}(E\setminus\{0\})$. The result is then a consequence of a known theorem (see e.g. [8] Satz 5.7, §5, Chapt. I, or [11] Chapt. II).
That $\mathcal{F}(E\setminus\{0\}) \subset \mathcal{B}(E\setminus\{0\})$ is obvious.
Suppose $A = B\cap E\setminus\{0\}$, $B \in \mathcal{B}(E)$, and $0 \in \overline{A}$. Then

$$A = \cup_{n=0}^{\infty} A_n \tag{37}$$
$$A_n = A\cap\Lambda_n\,, \quad \Lambda_n = \{x\in E: \tfrac{1}{n+1} < \|x\| \le \tfrac{1}{n}\} \quad for \quad n\ge 1 \tag{38}$$
$$A_0 = A\cap\{x\in E: \|x\| > 1\} \tag{39}$$

■

Remark 2.15 $\forall A\in\mathcal{B}(E\setminus\{0\})$ $N_t(A) = \lim_{N\to\infty} N_t(\cup_{n=0}^{N} A_n)$ *(with A, A_n as above)*

Proof of Theorem 2.13: That $\mathcal{F}(E\setminus\{0\})$ satisfies i), ii), iii) of definition 2.10 can easily be checked. That $N_t^A(\omega)\ge 0$ and satisfies i) in definition 2.11 is also easily verified.
Let us verify condition ii) in definition 2.11. Let $\{A_i\}_{i\in I\!N}$ given as in ii), then

$$N_t^{\cup_{i=1}^{\infty}A_i} = \sum_{0<s\le t} \mathbf{1}_{\cup_{i=1}^{\infty}A_i}(\Delta X_s) \tag{40}$$
$$= \sum_{0<s\le t}\sum_{i=1}^{\infty} \mathbf{1}_{A_i}(\Delta X_s)\,, \tag{41}$$

where the equality holds, as the jumps in A_i and A_j occur at different times if $i\ne j$. From Proposition 2.8 it follows that the sum in (40) is convergent, so that

$$N_t^{\cup_{i=1}^{\infty}A_i} = \sum_{i=1}^{\infty}\sum_{0<s\le t} \mathbf{1}_{A_i}(\Delta X_s) \tag{42}$$
$$\sum_{i=1}^{\infty} N_t^{A_i} \tag{43}$$

Hence $\Lambda \to N_t^{\Lambda}(\omega)$ is a σ -additive measure on $\mathcal{F}(E\setminus\{0\})$, for every fixed $\omega\in\Omega$. That $N_t^{\cdot}(\omega)$ is a σ -finite pre -measure, is seen immediately by taking $\Lambda_n = \{x\in E: \|x\| > \frac{1}{n}\}$ in definition 2.12.

Definition 2.16 *$N_t(\omega, \cdot) : \Lambda \to N_t^{\Lambda}(\omega)$ is a random measure on $(E, \mathcal{B}(E)$. It is the Poisson random measure of the Lévy process $(X_t)_{t\geq 0}$.*

Theorem 2.17 *The set function $\nu(\Lambda) := E[N_1^{\Lambda}(\omega)] \in I\!R$, $\Lambda \in \mathcal{F}(E \setminus \{0\})$, $\omega \in \Omega$ satisfies:*

$$\nu : \quad \mathcal{F}(E \setminus \{0\}) \to I\!R_+ \tag{44}$$

$$\Lambda \to E[N_1^{\Lambda}(\omega)] \tag{45}$$

and is a σ -finite pre -measure on $((E \setminus \{0\}), \mathcal{F}(E \setminus \{0\}))$

Corollary 2.18 *There is a unique σ -finite measure on the σ -algebra $\mathcal{B}(E \setminus \{0\})$*

$$\nu : \quad \mathcal{B}(E \setminus \{0\}) \to I\!R_+ \tag{46}$$

$$A \to E[N_1^{A}(\omega)] \tag{47}$$

which is the continuation to $\mathcal{B}(E \setminus \{0\})$ of the σ -finite pre- measure ν on the ring $((E \setminus \{0\}), \mathcal{F}(E \setminus \{0\}))$, given by Theorem 2.17.

Proof of Theorem 2.17:
As N_t^{Λ} is $\forall \Lambda \in \mathcal{F}(E \setminus \{0\})$ a Lévy process with bounded jumps (by 1), it follows (see e.g. Theorem 34 Chapt I §4 [38]) that $E[N_t^{\lambda}] < +\infty$ $\forall t \in I\!R_+$, $\forall \Lambda \in \mathcal{F}(E \setminus \{0\})$. The proof follows then the same lines as the proof of Theorem 2.13. ■

Proof of Corollary 2.18:
The proof is similar to the one of Corollary 2.14. ■

Remark 2.19 *Let A, $(A_n)_{n \in I\!N_0}$ be as in (37), then $\nu(A) = \lim_{N\to\infty} \nu(\cup_{n=0}^{N} A_n)$.*

In the last part of this Section we prove that the measure constructed in Corollary 2.18 is a Lévy measure, according to the following:

Definition 2.20 *A σ -finite positive measure ν on $(E \setminus \{0\}, \mathcal{B}(E \setminus \{0\}))$ is a <u>"Lévy measure"</u>, if there is a probability measure μ on $(E, \mathcal{B}(E))$ such that the Fourier transform $\hat{\mu}(F)$, $F \in E'$ satisfies*

$$\hat{\mu}(F) = \exp \int_{E\setminus\{0\}} exp(iF(x) - 1 - iF(x)\mathbf{1}_{\|x\|\leq 1})\nu(dx) \tag{48}$$

We call μ the "Poisson type measure" (associated) with (the) "Lévy measure" ν.

Theorem 2.21 *The σ -finite measure ν of Corollary 2.18 is a Lévy measure.*

Proof:
From the definition 2.1 of Lévy process it follows that the measure μ such that $\mu(A) = P(X_t \in A)$, $\forall A \in \mathcal{B}(E)$, is infinitely divisible. From the Lévy - Khinchine representation theorem for infinitely divisible laws on Banach spaces (see e.g. [4], [29]) it follows that $\mu = \mathcal{G} \star \delta_\alpha \star \mathcal{L}$, where $\star$ denotes the convolution, $\mathcal{G}$ is a centered Gaussian measure, δ_α is the delta measure in the point $\alpha \in E$ and $\mathcal{L}$ is a Poisson type probability measure with Lévy measure $\tilde{\nu}$. $\mathcal{G}$, δ_α, $\mathcal{L}$ are uniquely defined. Moreover, the Lévy measure $\tilde{\nu}$ is the unique σ -finite measure on $\mathcal{B}(E \setminus \{0\})$ such that for each $\delta > 0$, $\tilde{\nu}(\|x\| = \delta) = 0$ and $A \in \mathcal{B}(E)$ $\lim_{n\to\infty} n\mu^{\frac{1}{n}}(A \cap B_\delta^c) = \tilde{\nu}(A \cap B_\delta^c)$, where $B_\delta^c := \{x \in E : \|x\| > \delta\}$ (see e.g. [29] Theorem 5.7.3 or [4]). Moreover, we have

$$\lim_{n\to\infty} n\mu^{\frac{1}{n}}(A \cap B_\delta^c) = \lim_{n\to\infty} \sum_{k=0}^{n-1} P(X_{\frac{k+1}{n}} - X_{\frac{k}{n}} \in A \cap B_\delta^c) \tag{49}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} E[\mathbf{1}_{X_{\frac{k+1}{n}} - X_{\frac{k}{n}} \in A \cap B_\delta^c}] = E[N_1^{A \cap B_\delta^c}] = \nu(A \cap B_\delta^c) \tag{50}$$

■

Definition 2.22 *We call the measure ν of Theorem 2.21 "the Lévy measure of the Lévy process $(X_t)_{t\geq 0}$".*

Definition 2.23 *We call the random measure $q_t(\omega, \cdot) := N_t(\omega, \cdot) - t\nu(\cdot)$ "the compensated Poisson random measure of the Lévy process $(X_t)_{t\geq 0}$".*

3 Integration (of deterministic functions) with respect to random measures of Lévy processes on separable Banach spaces

In this Section we shall define the integration with respect to the compensated Poisson random measure $q_t(\omega, \cdot) := N_t(\omega, \cdot) - t\nu(\cdot)$, $\forall t \geq 0$, where N_t (resp. ν) is the Poisson random measure (resp. Lévy measure) of the Lévy process $(X_t)_{t\geq 0}$ on the separable Banach space $(E, \mathcal{B}(E))$. (In [42] the whole approach is extended to the case of random functions.) We shall consider here integration of deterministic functions $f : E \to F$, f being $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ -measurable, where $(F, \mathcal{B}(F))$ is a separable Banach space with norm $\|\cdot\|_F$. We will define different kinds of integrability conditions , strong p -integrability and simple p -integrability, $p \geq 1$. Moreover we will prove that functions which are Bochner integrable w.r.t. the Lévy measure ν on $E \setminus \{0\}$ are strong 1 -integrable, and functions with values on separable Banach spaces $(F, \mathcal{B}(F))$ of type 2, which satisfy the condition $\int \|f(x)\|^2 \nu(dx) < \infty$ are strong 2 -integrable, the strong p -integrability being equivalent to the simple p -integrability, under the above conditions. Moreover we will introduce the notion of simple integral w.r.t. q_t and prove that under the above conditions a function is simply integrable w.r.t. q_t and the simple integral coincides with the simple or strong p -integral, $p = 1, 2$. In the following Section we will prove that in these two cases the Lévy -Ito decomposition of a Lévy process $(X_t)_{t \geq 0}$ (into a jump semimartingale and a continuous semimartingale with a centered Brownian motion being the martingale part) holds, the jump martingale being a strong p -integral of the function $f(x) = x$ with respect to the compensated Poisson random measure of the Lévy process $(X_t)_{t \geq 0}$.

Let us first introduce the integration w.r.t. Poisson random measure N_t on the field $\mathcal{F}(E \setminus \{0\})$ and recall separately the definition of Bochner integral, in particular applied to the Lévy measure ν on the σ -algebra $\mathcal{B}(E \setminus \{0\})$.

Definition 3.1 *Let $\Lambda \in \mathcal{F}(E \setminus \{0\})$, $f : E \to F$ be $\mathcal{F}(E \setminus \{0\})/\mathcal{B}(F)$ - measurable and bounded on $E \setminus \{0\} \cap \Lambda$, then*

$$\int_\Lambda f(x)\, N_t(\omega, dx) = \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_\Lambda(\Delta X_s) \tag{51}$$

Remark 3.2 *Let $\Lambda \in \mathcal{F}(E \setminus \{0\})$, $f : E \to F$ be $\mathcal{F}(E \setminus \{0\})/\mathcal{B}(F)$ - measurable and bounded on $E \setminus \{0\} \cap \Lambda$, then*
i)

$$\| \int_\Lambda f(x)\, N_t(\omega, dx)\, \|_F \leq \sum_{0<s\leq t} \|f(\Delta X_s)(\omega)\|_F\, \mathbf{1}_\Lambda(\Delta X_s)(\omega) \tag{52}$$

ii) $\int_\Lambda f(x)\, N_t(\omega, dx)$ *, and* $X_t - \int_\Lambda f(x)\, N_t(\omega, dx)$ *are adapted processes with values in F, and have independent stationary increments.*
iii)

$$\sum_{0<s\leq t} f(\Delta X_s)(\omega)\, \mathbf{1}_\Lambda(\Delta X_s)(\omega) = \sum_{n \in I\!N} f(\Delta X_{T_n^\Lambda}) \mathbf{1}_{T_n^\Lambda \leq t}\,, \tag{53}$$

where T_n^Λ are defined in (10), (11).
From ii) and iii) one has the following property:
iv) $\int_\Lambda f(x)\, N_t(\omega, dx)$ *is a Lévy processes.*
v) Like in Theorem 39, Chapt I, [38], where the independence of the corresponding characteristic functions is analyzed, the following can be proven: if Λ_1, $\Lambda_2 \in \mathcal{F}(E \setminus \{0\})$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, $f_1 : E \to E$, $f_2 : E \to E$ are bounded in Λ_1 resp. Λ_2, then $\int_{\Lambda_1} f_1(x)\, N_t(\omega, dx)$ *is independent of* $\int_{\Lambda_2} f_2(x)\, N_t(\omega, dx)$.

Definition 3.3 *Let $\Lambda \in \mathcal{F}(E \setminus \{0\})$, $f : E \to F$ be $\mathcal{F}(E \setminus \{0\})/\mathcal{B}(F)$ - measurable and bounded on $E \setminus \{0\} \cap \Lambda$, then the natural integral w.r.t. the compensated Poisson random measure $N_t(dx) - t\nu(dx)$ is*

$$\int_\Lambda f(x)\,(N_t(\omega, dx) - t\nu(dx)) = \sum_{0<s\le t} f((\Delta X_s)(\omega))\,\mathbf{1}_\Lambda(\Delta X_s) - t\int_\Lambda f(x)\nu(dx) \tag{54}$$

where the last term is understood as a Bochner integral.

We remind here the definition and properties of Bochner integrals. To this end let us define the set of "simple functions" $\mathcal{S}(E/F)$:

Definition 3.4 *A function f belongs to the sets $\mathcal{S}(E/F)$ of simple functions , if $f : E \setminus \{0\} \to F$ is such that*

$$f(x) = \sum_{k=1}^{N} a_k \mathbf{1}_{A_k}, \quad A_k \in \mathcal{F}(E \setminus \{0\}), \tag{55}$$

with $N \in \mathbb{N}$, $a_k \in F$, $k \in (1, ...N)$. If $E = F$, we write $\mathcal{S}$ instead of $:= \mathcal{S}(E/E)$

The Bochner integral w.r.t. any σ -finite measure ν on $\mathcal{B}(E \setminus \{0\})$ of a function $f : E \to F$, which is $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ measurable, is well defined on every $\Lambda \in \mathcal{B}(E \setminus \{0\})$ (see e.g. [52], Chapt. V, §5).
We recall here its Definition.

Definition 3.5 *Let $f : \{E \setminus 0\} \to F$ be a "simple function" like in equation (55). For any $\Lambda \in \mathcal{B}(E \setminus \{0\})$ the Bochner integral of f w.r.t. ν on Λ is*

$$\int_\Lambda f(x)\nu(dx) = \sum_{k=1}^{N} a_k \nu(A_k \cap \Lambda) \tag{56}$$

Let $f : E \to F$ be $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ measurable , f is said to be Bochner -integrable w.r.t. ν, if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ a.s. and

$$\lim_{n\to\infty} \int \|f_n - f\|_F\, \nu(dx) = 0, \tag{57}$$

The Bochner -integral of f on any $\Lambda \in \mathcal{B}(E \setminus \{0\})$ is

$$\int_\Lambda f(x)\,\nu(dx) = \lim_{n\to\infty} \int_\Lambda f_n(x)\,\nu(dx) \tag{58}$$

and is independent of the sequence $f_n \to f$, as long as (57) holds.

Remark 3.6 *Let $f : E \to F$ be $\mathcal{B}(E\backslash\{0\})/\mathcal{B}(F)$ measurable , then the following properties hold:*

i) there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions converging ν -a.s. to f (see e.g. Proof of Theorem (B.J. Pettis) Chapt.V, §4, [52]);

ii) if the sequence $\{f_n\}_{n\in\mathbb{N}}$ in i) satisfies (57), then the limit in (58) exists, and does not depend on the choice of $\{f_n\}_{n\in\mathbb{N}}$, i.e. the Bochner -integral is well defined (see e.g. Chapt.V, §5, [52]);

iii)) f is Bochner -integrable w.r.t. ν iff $\int \|f(x)\|_F \,\nu(dx) < \infty$ (Theorem (S.Bochner) e.g. [52], Chapt. V, §5,);

iv) $\forall\Lambda \in \mathcal{B}(E\setminus\{0\}$ $\|\int_\Lambda f(x)\nu(dx)\|_F \leq \int_\Lambda \|f(x)\|_F\nu(dx)$ (e.g., Corollary 1 in Chapt.V, §5, [52]).

We are interested in the following set of functions. Let $p \geq 1$,

$$M_\nu^p(E/F) := \{f : E \to F \quad \mathcal{B}(E\backslash\{0\})/\mathcal{B}(F) \quad -measurable, \quad \int \|f(x)\|^p \,\nu(dx) < \infty\} \tag{59}$$

Remark 3.7 *A function $f : E \to F \quad \mathcal{B}(E\setminus\{0\})/\mathcal{B}(F) \quad -$ measurable is Bochner integrable w.r.t. ν on $E\setminus\{0\}$ if and only if $f \in M_\nu^1(E/F)$*

Similarly as in the proof of the "if -part" of a Theorem (S.Bochner) in [52], Chapt. V, §5 the following can be proved

Proposition 3.8 *For any σ -finite measure ν on $(E\setminus\{0\}, \mathcal{B}(E\setminus\{0\})$ and for any $f \in M_\nu^p(E/F)$, there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions converging ν -a.s. to f, such that*

$$\lim_{n\to\infty} \int \|f_n - f\|_F^p \,d\nu = 0, \tag{60}$$

Proof of Proposition 3.8:
Let $f \in M_\nu^p(E/F)$ and let $\{g_n\}_{n\in\mathbb{N}} \in \mathcal{S}(E/F)$ be a sequence converging ν -a.s. to f. Let

$$f_n(x) = g_n(x) \quad if \quad \|g_n(x)\| \leq \|f(x)\|(1 + 2^{-n}) \tag{61}$$

$$f_n(x) = 0 \quad if \quad \|g_n(x)\| > \|f(x)\|(1 + 2^{-n}). \tag{62}$$

Then $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{S}(E/F)$, $\|f_n(x)\|^p \leq \|f(x)\|^p(1 + 2^{-n})^p$, $\lim_{n\to 0} \|f_n(x) - f(x)\|^p = 0$ ν -a.s.. Thus applying the Lebesgue dominated convergence theorem to $\|f_n(x) - f(x)\|^p \leq 4\|f(x)\|^p(1 + 2^{-n})^p$, (60) follows. ■

We define now the integral w.r.t. the compensated Poisson random measure $q_t := N_t - t\nu$ of a Lévy -process $(X_t)_{t\geq 0}$. To this purpose we give the following definition:

Definition 3.9 *Let $p \geq 1$. $L_p^F(\Omega, \mathcal{F}, P)$ is the space of F -valued random variables, such that $E\|Y\|^p = \int \|Y\|^p dP < \infty$. We denote by $\|\cdot\|_p^F$ (or simply $\|\cdot\|_p$ when $E = F$) the quasi -norm ([29]) given by $\|Y\|_p^F = (E\|Y\|_F^p)^{1/p}$. Given $(Y_n)_{n\in\mathbb{N}}, Y \in L_p^F(\Omega, \mathcal{F}, P)$, we write $\lim_{n\to\infty}^p Y_n = Y$ if $lim_{n\to\infty}\|Y_n - Y\|_p^F = 0$*

Definition 3.10 *Let $f \in \mathcal{S}(E/F)$ be of the form*

$$f(x) = \sum_{k=1}^{N} a_k \mathbf{1}_{A_k}(x)\,, a_k \in E\setminus\{0\},\ A_k \in \mathcal{F}(E\setminus\{0\}),\ A_k \cap A_j = \emptyset \, if\, k \neq j. \tag{63}$$

The integral of f w.r.t. the random measure q_t on any set $\Lambda \in \mathcal{B}(E\setminus\{0\})$ is defined by

$$\int_\Lambda f(x) q_t(dx) := \sum_{k=1}^{N} a_k q_t(A_k \cap \Lambda). \tag{64}$$

Definition 3.11 *Let $p \geq 1$ We say that $f : E \to F$, which is $\mathcal{B}(E\setminus\{0\})/\mathcal{B}(F)$ -measurable, is strong p -integrable on $\Lambda \in \mathcal{B}(E\setminus\{0\})$ w.r.t. the random measure q_t if the limit*

$$\int_\Lambda f(x) q_t(dx) := lim_{n\to\infty}^p \int_\Lambda f_n(x) q_t(dx) \tag{65}$$

exists for any sequence $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{S}$ which satisfies the condition in Proposition 3.8, and does not depend on the choice of the sequence $\{f_n\}_{n\in\mathbb{N}}$.

Remark 3.12 *Let f be strong p -integrable. Then $\forall \Lambda \in \mathcal{B}(E\setminus\{0\})$*

$$E[\int_\Lambda f(x) q_t(dx)] = 0 \tag{66}$$

In fact, by definition of Bochner integral of random variables from $(\Omega, \mathcal{F}_\infty, P)$ to $(F, \mathcal{B}(F))$, one has

$$E[\int_\Lambda f(x) q_t(dx)] = \lim_{n\to\infty} E[\int_\Lambda f_n(x) q_t(dx)] = 0 \tag{67}$$

as

$$\lim_{n\to\infty} \| \int_\Lambda f(x) q_t(dx) - \int_\Lambda f_n(x) q_t(dx) \| = 0 \quad P-a.s. \tag{68}$$

and $\int_\Lambda f_n(x) q_t(dx) \in \mathcal{S}(E/F)$.

Remark 3.13 *Let f, g be strong p -integrable. For any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is strong p -integrable and we have that $\forall \Lambda \in \mathcal{B}(E\setminus\{0\})$*

$$\alpha \int_\Lambda f(x) q_t(dx) + \beta \int_\Lambda g(x) q_t(dx) = \int_\Lambda (\alpha f(x) + \beta g(x)) q_t(dx) \tag{69}$$

Theorem 3.14 *Let $f \in M^1_\nu(E/F)$, then f is strong 1 -integrable w.r.t. q_t. Moreover*

$$E[\| \int_\Lambda f(x) q_t(dx) \|] \le 2t \int_\Lambda \|f(x)\| \nu(dx) \quad \forall \Lambda \in \mathcal{B}(E \setminus \{0\}) \tag{70}$$

Theorem 3.15 *Suppose $(F, \mathcal{B}(F))$ is a separable Banach space of type 2. Let $f \in M^2_\nu(E/F)$, then f is strong 2 -integrable w.r.t. q_t. Moreover*

$$E[\| \int_\Lambda f(x) q_t(dx) \|^2] \le 4K_2 t \int_\Lambda \|f(x)\|^2 \nu(dx) \quad \forall \Lambda \in \mathcal{B}(E \setminus \{0\}) \tag{71}$$

where K_2 is the constant K_p, $p = 2$ in the Definition 1.1 (of type p Banach spaces).

Theorem 3.16 *Suppose $(F, \mathcal{B}(F)) := (H, \mathcal{B}(H))$ is a separable Hilbert space. Let $f \in M^2_\nu(E/H)$, then f is strong 2 -integrable w.r.t. q_t. Moreover*

$$E[\| \int_\Lambda f(x) q_t(dx) \|^2] = t \int_\Lambda \|f(x)\|^2 \nu(dx) \quad \forall \Lambda \in \mathcal{B}(E \setminus \{0\}) \tag{72}$$

Proof of Theorem 3.14:
First we remark that given $f \in \mathcal{S}(E/F)$ of the form (63) the inequality (70) holds. In fact, given f as in (63), it follows from Theorem 2.7

$$E[\| \textstyle\int_\Lambda f(x) q_t(dx) \|] \le tE[\sum_{k=1}^N \|a_k\| \|N_t(A_k \cap \Lambda) - \nu(A_k \cap \Lambda)\|] \tag{73}$$

$$\le tE[\textstyle\sum_{k=1}^N \|a_k\| [|N_t(A_k \cap \Lambda)| + \nu(A_k \cap \Lambda)]] \tag{74}$$

$$= 2t \textstyle\sum_{k=1}^N \|a_k\| \nu(A_k \cap \Lambda) \tag{75}$$

$$= 2t \textstyle\int_\Lambda \|f(x)\| \nu(dx) \quad \forall \Lambda \in \mathcal{B}(E \setminus \{0\}), \tag{76}$$

where in the latter equality we used the definition of ν -integral, (63) and

$$\|\mathbf{1}_\Lambda f(x)\| = \sum_{k=1}^N \|a_k\| \mathbf{1}_{A_k \cap \Lambda}. \tag{77}$$

Let $f \in M^1_\nu(E/F)$, and let $\{f_n\}_{n \in I\!N} \in \mathcal{S}(E/F)$ be a sequence satisfying the condition in Proposition 3.8. Then, for all $\Lambda \in \mathcal{B}(E \setminus \{0\})$:

$$E[\| \int_\Lambda (f_n(x) - f_m(x)) q_t(dx) \|] \le 2tE \int_\Lambda \|f_n(x) - f_m(x)\| \nu(dx) \tag{78}$$

so that $\int_\Lambda f_n(x) q_t(dx)$ is a Cauchy sequence in $L^F_1(\Omega, \mathcal{F}, P)$ and the limit (65) exists for $p = 1$. (Moreover the limit does not depend on the choice of the sequence $\{f_n\}_{n \in I\!N}$).

Let us prove that inequality (70) holds for any $f \in M^1_\nu(E/F)$. Using the triangle inequality and Remark 3.6, iv) we get:

$$E[\|\int_A f(x)q_t(dx)\|] \leq E[\|\int_A(f(x) - f_n(x)q_t(dx)\| \tag{79}$$
$$+E[\|\int_A f_n(x)q_t(dx)\|] \tag{80}$$
$$\leq E[\|\int_A(f(x) - f_n(x))q_t(dx)\|] \tag{81}$$
$$+E[\int_A \|f_n(x)\|t\nu(dx)] \tag{82}$$

(because of (73)). Moreover we have

$$\lim_{n\to\infty} E[\int_A \|f_n(x)\|\nu(dx)] = E[\int_A \|f(x)\|\nu(dx)] \tag{83}$$

because

$$E[\int_A |\|f_n(x)\| - \|f(x)\||\nu(dx)] \leq E[\int_A \|f_n(x) - f(x)\|\nu(dx)] \tag{84}$$

so that (70) holds. ■

Proof of Theorem 3.15:
First we remark that given $f \in \mathcal{S}(E/F)$ of the form (63) the inequality (71) holds. In fact, given f as in (63), it follows from Theorem 2.7

$$E[\|\int_\Lambda f(x)q_t(dx)\|^2] \leq K_2 t E[\sum_{k=1}^N \|a_k\|^2\nu(A_k\cap\Lambda)] = K_2 t \int_\Lambda \|f(x)\|^2\nu(dx) \quad \forall\Lambda \in \mathcal{B}(E\backslash\{0\}) \tag{85}$$

in the first inequality we use the assumption that f is of type 2 and $E[\|N_t(A_k \cap \Lambda) - t\nu(A_k \cap \Lambda)\|^2] = t\nu(A_k \cap \Lambda)$. The equality in the r. h. s. of (85) follows from (77) which implies that $\|\mathbf{1}_\Lambda f\|^2 = \sum_{k=1}^N \|a_k\|^2 \mathbf{1}_{A_k\cap\Lambda}$

Let $f \in M^2_\nu$, and let $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{S}(E/F)$ be a sequence satisfying the condition in Theorem 3.8. Then, for all $\Lambda \in \mathcal{B}(E \setminus \{0\})$, using (85) we obtain

$$E[\|\int_\Lambda (f_n(x) - f_m(x))q_t(dx)\|^2] \leq tK_2 E \int_\Lambda \|f_n(x) - f_m(x)\|^2\nu(dx) \tag{86}$$

so that $\int_\Lambda f_n(x)q_t(dx)$ is a Cauchy sequence in $L^E_2(\Omega, \mathcal{F}, P)$ and the limit (65) for $p = 2$ exists. Moreover the limit does not depend on the choice of the sequence $\{f_n\}_{n\in\mathbb{N}}$. Moreover we have, using the triangle inequality and (85)

$$E[\|\int_A f(x)q_t(dx)\|^2] \leq 2E[\|\int_A(f(x) - f_n(x)q_t(dx)\|^2 \tag{87}$$
$$+2E[\|\int_A f_n(x)q_t(dx)\|^2] \tag{88}$$
$$\leq 2E[\|\int_A(f(x) - f_n(x))q_t(dx)\|^2] \tag{89}$$

$$+2K_2E[\int_A \|f_n(x)\|^2 t\nu(dx)] \quad (90)$$
$$\leq 2E[\| \int_A (f(x)-f_n(x))q_t(dx)\|^2] \quad (91)$$
$$+4K_2E[\int_A \|f_n(x)-f(x)\|^2 t\nu(dx)] \quad (92)$$
$$+4K_2E[\int_A \|f(x)\|^2 t\nu(dx)] \quad (93)$$

so that (71) holds. ■

Proof of Theorem 3.16:
First we remark that given $f \in \mathcal{S}(E/F)$ of the form (63) the equality (72) holds. In fact, given f as in (63), it can be easily checked that

$$E[\| \int_\Lambda f(x)q_t(dx)\|^2] = tE[\sum_{k=1}^{N} \|a_k\|^2 \nu(A_k \cap \Lambda)] = t\int_\Lambda \|f(x)\|^2 \nu(dx) \quad \forall \Lambda \in \mathcal{B}(E\setminus\{0\}) \quad (94)$$

The rest of the proof can be done by the same argument as in the proof of Theorem 3.15. ■

Proposition 3.17 *Let $p \geq 1$, f be p -strong integrable and $f \in M_\nu^p(E/F)$. For all $\Lambda \in \mathcal{F}(E\setminus\{0\})$ the strong p -integral of f coincides with the natural integral of f, i.e.*

$$\int_\Lambda f(x)q_t(dx) = \sum_{0<s<t} f(\Delta X_s)\mathbf{1}_{\Delta X_s \in \Lambda} - t\int_\Lambda f(x)\nu(dx) \quad P-a.s. \quad (95)$$

Proof:
Let $f \in \mathcal{S}(E/F)$, like in (55). From (9) it follows that $\forall \Lambda \in \mathcal{F}(E\setminus\{0\})$

$$\int_\Lambda f(x)q_t(dx) = \sum_{k=1}^N a_k(N_t(A_k\cap\Lambda) - t\nu(A_k\cap\Lambda)) \quad (96)$$
$$= \sum_{0<s<t}\sum_{k=1}^N a_k \mathbf{1}_{\Delta X_s\in A_k\cap\Lambda} - \sum_{k=1}^N a_k\nu(A_k\cap\Lambda) \quad (97)$$
$$= \sum_{0<s<t} f(\Delta X_s)\mathbf{1}_{\Delta X_s\in\Lambda} - t\int_\Lambda f(x)\nu(dx) \quad (98)$$

Let $f \in M_\nu^p(E/F)$, $\{f_n\}_{n\in I\!N} \in \mathcal{S}(E/F)$ be a sequence satisfying the condition in Proposition 3.8. Let N be such that $\nu(N)=0$ and $\lim_{n\to 0}\|f_n(x)-f(x)\|=0$ $\forall x \in N^c$. Let $B = \{\omega \in \Omega : N_t(N)(\omega) > 1)\}$. Then $P(B) \leq E[N_t(N)]$ $= \nu(N) = 0$. It follows

$$\lim_{n\to\infty}\sum_{0<s<t}(f_n(\Delta X_s(\omega)) - f(\Delta X_s(\omega)))\mathbf{1}_{\Delta X_s(\omega)\in\Lambda} = 0 \quad \forall\omega\in B^c \quad (99)$$

Moreover, since $\Lambda \in \mathcal{F}(E\setminus\{0\})$ we have $\nu(\Lambda) < \infty$ and using (60) and Hölder inequality

$$\lim_{n\to\infty}\|\int_\Lambda (f_n(x)-f(x))\nu(dx)\|^p = 0 \quad (100)$$

■

Definition 3.18 *Let $p \geq 1$. We say that $f : E \to F$, which is $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ - measurable, is* <u>*simply p -integrable*</u> *w.r.t. the random measure q_t if for any sequence $\delta_n > 0$, which converges to zero when $n \to \infty$, the limit*

$$\int_{0<\|x\|\leq 1} f(x)(N_t(x)-t\nu(dx)) := lim^p_{n\to\infty} \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}}(\Delta X_s) - t\int_{\Lambda_{\delta_n}} f(x)\nu(dx) \tag{101}$$

exists with

$$\Lambda_{\delta_n} := \{x \in E \setminus \{0\} : \delta_n < \|x\| \leq 1\}\,. \tag{102}$$

The <u>*simple p -integral*</u> *on a set $\Lambda \in \mathcal{F}(E\setminus\{0\}$ coincides with the natural integral in Definition 3.3. On a set $\Lambda \in \mathcal{B}(E \setminus \{0\}$ it is defined as*

$$\int_{\Lambda} f(x)(N_t(x)-t\nu(dx)) := lim^p_{n\to\infty} \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}\cap\Lambda}(\Delta X_s) - t\int_{\Lambda_{\delta_n}\cap\Lambda} f(x)\nu(dx) \tag{103}$$

(and does henceforth not depend on the choice of the sequence δ_n satisfying the above hypothesis)

Proposition 3.19 *Let $f \in M^1_\nu(E/F)$, then f is simply 1 -integrable and the simple 1 -integral coincides with the strong 1 -integral.*

Proposition 3.20 *Let $f \in M^2_\nu(E/F)$, F a separable Banach space of type 2, then f is simply 2 -integrable. The simple 2 -integral coincides with the strong 2 -integral.*

Corollary 3.21 *Let $f \in M^2_\nu(E/H)$, and H a separable Hilbert space, then f is simply 2 -integrable. The simple 2 -integral coincides with the strong 2 -integral.*

Proof of Proposition 3.19:
Let $f \in M^1_\nu(E/F)$, from inequality (70) and Proposition 3.17 it follows

$$E[\| \textstyle\int_\Lambda f(x)q_t(dx) - \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}\cap\Lambda}(\Delta X_s) \tag{104}$$

$$-t \textstyle\int_{\Lambda_{\delta_n}\cap\Lambda} f(x)\nu(dx)\|] \tag{105}$$

$$= E[\| \textstyle\int_\Lambda f(x)q_t(dx) - \int_{\Lambda_{\delta_n}\cap\Lambda} f(x)q_t(dx)\|] \tag{106}$$

$$\leq 2 \textstyle\int_\Lambda \|f(x) - \mathbf{1}_{\Lambda_{\delta_n}}(x)f(x)\|\nu(dx) \tag{107}$$

$$\leq 2 \textstyle\int_{\Lambda\cap\{0<\|x\|\leq\delta_n\}} \|f(x)\|\nu(dx) \tag{108}$$

which converges to zero when $n \to \infty$. ∎

Proof of Proposition 3.20 and Corollary 3.21: the proof is similar to the proof of Proposition 3.19, the Corollary is an immediate consequence. ∎

Definition 3.22 *We say that $f : E \to F$, which is $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ - measurable, is simply integrable w.r.t. the random measure q_t if for any sequence $\delta_n > 0$, which converges to zero when $n \to \infty$, the limit*

$$\int_{0<\|x\|\leq 1} f(x)(N_t(x)-t\nu(dx)) := lim_{n\to\infty} \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}}(\Delta X_s) - t\int_{\Lambda_{\delta_n}} f(x)\nu(dx) \tag{109}$$

exists a.s., with Λ_{δ_n} defined in (102).
The simple integral on a set $\Lambda \in \mathcal{F}(E \setminus \{0\}$ coincides with the natural integral in Definition 3.3. On a set $\Lambda \in \mathcal{B}(E \setminus \{0\}$ it is defined as

$$\int_{\Lambda} f(x)(N_t(x)-t\nu(dx)) := lim_{n\to\infty} \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}\cap\Lambda}(\Delta X_s) - t\int_{\Lambda_{\delta_n}\cap\Lambda} f(x)\nu(dx) \tag{110}$$

where the sequence converges a.s...

Remark 3.23 *We remark that the convergence in this definition is a.s., whereas in Definition 3.18 it was in the L_p^F -sense.*

Remark 3.24 *Let f be simply integrable. Then for all $\Lambda \in \mathcal{B}(E \setminus \{0\})$*

$$E[\int_{\Lambda} f(x)q_t(dx)] = 0 \tag{111}$$

Let f, g be simply integrable. For any $\alpha, \beta \in I\!R$, $\alpha f + \beta g$ is simply integrable and we have that $\forall \Lambda \in \mathcal{B}(E \setminus \{0\})$

$$\alpha \int_{\Lambda} f(x)q_t(dx) + \beta \int_{\Lambda} g(x)q_t(dx) = \int_{\Lambda} (\alpha f(x) + \beta g(x))q_t(dx) \tag{112}$$

Proposition 3.25 *If f is simply p -integrable for some $p \geq 1$, then it is simply integrable, and the simple p - integral coincides with the simple integral.*

Proof of Proposition 3.25:

As f is simply p -integrable for some $p \geq 1$, it follows that for any sequence $\{\delta_n\}$, such that $\delta_n \to 0$ when $n \to \infty$

$$\int_{0<\|x\|\leq 1} f(x)(N_t(x)-t\nu(dx)) := lim_{n\to\infty} \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}}(\Delta X_s) - t\int_{\Lambda_{\delta_n}} f(x)\nu(dx)\,, \tag{113}$$

the convergence being in probability. From a theorem of Ito -Nisio (Theorem 3.1, [21])(see also [46]) it follows that the limit exists a.s. and coincides with the p -simple integral of f w.r.t. q_t. ■

Corollary 3.26 *Let $f : E \to F$ be $\mathcal{B}(E\backslash\{0\})/\mathcal{B}(F)$ -measurable, $f \in M_\nu^1(E/F)$. f is simply integrable. The simple integral coincides with the simple 1 -integral and strong 1 -integral.*

Corollary 3.27 *Let $f : E \to F$ be $\mathcal{B}(E\backslash\{0\})/\mathcal{B}(F)$ -measurable, $f \in M_\nu^2(E/F)$, F a separable Banach space of type 2. f is simply integrable. The simple integral coincides with the simple 2 -integral and the strong 2 -integral.*

Corollary 3.28 *Let $f : E \to F$ be $\mathcal{B}(E\backslash\{0\})/\mathcal{B}(F)$ -measurable, $f \in M_\nu^2(E/F)$, F a separable Hilbert space. The simple integral coincides with the simple 2 -integral and the strong 2 -integral.*

Proof of the Corollaries 3.26, 3.27, 3.28:

The proof follows from Proposition 3.25 and Proposition 3.19, resp. Proposition 3.20, Corollary 3.21. ■

Proposition 3.29 *If f is simply integrable and for some $p \geq 1$ fixed*

$$sup_{n \geq N} E[\| \sum_{0<s\leq t} f((\Delta X_s))\, \mathbf{1}_{\Lambda_n}(\Delta X_s) - t \int_{\Lambda_n} f(x)\nu(dx)\|^p] < \infty \tag{114}$$

then f is simply p -integrable.

Proof of Proposition 3.29:

The statement of Proposition 3.29 follows from a theorem of Hoffmann -Joergensen ([16] Theorem 5.5 Chap. II). ■.

Remark 3.30 *The simple integrability of a function f does not imply in general that the function f is Bochner integrable on $E \setminus \{0\}$ w.r.t. ν (hence simply integrability does not imply strong 1 -integrability). In fact, the function $f(x) = x$ is simply integrable w.r.t. any compensated Poisson Random measure $N_t - t\nu$ as results from the proof of the Lévy - Khinchine formula on separable Banach spaces (see e.g.[4], [29])) but it is not true that f is Bochner integrable on $E \setminus \{0\}$ for all Lévy measures ν (i.e. there exist Lévy measures ν such that $f(x) = x \notin M_\nu^1$), see e.g. [4].*

Remark 3.31 *For previous definitions of stochastic integrals with respect to general "abstract" martingales on Banach spaces, see e.g. [6],[12], [26], [33],[34], [36],[39],[40], [41], [50], [51] and, for the case of Hilbert spaces, e.g. [24], [37]. For previous definitions of stochastic integrals of real valued functions with respect to general "abstract" martingale measures on Banach spaces, see e.g. [17], [49]. The main point of the present paper, in this context, is to define stochastic integrals of Banach valued functions with respect to a "concretely constructed" Banach valued Lévy noise.*

4 The Lévy -Ito decomposition theorem on separable Banach spaces of type 2

We have the following

Theorem 4.1 Lévy -Ito decomposition theorem on separable Banach spaces *Let $(X_t)_{t\geq 0}$ be a Lévy -process on a separable Banach space $(E, \mathcal{B}(E))$, and ν the corresponding Lévy measure (according to Definition 2.22). Suppose $N_t(\omega, dx)$ is the Poisson random measure and respectively $q_t(\omega, dx) := N_t(\omega, dx) - t\nu(dx)$ the compensated Poisson random measure associated to the Lévy process $(X_t)_{t\geq 0}$. Suppose the following condition holds*

c) *E is a separable Banach space of type* 2, *and*

$$\int_{\{E\backslash 0\}} min(1, \|x\|^2)\, \nu(dx) < \infty\,. \tag{115}$$

Then for all $K > 0$, there is $\alpha_K \in E$ such that $\forall t \geq 0$

$$X_t = B_t + \int_{\|x\|<K} x(N_t(dx) - t\nu(dx)) + \alpha_K t + \int_{\|x\|\geq K} xN_t(dx) \quad P-a.s. \tag{116}$$

(where we omit here for simplicity to write the dependence on $\omega \in \Omega$), where $(B_t)_{t\geq 0}$ is an E -valued Brownian motion with 0 -mean. For all $\Lambda \in \mathcal{F}(E\setminus\{0\})$, $(B_t)_{t\geq 0}$ is independent of $(N_t^{\Lambda})_{t\geq 0}$, (with the notation $N_t^{\Lambda}(\omega) := N_t(\omega, \Lambda)$). The integral $\int_{\|x\|\geq K} x(N_t(dx) - t\nu(dx))$ is the strong 2 -integral of the function $f(x) = x$ w.r.t. q_t.

Remark 4.2 *Let μ be such that $\mu(A) = P(X_1 \in A)$, $\forall A \in \mathcal{B}(E)$. Let us take $K = 1$ in the decomposition (116). ¿From Theorem 2.21 and the Lévy -Khinchine representation theorem for infinitely divisible laws on separable Banach spaces (see e.g. [29] Theorem 5.7.3 or [4]) it follows that $\mu = \mathcal{G} \star \delta_{\alpha_1} \star \mathcal{L}$, $\mathcal{G}$ is the distribution of the Brownian motion $(B_t)_{t\geq 0}$, $\mathcal{L}$ is the Poisson type probability measure associated with the Lévy measure ν, and $\alpha_1 \in E$.*

In order to prove Theorem 4.1 we need first to prove some further results (Propositions 4.3, 4.4).

Proposition 4.3 *Let $(X_t)_{t\geq 0}$ be a Lévy -process on $(E, \mathcal{B}(E))$ with jumps bounded by a constant $K > 0$ P -a.s., i.e.*

$$\|\Delta X_s\| \leq K \quad \forall s \geq 0, P - a.s. \tag{117}$$

then

$$E[\|X_t\|^n] < \infty \quad \forall n \in I\!N, t \geq 0 \tag{118}$$

The proof can be done according to the proof proposed in [38], Theorem 34, Chapt. I, for the case $E = I\!R$, which uses the following result, that we generalize to the case where E is a separable Banach space:

Proposition 4.4 *Let $(X_t)_{t\geq 0}$ be a Lévy -process on $(E, \mathcal{B}(E))$, and T be a stopping time, then the process $(Y_t)_{t\geq 0}$ with*

$$Y_t := X_{T+t} - X_T \tag{119}$$

is a Lévy process adapted to $(\mathcal{H}_t)_{t\geq 0}$ with $\mathcal{H}_t := \mathcal{F}_{t+T}$

We recall the following

Definition 4.5 *If T is a stopping time, then*

$$\mathcal{F}_T := \{\Gamma \in \mathcal{F}_\infty / \Gamma \cap \{T \leq t\} \in \mathcal{F}_t\} \tag{120}$$

Proof of Proposition 4.4:
For each $F \in E'$, with E' the topological dual of E, i.e. the set of all linear continuous mappings from E into $I\!R$, $(F(X_t))_{t\geq 0}$ is a Lévy process on $I\!R$. It follows from, e.g., Theorem 32 Chapt. I [38], that $(F(Y_t)))_{t\geq 0}$ is a Lévy process adapted to $(\mathcal{H}_t)_{t\geq 0}$. As any probability measure on $(E, \mathcal{B}(E))$ is uniquely determined by its Fourier transform and hence by its finite -dimensional distributions, i.e. as the distribution of X_t is uniquely determined by the distributions of $F(X_t)$, $\forall F \in E'$, the statement in Proposition 4.4 follows. ■

Proof of Proposition 4.3
Let

$$T^1 := inf\{s > 0 : \|X_s\| \geq K\}\,, \tag{121}$$

$$T^{n+1} := inf\{s > T^n : \|X_s - X_{T_n}\| \geq K\}\,, n \in I\!N \tag{122}$$

then, similarly as in the proof of Proposition 2.9, it can be proven by induction that $\{T^n_\Lambda\}_{n\in I\!N}$ is a strictly increasing sequence of stopping times. As by hypothesis $\Delta X_{T_n} \leq K$ $\forall n \in I\!N$, P -a.s., it follows $sup_s\|X^{T_n}_s\| \leq 2nK$ P -a.s., where $X^{T_n}_s = X_{T_n \cap s}$. Moreover, because of Proposition 4.4, it follows that $E[\exp^{-T_n}]$ $= E[e^{-T_1}]^n$. So that

$$P(\|X_t\| > 2nK) \leq P(T_n < t) \leq E[\mathbf{1}_{T_n<t}e^{t-T_n}] \leq e^t E[e^{-T_n}] = e^t E[e^{-T_1}]^n \tag{123}$$

This implies that X_t has exponential moments and hence moments of all orders. In fact

$$E[e^{\alpha\|X_t\|}] = \alpha \int_0^\infty e^{\alpha s} P(\|X_t\| > s) ds \tag{124}$$

$$\leq \alpha e^t \int_0^\infty e^{\alpha s} E[e^{-T_1}]^{\frac{s}{2K}} ds \leq e^t \quad \forall \alpha \in \mathbb{R}_+ , \tag{125}$$

which implies in particular (118). ∎

Theorem 4.6 *Let $(X_t)_{t\geq 0}$ be a Lévy process with jumps bounded by a constant $K > 0$, P -a.s.. Suppose condition C) in theorem 4.1 is satisfied. Let*

$$B_t := X_t - E[X_t] - \int_{\|x\|\leq K} x(N_t(dx) - t\nu(dx)) , \tag{126}$$

then $(B_t)_{t\geq 0}$ is an E -valued Brownian motion with zero mean.

Proof of Theorem 4.6:
Let

$$B_t^n := X_t - E[X_t] - \int_{x\in\Lambda_n} x(N_t(dx) - t\nu(dx)) \tag{127}$$

with

$$\Lambda_n := \{x \in E \setminus \{0\} : \frac{1}{n} < \|x\| \leq K\} \tag{128}$$

From the results in the previous Section it follows that $(B_t^n)_{t\geq 0}$ are Lévy processes with zero mean and bounded jumps. This implies in particular that the (B_t^n) are E -valued martingales (see [33] , Chapt.2, §8, for the definition of E -valued martingale) with finite moments of all orders. $(\|B_t^n\|)_{t\geq 0}$ are then positive sub -martingales (Lemma 8.11,Chapt. II, Part I, [33]), so that Doob's inequality holds. In particular:

$$E[(sup_{t\geq 0}\|B_t^n\|)^2] \leq 4 sup_{t\geq 0} E[\|B_t^n\|^2] \leq 4E[\|X_t - E[X_t]\|^2] + 4\int_{\|x\|\leq K} \|x\|^2 t\nu(dx)). \tag{129}$$

B_t^n converge when $n \to \infty$ in $L_2^E(\Omega, P)$ to B_t uniformly in t on compacts. Moreover there is a subsequence converging P -a.s. uniformly in t on compacts, so that $(B_t)_{t\geq 0}$ is continuous.
For any $F \in E'$, it follows that $(F(B_t))_{t\geq 0}$ is a Lévy process, which is P -a.s. continuous, and hence a real -valued Brownian motion. It follows that $(B_t)_{t\geq 0}$ is an E -valued Brownian motion (see e.g. [29], Proposition 5.2.3, which implies that if the finite dimensional distributions of a probability measure ρ on a Banach space E are Gaussian, so is ρ). ∎

Proof of Theorem 4.1:
Let $J_t = \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\|\Delta X_s\|\geq K} = \int_{\|x\|\geq K} x N_t(dx)$, then, by Remark 3.2, $(J_t)_{t\geq 0}$ and $(X_t - J_t)_{t\geq 0}$ are Lévy processes. Moreover, from Proposition 4.3 it follows that $E[\|X_t - J_t\|^n] < \infty \; \forall n \in \mathbb{N} \; t \geq 0$. Let $\alpha_K := E[X_1 - J_1]$, (where the

expectation is a Bochner integral from $(\Omega, \mathcal{F}, P)$ to $(E, \mathcal{B}(E))$) then $E[X_t - J_t] = \alpha_K t$. From Theorem 4.6 it follows that $X_t - J_t - t\alpha_K - \int_{\|x\|<K} x(N_t(dx) - t\nu(dx))$ is an E -valued Brownian motion with zero mean.
Moreover, from Remark 3.2 v) one has, that $\forall n \in I\!N$, (B_t^n) in the proof of Theorem 4.6 is independent of N_t^Λ, $\forall \Lambda \in \mathcal{F}(E \setminus \{0\})$. Taking the limits this is seen to hold also for B_t. ■

Remark 4.7 *As the main work in this paper was already finished, we learned from J. Rosinski and W. A. Woyczynski that the Lévy -Ito decomposition theorem is also stated in [12]. [12] contains however only a rather sketchy proof, without a direct expression of the "Lévy part"* $\int_{0<\|x\|\leq 1} \int x\,[N_t(dx) - t\nu(dx)]$ *as a special case of a (deterministic) stochastic integral, in the sense of Definition 3.11.* ($\int_{0<\|x\|\leq 1} \int x\,[N_t(dx) - t\nu(dx)]$ *is in [12] the limit in* $L_p^E(\Omega, \mathcal{F}, P)$, $p > 1$ *of* $\int_{\frac{1}{n}<\|x\|\leq 1} \int x\,[N_t(dx) - t\nu(dx)]$, *when* $n \to \infty$). *Let us describe a proof of the Lévy -Ito decomposition theorem along the lines of [12], however completing it as much as possible with precise references. Let* ν *be a Lévy measure on the separable Banach space* $(E, \mathcal{B}(E))$. *By [29] (Prop. 5.4.5, iii, p. 76), given any sequence* $\delta_n \downarrow 0$, *the sequence of probability measures* $\rho_n(\cdot) := e^{-\nu(E)} \sum_{k=0}^\infty \frac{\nu^k(\cdot \cap \|x\| > \delta_n)}{k!}$ *is such that there exist points* $x_n \in E$, *such that* $(\rho_n \star \delta_{x_n})(\cdot)$ *contains a subsequence which converges weakly as* $\delta_n \downarrow 0$. *From Corollary 5.4.6 [29] it follows that* $x_n := \int_{\delta_n<\|z\|\leq 1} zt\nu(dz)$. *Moreover the Fourier transform* $\widehat{(\rho_n \star \delta_{x_n})}(k)$ *of* $(\rho_n \star \delta_{x_n})(\cdot)$ *converges itself for* $n \to \infty$ *point wise to* $\exp(\int_{E\setminus 0} e^{i<k,x>} - 1 - i < x, k >)\nu(dx)$ *([29], Theorem 5.4.8, ii), p. 78). This identifies the weak limit of* $\rho_n \star \delta_{x_n}(\cdot)$, *as* $n \to \infty$. *Moreover by [29] (Theorem 5.3.6., p. 70)* $\nu(\cdot \cap \|x\| > \delta_n)$ *has a weakly convergent subsequence, the weak limit being* $\nu(\cdot)$. *Suppose now that* ν *is the Lévy measure of the L évy process* X_t *(according to definition 2.22). From the proof of the Theorem 2.21 (equations (49), (50)) it follows that* $\tilde{\nu}(\cdot) := \nu(\cdot \cap \|x\| \leq 1)$ *is the Lévy measure of the Lévy process* $X_t - J_t - E[X_t - J_t]$, *with* $J_t = \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\|\Delta X_s\| \geq 1}$. *The distribution of* $S_t^n := \int_{1/n<\|x\|\leq 1} x(N_t(dx) - t\nu(dx))$ *is given by* $\rho_n(\cdot \cap \|x\| \leq 1) \star \delta_{\int_{\frac{1}{n}<\|z\|\leq 1} zt\nu(dz)}(dx)$ *and by the above arguments converges weakly , as* $n \to \infty$, *to a random variable* Y_t *with Fourier transform* $\exp(\int_{0<\|x\|\leq 1} (e^{i<k,x>} - 1 - i < x, k >)\nu(dx)$. *From the Ito -Nisio theorem [21] it follows that* S_t^n *converges a.s., for* $n \to \infty$. *It follows (see e.g. [7], pag. 72)*

$$\sup_{n \in I\!N} E[\|S_t^n\|^p \leq E[\|X_t - J_t - E[X_t - J_t]\|^p] \quad \forall p \geq 1 \tag{130}$$

From this one can deduce by a theorem of [16] (theorem II.5.5) that S_t^n *converges in* $L_p^E(\Omega, P)$ *to* Y_t *and, for any* $a > 0$:

$$P[sup_{0\leq r\leq t} \|S_r^n - Y_r\| > a] \leq a^{-p} E[\|S_t^n - Y_t\|^p] \quad \forall p > 1 \tag{131}$$

so that there is a subsequence n_k *for which* $X_t - J_t - E[X_t - J_t] - S_r^{n_k}$ *converges a.s. uniformly in* $r \in [0, t]$, *when* $k \to \infty$. *From this one can deduce the Lévy*

-Ito decomposition, with $X_t - J_t - Y_t$ *the Gaussian part and* $Y_t + \int_{\|x\|>1} xN_t(dx)$ *the Lévy part, like it was done in our Proof of Theorem 4.1.*

The difference between our proof of Theorem 4.1 and the proof of the Lévy -Ito decomposition in [12] is that in our proof $\int_{0<\|x\|\leq 1} \int x\,[N_t(dx) - t\nu(dx)]$ *is the strong* 2 *-integral (our Definition 3.11) of the function* $f(x) = x$*. (130) together with Proposition 3.29 implies that* $\int_{0<\|x\|\leq 1} \int x\,[N_t(dx) - t\nu(dx)]$ *is (in terms of our Definition 3.18) simply* p *-integrable* $\forall p \geq 1$.

Dedication: The second author would like to dedicate this work to the memory of her father Ulrich Rüdiger.

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