Gibbs states on loop lattices: existence and a priori estimates

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Running title: Gibbs states on loop lattices

Abstract. We prove existence and uniform a priori estimates for Euclidean Gibbs states of quantum lattice systems with unbounded spins. These results substantially extend all previous existence results. Detailed proofs are contained in [3].

États de Gibbs sur des réseaux de lacets: existence et estimations à priori

Résumé. Des estimations à priori et l'existence des états de Gibbs pour des systèmes quantiques sur réseaux avec spins non bornés sont données. Ces résultats étendent substanciellement tous les résultats d'existence précédents. Les preuves détaillées sont contenues dans [3].

Version française abrégée

Nous étudions un système quantique d'oscillateurs anharmoniques (de masse $\mathfrak{m} > 0$ et avec spins non bornés $x_k \in \mathbb{R}$) distribués sur le réseau $\mathbb{Z}^d$. Le système est décrit par un Hamiltonien $\mathbb{H}$ donné heuristiquement par la formule (1) ci-dessous. Les interactions à deux points sont données par des fonctions symétriques qui satisfont à une condition de croissance polynomiale. Les autointeractions harmoniques sont données par $\frac{1}{2}a^2x_k^2$ avec intensité $a > 0$. Les autointeractions anharmoniques $V_k \in C^2_{b,loc}(\mathbb{R} \to \mathbb{R})$ satisfont à une condition de coercivité et à une condition de restriction de croissance. Ces conditions sont satisfaites dans le cas de nombreuses interactions d'importance physique.

Nous décrivons les propriétés d'équilibre de ces systèmes quantiques au moyen des états de Gibbs qui sont donnés par des mesures de Gibbs (≪euclidiennes≫) μ_β associées avec $\mathbb{H}$ et une température inverse $\beta > 0$ [1, 2]. La définition rigoureuse de μ_β est comme suit. Soit $S_\beta \cong [0, \beta]$ le cercle avec mesure de Lebesgue $d\tau$. La variable de spin associée à $k \in \mathbb{Z}^d$ prend ses valeurs dans $L^r_\beta := L^r(S_\beta, d\tau)$, $r \geq 1$, resp. $(C^\alpha_\beta = C^\alpha(S_\beta), \alpha \geq 0)$ $C_\beta = C(S_\beta)$ (espaces de lacets $\omega_k : S_\beta \to \mathbb{R}$ intégrables resp. (Hölder) continus). Les espaces $\Omega_\beta^{-p,r}$ resp. $C_\beta^{-\beta,\alpha}$ sont définis ci-dessous. Les sous-ensembles des configurations resp. distributions tempérées sont donnés par $\Omega^t_\beta := \cup_{p\geq 1}\Omega_\beta^{-p,R}$ resp. $\mathcal{M}^t_\beta := \left\{\mu \,\middle|\, \exists p = p(\mu) > d/2 : \mu\left(\Omega_\beta^{-p,R}\right) = 1\right\}$ (pour un R approprié). Les mesures de Gibbs μ_β sont décrites par leur specification locale $\{\pi_{\beta,\Lambda} | \Lambda \subset\subset Z^d\}$ (cf. [4,5]). $\pi_{\beta,\Lambda}$ est défini comme noyau stochastique sur $(\Omega^t_\beta, B(\Omega^t_\beta))$: $\forall B \in \mathcal{B}(\Omega^t_\beta) := \mathcal{B}((C_\beta)^{Z^d}) \cap \Omega^t_\beta$, $\forall \xi \in \Omega^t_\beta$ et est donné par (3), (4) ci-dessous. Nous définissons l'ensemble $\mathcal{G}^t_\beta$ d'états de Gibbs tempérées comme les $\mu_\beta \in \mathcal{M}^t_\beta$ qui satisfont aux équations DLR $\mu_\beta \pi_{\beta,\Lambda} = \mu_\beta$.

Proposition 1 donne l'identité de $\mathcal{G}^t_\beta$ avec $\mathcal{M}^t_b$, un ensemble de measures qui consiste d'éléments μ dans $\mathcal{M}^t_\beta$ qui satisfont à la formule d'integration par parties

$$\int_{\Omega_\beta} \partial_i f \, d\mu = -\int_{\Omega_\beta} f b_i \, d\mu$$

pour tous les f appartenant à un certain sous-ensemble de fonctions dans $(C_\beta)^{\mathbb{Z}^d}$, et pour toutes les directions h_i dans une base orthonormale de $\mathcal{H}_\beta := l^2(\mathbb{Z}^d \to L^2_\beta)$. b_i est la dérivée logarithmique partielle dans la direction h_i.

Le théorème 1 montre que $\mathcal{G}^t_\beta \neq \emptyset$. Le théorème 2 montre que chaque $\mu_\beta \in \mathcal{G}^t_\beta$ a son support dans un ensemble de lacets Höldériens, et des estimations de ces moments sont données. Les preuves utilisent un analogue infini-dimensionnel de la méthode des fonctions de Lyapunov. Les résultats contiennent comme cas particuliers tous les résultats connus sur l'existence et les estimations à priori des états de Gibbs euclidiens pour les crystaux quantiques. De plus, le cas d'une croissance superquadratique des potentiels d'integration est inclu.

1 Quantum crystals and Euclidean Gibbs measures

We study an interacting system of quantum anharmonic oscillators (of mass $\mathfrak{m} > 0$ and with unbounded spins $x_k \in \mathbb{R}$) on a lattice $\mathbb{Z}^d (\subset \mathbb{R}^d)$, $d \in \mathbb{N}$, which is described by the heuristic Hamiltonian

$$\mathbb{H} := -\frac{1}{2\mathfrak{m}} \sum_{k \in \mathbb{Z}^d} \frac{d^2}{dx_k^2} + \frac{a^2}{2} \sum_{k \in \mathbb{Z}^d} x_k^2 + \sum_{\{k,j\} \subset \mathbb{Z}^d} W_{\{k,j\}}(x_k, x_j) + \sum_{k \in \mathbb{Z}^d} V_k(x_k), \tag{1}$$

(see e.g. [6] for physical motivations). We specify the assumptions on the system (1) as follows:

A$_1$**)** The two-particle interactions (taken over all unordered pairs $\{k,j\} \subset \mathbb{Z}^d$, $k \neq j$) are given by symmetric functions $W_{\{k,j\}} \in C^2_{b,pol}(\mathbb{R}^2 \to \mathbb{R})$ satisfying the *polynomial growth* condition: $\exists R \geq 2$, $\exists J_{\{k,j\}} \geq 0$, $\forall x_k, x_j \in \mathbb{R}$

$$|\partial_k^{(l)} W_{\{k,j\}}(x_k, x_j)| \leq J_{\{k,j\}}(1 + |x_k| + |x_j|)^{R-l}, \quad l = 0, 1, 2,$$

where $\partial_k^{(l)}$ denotes the l-th derivative w.r.t. coordinate x_k.

A$_2$**)** The matrix $J = \{J_{\{k,j\}}\}$ is *fastly decreasing*, that is, $\forall p \in \mathbb{N}$:

$$\|J\|_p := \sup_{k \in \mathbb{Z}^d} \left\{ \sum_{j \in \mathbb{Z}^d \setminus \{k\}} J^2_{\{k,j\}} (1 + |k - j|)^{2p} \right\}^{1/2} < \infty.$$

A$_3$**)** The harmonic self-interactions are given by $a^2 x_k^2/2$ with an intensity $a > 0$. The anharmonic self-interactions $V_k \in C^2_{b,loc}(\mathbb{R} \to \mathbb{R})$ satisfy the *coercivity* estimate: $\exists A_1, B_1, \sigma > 0$, $\forall k \in \mathbb{Z}^d$, $\forall x_k \in \mathbb{R}$

$$V_k'(x_k)x_k \geq A_1^{-1} \left[|x_k|^{R+\sigma} + |V_k'(x_k)| + |V_k''(x_k)x_k| \right] - B_1$$

and the *growth* condition: $\exists A_2, B_2 > 0$, $\forall k \in \mathbb{Z}^d$, $\forall x_k \in \mathbb{R}$

$$|V_k''(x_k)| \leq A_2 \left[|x_k|^{R-1} + |V_k'(x_k)| \right] + B_2.$$

We note that the above assumptions are fulfilled for many classes of interactions of physical relevance.

A mathematical description of equilibrium properties of quantum systems is carried out in terms of their temperature (i.e., Gibbs) states. We will take the Euclidean (i.e., path space) approach, see e.g. [1, 2] and the references therein. Therewith the *Euclidean Gibbs measures* μ_β associated with the lattice system (1) at the inverse temperature $\beta > 0$ are rigorously defined as follows:

Let $S_\beta \cong [0,\beta]$ be a circle with Lebesgue measure $d\tau$. As the *single spin spaces* for every $k \in \mathbb{Z}^d$ we will use the spaces $L^r_\beta := L^r(S_\beta, d\tau)$, $r \geq 1$, resp. ($C^\alpha_\beta := C^\alpha(S_\beta)$, $\alpha \geq 0$) $C_\beta := C(S_\beta)$ of integrable resp. (Hölder) continuous loops $\omega_k : S_\beta \longrightarrow \mathbb{R}$. As the *configuration space* for the infinite volume system (1) we define the temperature loop lattices

$$\Omega^{-p,r}_\beta \ : \ = \left\{ \omega \in (C_\beta)^{\mathbb{Z}^d} \,\middle|\, ||\omega||^2_{-p,r} := \sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2p} |\omega_k|^2_{L^r_\beta} < \infty \right\},$$
$$\mathcal{C}^{-p,\alpha}_\beta \ : \ = \left\{ \omega \in (C^\alpha_\beta)^{\mathbb{Z}^d} \,\middle|\, ||\omega||^2_{-p,\alpha} := \sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2p} |\omega_k|^2_{C^\alpha_\beta} < \infty \right\}$$
.

Thereafter, we define the subsets of *tempered configurations* resp. of *tempered distributions* by

$$\Omega^t_\beta := \bigcup\nolimits_{p\geq 1} \Omega^{-p,R}_\beta \text{ resp. } \mathcal{M}^t_\beta := \left\{ \mu \,\middle|\, \exists p = p(\mu) > d/2 : \mu\left(\Omega^{-p,R}_\beta\right) = 1 \right\}. \tag{2}$$

The Euclidean Gibbs measures μ_β are described by their local specifications $\left\{\pi_{\beta,\Lambda} \,\middle|\, \Lambda \Subset \mathbb{Z}^d\right\}$ (cf. [4, 5]). Let γ_β be a centered Gaussian measure on C_β with correlation operator $\mathbb{A}^{-1}_\beta$, where $\mathbb{A}_\beta := -\mathfrak{m}d^2/d\tau^2 + a^2\mathbf{1}$ is considered as the self-adjoint operator in the Hilbert space L^2_β. For $\Lambda \Subset \mathbb{Z}^d$, $\pi_{\beta,\Lambda}$ is defined as a stochastic kernel on $(\Omega^t_\beta, \mathcal{B}(\Omega^t_\beta))$: $\forall B \in \mathcal{B}(\Omega^t_\beta) := \mathcal{B}((C_\beta)^{\mathbb{Z}^d}) \cap \Omega^t_\beta$, $\forall \xi \in \Omega^t_\beta$

$$\pi_{\beta,\Lambda}(B|\xi) := Z^{-1}_{\beta,\Lambda}(\xi) \int\limits_{\Omega_{\beta,\Lambda}} \exp\left\{ -\mathcal{I}^{V,W}_{\beta,\Lambda}(\omega|\xi) \right\} \mathbf{1}_\Delta(\omega_\Lambda, \xi_{\Lambda^c}) \prod_{k\in\Lambda} d\gamma_\beta(\omega_k). \tag{3}$$

Here $\omega_\Lambda := (\omega_k)_{k\in\Lambda} \in (C_\beta)^\Lambda$, $Z_{\beta,\Lambda}(\xi)$ is a normalization factor, and

$$\mathcal{I}^{V,W}_{\beta,\Lambda}(\omega|\xi) := \int\limits_{S_\beta} \left[\sum_{k\in\Lambda} V_k(\omega_k) + \sum_{\{k,j\}\subset\Lambda} W_{\{k,j\}}(\omega_k,\omega_j) + \sum_{k\in\Lambda, j\in\Lambda^c} W_{\{k,j\}}(\omega_k,\xi_j) \right] d\tau. \tag{4}$$

Obviously, (3), (4) make sense because of ($\mathbf{A}_{1-3}$). We define the set $\mathcal{G}^t_\beta$ of tempered Euclidean Gibbs states as those $\mu \in \mathcal{M}^t_\beta$ which satisfy the DLR equations $\mu_\beta \pi_{\beta,\Lambda} = \mu_\beta$, $\forall \Lambda \Subset \mathbb{Z}^d$.

2 Main results

We start with an integration by parts (IbP) description of $\mu_\beta \in \mathcal{G}^t_\beta$. We fix the orthonormal basis $h_i := \{\delta_{k-j}\varphi_n\}_{j\in\mathbb{Z}^d}$, $i = (k,n) \in \mathbb{Z}^{d+1}$ in $\mathcal{H}_\beta := l^2(\mathbb{Z}^d \to$

L^2_β), where $\{\varphi_n\}_{n\in\mathbb{Z}} \subset C^\infty_\beta$ is respectively the complete orthonormal system of eigenvectors of the operator $\mathbb{A}_\beta$ in L^2_β (i.e., $\mathbb{A}_\beta\varphi_n = \lambda_n\varphi_n$ with $\lambda_n = (2\pi n/\beta)^2\mathfrak{m} + a^2$).

We define the *partial logarithmic derivatives* along directions h_i, $i = (k,n) \in \mathbb{Z}^{d+1}$,

$$b_i(\omega) := -(\mathbb{A}_\beta\varphi_n, \omega_k)_\beta - (F_k^{V,W}(\omega), \varphi_n)_\beta, \quad \omega \in \Omega^t_\beta, \tag{5}$$

where $F_k^{V,W} : \Omega^t_\beta \to L^1_\beta$ is given by

$$F_k^{V,W}(\omega) := V_k'(\omega_k) + \sum_{j\neq k} \partial_k W_{\{k,j\}}(\omega_k, \omega_j). \tag{6}$$

Fixed $i = (k,n) \in \mathbb{Z}^{d+1}$, $p > d/2$, we define the set $C^1_{\mathsf{dec},i}(\Omega_\beta^{-p,R_W})$ of all functions $f : \Omega_\beta^{-p,R_W} \to \mathbb{R}$ which are bounded and continuous together with their partial derivatives $\partial_i f := \partial_{h_i} f$ in the direction h_i and, moreover, fit the extra decay condition

$$|f(\omega)| \le C_k(f)\left(1 + |\omega_k|_{L^1_\beta} + |F_k^{V,W}(\omega)|_{L^1_\beta}\right)^{-1}, \ \omega \in \Omega_\beta^{-p,R_W}. \tag{7}$$

Proposition 1 (IbP Characterization) *Let $\mathcal{M}^t_b$ denote the set of all probability measures μ on Ω^t_β which for some $p > d/2$ satisfy the temperedness condition (2) and the (IbP)-formula*

$$\int_{\Omega_\beta} \partial_i f(\omega)\, d\mu(\omega) = -\int_{\Omega_\beta} f(\omega) b_i(\omega)\, d\mu(\omega) \tag{8}$$

for all functions $f \in C^1_{\mathsf{dec},i}(\Omega_\beta^{-p,R_W})$ and directions h_i, $i = \mathbb{Z}^{d+1}$. Then $\mathcal{G}^t_\beta = \mathcal{M}^t_b$.

The measures given by the local specifications $\pi_{\beta,\Lambda}$ satisfy Propositions 1, but only in directions h_i, $i = (k,n)$, $k \in \Lambda \Subset \mathbb{Z}^d$, $n \in \mathbb{Z}$. Since under our assumptions b_i are continuous locally bounded functions on Ω_β^{-p,R_W}, the latter means that every accumulation point of the family $\left\{\pi_{\beta,\Lambda}(d\omega|\xi)\,\middle|\,\Lambda \Subset \mathbb{Z}^d, \xi \in \Omega^t_\beta\right\}$ is Gibbs.

Theorem 1 (Existence of Tempered Gibbs States) *Let assumptions ($\mathbf{A}_{1-3}$) on the potentials V_k and $W_{\{k,j\}}$ be fulfilled. Then for all values of mass $\mathfrak{m} > 0$ and temperature $\beta > 0$*

$$\mathcal{G}^t_\beta \neq \emptyset.$$

Theorem 2 (A Priori Estimates on Tempered Gibbs States) *Let assumptions* ($\mathbf{A}_{1-3}$) *be fulfilled. Then every* $\mu_\beta \in \mathcal{G}^t_\beta$ *is supported by the set of Hölder loops* $\bigcap_{\substack{p>d/2 \\ 0\le\alpha<1/2}} \mathcal{C}_\beta^{-p,\alpha}$. *Moreover,* $\forall Q \geq 1$

$$(i) \quad \sup_{\substack{\mu_\beta \in \mathcal{G}^t_\beta \\ k\in\mathbb{Z}^d}} \int_{\Omega_\beta} |\omega_k|^Q_{C^\alpha_\beta} d\mu_\beta(\omega) < \infty,$$

and thus

$$(ii) \quad \sup_{\substack{\mu_\beta \in \mathcal{G}^t_\beta \\ \tau\in S_\beta, k\in\mathbb{Z}^d}} \int_{\Omega_\beta} |\omega_k(\tau)|^Q d\mu_\beta(\omega) < \infty.$$

The key point of the proofs is that (according to Proposition 1) μ_β resp. $\pi_{\beta,\Lambda}$ are viewed as the solutions of an infinite system of first order partial differential equations. Due to the pointwise coercivity and growth assumptions ($\mathbf{A}_{1-3}$) on the potentials V_k, $W_{\{k,j\}}$, the corresponding vector fields $b_i, i \in \mathbb{Z}^{d+1}$, also possess certain *coercivity properties* w.r.t. the tangent space $\mathcal{H}_\beta$. This allows us to apply an analog of the *Lyapunov function method* well known from finite dimensional PDE's (cf. [3]).

3 Concluding remarks

Among the particular novelities of our results are the following:

1) We emphasize that Theorems 1, 2 include all known results on the existence and a priori estimates for Euclidean Gibbs states for quantum crystals as particular cases.

2) The case of superquadratic growth of the interaction potentials is included. Note that this case does not admit an application of the well-known superstability bounds of Ruelle [7].

3) Many-particle interactions can be considered in the same way (cf. [3]). Also the case $\sigma = 0$ in assumption ($\mathbf{A}_3$) is allowed by taking constant $A_1 > 0$ small enough.

4) The a priori estimates described above are uniform w.r.t. all Gibbs measures from $\mathcal{G}^t_\beta$.

The precise relation to already known results is discussed in [3] in detail.

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