Semilinear perturbations of harmonic spaces, Liouville property and a boundary value problem

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Abstract. Let $(X, \mathcal{H})$ be a $\mathcal{P}$-harmonic Bauer space and let Ψ be a Borel measurable function satisfying conditions (A) through (D) of section 2. For every Kato family M of potential kernels on X let ${}^M\mathcal{U}(X)$ denote the set of all real continuous functions on X such that $u + K_D^M \Psi(\cdot, u) \in \mathcal{H}(D)$ for all open relatively compact subset D of X. We study the existence of a non-trivial function in ${}^M\mathcal{U}(X)$ which is dominated by a given positive harmonic function on X. If X is a Greenian domain of $\mathbb{R}^d$, μ is a positive Kato measure on X, we apply our study to derive a characterization of finite positive measures ν on the minimal Martin boundary $\partial_1^M X$ for which the boundary value problem $\Delta u = \Psi(\cdot, u)\mu$ in X and $u = \nu$ on $\partial_1^M X$ is solvable.

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1 Introduction

Let us consider a $\mathcal{P}$-harmonic Bauer space $(X, \mathcal{H})$ in the sense of [5], a Kato family $M = (K_\Omega^M)_\Omega$ of potential kernels on X, and denote by ${}^M\mathcal{H}$ the corresponding perturbed harmonic sheaf on X (see [4] or [9, Sect.7]). Given a positive harmonic function h on X, A. Grigor'yan and W. Hansen studied in [9] the existence of a non-trivial M-harmonic function u on X (i.e., $u \in {}^M\mathcal{H}(X)$) which is dominated by h. Using h-thick sets, they gave necessary and sufficient conditions on M under which the following property holds:

$$u \in {}^M\mathcal{H}(X) \text{ and } |u| \leq h \;\Rightarrow\; u \equiv 0 \text{ on } X.$$

If moreover the constants are harmonic, under these conditions the constant zero is the only bounded M-harmonic function on X. So, this property can be seen as an extension of the well known Liouville theorem in potential theory related to the Laplacian on $\mathbb{R}^d$.

In this paper we discuss the same problem in a non-linear setting. More precisely, we consider the semilinear perturbation of $(X, \mathcal{H})$ generated by the pair (M, Ψ) where Ψ is a real-valued Borel measurable function on $X \times \mathbb{R}$ satisfying some conditions (see Section 2). A standard example of Ψ is given by

$$\Psi(x,t) = \gamma(x)t|t|^{\alpha-1}$$

where α is a real > 1 and γ is a positive Borel measurable function on X such that γ and $1/\gamma$ are bounded.

For every open subset $\Omega \subset X$ let ${}^M\mathcal{U}(\Omega)$ denote the set of all continuous functions u on Ω such that

$$u + K_D^M \Psi(\cdot, u) \in \mathcal{H}(D)$$

for every open relatively compact subset D with $\overline{D} \subset \Omega$. In the linear setting (i.e., $\Psi(x,t) = t$) we have

$${}^M\mathcal{H}(\Omega) = {}^M\mathcal{U}(\Omega),$$

but in general ${}^M\mathcal{U}(\Omega)$ has not to be a linear space. However, the convergence property of Bauer is fulfilled for every sequence in ${}^M\mathcal{U}(\Omega)$ (see Theorem 2.5).

In section 3 we study the Liouville property for $(X, {}^M\mathcal{U})$ and we prove that ${}^M\mathcal{U}(X)$ contains non-trivial functions dominated by a given positive harmonic function h if and only if M decomposes into a sum

$$M = N + P$$

of two families of potential kernels such that N is supported by a non-h-thick set and $K^P\Psi(\cdot, h)$ is a potential on X. This result extends Theorem 7.19 in [9] to the semilinear perturbations considered in this paper. Let us note that after some preparations in section 2 and in the first part of section 3 we may use similar techniques as in [9].

Denote by Δ the Laplace operator in $\mathbb{R}^d$ and suppose that Ω is a smooth bounded domain of $\mathbb{R}^d$. A characterization of all positive finite measures ν on $\partial\Omega$ for which the boundary value problem

$$\begin{aligned} \Delta u &= u^\alpha \quad \text{in } \Omega, \\ u &= \nu \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

has a solution is known (see [17] for $\alpha = 2$, [7] for $1 < \alpha \leq 2$, [19] for $\alpha > 2$). Given a Greenian domain X of $\mathbb{R}^d$ and a positive Kato measure μ on X, we investigate the problem

$$\begin{aligned} \Delta u &= \Psi(\cdot, u)\mu \quad \text{in } X, \\ u &= \nu \quad \text{on } \partial_1^M X \end{aligned} \tag{1.2}$$

where $\partial_1^M X$ denotes the minimal Martin boundary of X and ν is a positive finite measure on $\partial_1^M X$. We give necessary and sufficient conditions under which (1.1) is solvable. These conditions are connected to minimal thin sets (see Theorem 5.1) and they are completely different from conditions given in [17, 7, 19] which are based on the theory of capacity. Moreover, we do not assume any smoothness conditions on X and our results hold true in the parabolic setting as well.

2 Basic facts on semilinear perturbations

Let X be a locally compact space with countable base. For every open subset Ω of X let $\mathcal{B}(\Omega)$ ($\mathcal{C}(\Omega)$ resp.) be the set of all Borel measurable numerical (continuous real resp.) functions on Ω. Given any set $\mathcal{F}$ of numerical functions, $\mathcal{F}_b$ ($\mathcal{F}^+$ resp.) will denote the set of all $f \in \mathcal{F}$ which are bounded (positive resp.).

In all this paper $\mathcal{H}$ is defined to be a harmonic sheaf on X so that $(X, \mathcal{H})$ is a $\mathcal{P}$-harmonic Bauer space (see [5]). As usual, functions in $\mathcal{H}(\Omega)$ are called harmonic on Ω and $\mathcal{S}(\Omega)$ ($\mathcal{P}(\Omega)$ resp.) denotes the set of all superharmonic functions (potentials resp.) on Ω.

We denote by $\mathcal{O}$ the collection of all open relatively compact subsets of X and let $\mathcal{O}_r$ be the set of all elements of $\mathcal{O}$ which are regular: An open relatively compact subset $\Omega \subset X$ is regular if and only if every $f \in \mathcal{C}(\partial\Omega)$ possesses a unique continuous extension $H_\Omega f$ on $\overline{\Omega}$ such that

$$H_\Omega f \in \mathcal{H}(\Omega) \text{ and } H_\Omega f \geq 0 \text{ if } f \geq 0.$$

For every $x \in \Omega \in \mathcal{O}_r$ the map $f \to H_\Omega f(x)$ defines a positive Radon measure on $\partial\Omega$ which will be denoted by μ_x^Ω. It is called the harmonic measure of x with respect to Ω. For $x \in \Omega^c := X \backslash \Omega$ we define $\mu_x^\Omega := \varepsilon_x$ where ε_x denotes the Dirac measure concentrated at x. The harmonic kernel $H_\Omega = (\mu_x^\Omega)$ is given by

$$H_\Omega f(x) = \int_X f(y)\, d\mu_x^\Omega(y) \qquad (x \in X)$$

for every $f \in \mathcal{B}(X)$ such that the integral makes sense.

Let A be a subset of X. For every $s \in \mathcal{S}^+(X)$ we consider the *reduit* $R^A s$ of s on A, which is defined by

$$R^A s := \inf\{v : v \in \mathcal{S}^+(X); v \geq s \text{ on } A\}.$$

The *balayage* of s on A is given by $\hat{R}^A s(x) := \liminf_{y \to x} R^A s(y)$ for all $x \in X$. To each $x \in X$ there corresponds a unique measure ε_x^A on X such that

$$R^A p(x) = \int_X p(y) \varepsilon_x^A(dy)$$

for all continuous potential p on X (see [3] or [5]). If A is the complement of a set $\Omega \in \mathcal{O}_r$ then $R^A s = H_\Omega s$ for all $s \in \mathcal{S}^+(X)$, which implies that $\varepsilon_x^A = \mu_x^\Omega$ for all $x \in X$. We shall denote again by H_Ω the kernel given by

$$H_\Omega = (\varepsilon_x^{X \backslash \Omega})_{x \in X}$$

for any open (not necessarily regular) set $\Omega \in \mathcal{O}$.

Following [4] the convex cone of Kato families of potential kernels is denoted by $\mathcal{M}^+(\mathcal{H})$. Recall that $M = (K_\Omega^M)_{\Omega\in\mathcal{O}} \in \mathcal{M}^+(\mathcal{H})$ means that for every $\Omega \in \mathcal{O}$, K_Ω^M is a kernel such that for every $f \in \mathcal{B}_b^+(\Omega)$, $K_\Omega^M f$ is a bounded continuous potential on Ω which is harmonic outside $\overline{\{f > 0\}}$, and

$$K_\Omega^M f - K_{\Omega'}^M f \in \mathcal{H}(\Omega \cap \Omega') \text{ for all } \Omega' \in \mathcal{O},\ f \in \mathcal{B}_b(\Omega \cup \Omega').$$

Let Ψ be a Borel measurable real-valued function on $X \times \mathbb{R}$ and assume that the following properties are satisfied:

A. The function $\Psi(\cdot, 1)$ is locally bounded on X.

B. For every $x \in X$, $\Psi(x, \cdot)$ is odd, continuous, and increasing on $\mathbb{R}$.

C. $\Psi(x, t+s) \geq \Psi(x,t) + \Psi(x,s)$ for all $t, s \geq 0$, $x \in X$.

D. There exists $\kappa > 0$ such that $\Psi(x, 2t) \leq \kappa\, \Psi(x,t)$ for all $t \geq 0$, $x \in X$.

We easily see that the above conditions hold true for every function of the form

$$\Psi(x,t) = \gamma(x)\, \mathrm{sgn}(t) \psi(|t|)$$

where ψ is an N-function satisfying the Δ_2-condition (see, e.g, [14]) and $\gamma \in \mathcal{B}^+(X)$ such that γ and $1/\gamma$ are locally bounded on X. Here, $\mathrm{sgn}(t) = 1$ if $t > 0$, $\mathrm{sgn}(0) = 0$ and $\mathrm{sgn}(t) = -1$ if $t < 0$.

In the sequel, M will always denote an element of $\mathcal{M}^+(\mathcal{H})$. For every $\Omega \in \mathcal{O}$ and every $f \in \mathcal{B}(\Omega)$ we define

$$T_\Omega^M f := K_\Omega^M \Psi(\cdot, f)$$

provided $K_\Omega^M \Psi(\cdot, f)$ has a sense. Therefore, $T_\Omega^M(\mathcal{B}_b^+(\Omega)) \subset \mathcal{P}(\Omega) \cap \mathcal{C}_b(\Omega)$, and

$$T_\Omega^M = T_D^M + H_D T_\Omega^M \tag{2.1}$$

whenever $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$.

Proposition 2.1. *Let $\Omega \in \mathcal{O}$ and let f, g be two real-valued functions in $\mathcal{B}(\Omega)$ such that $T_\Omega^M|f|$ and $T_\Omega^M|g|$ are finite potentials on Ω. Assume moreover that $s := f - g + T_\Omega^M f - T_\Omega^M g \in \mathcal{S}(\Omega)$. Then the following statements are equivalent:*

(a) *$s \geq 0$ on Ω;*

(b) *$f - g \geq 0$ on Ω;*

(c) $\liminf_{x \longrightarrow z} [f(x) - g(x)] \geq 0$ *for all $z \in \partial\Omega$.*

Proof. Clearly (b) implies (c). To prove that (c) yields (a) it is enough to recall that every superharmonic function on Ω which is bounded below by a difference of two potentials on Ω is positive.

Suppose now that (a) holds and let $h := \Psi(\cdot, f) - \Psi(\cdot, g)$. Since $\Psi(x, \cdot)$ is increasing on $\mathbb{R}$ for every $x \in \Omega$ we see that the positive superharmonic function

$$w := s + K_\Omega^M(h^-)$$

dominates $K_\Omega^M(h^+)$ on the set $\{h^+ > 0\}$. Therefore $w \geq K_\Omega^M(h^+)$ on Ω by the same arguments as in [4, Proposition 2.4]. Consequently

$$f - g = w - K_\Omega^M(h^+) \geq 0 \text{ on } \Omega.$$

□

The following theorem is recently shown in [1] for a general setting. We give here the proof for the convenience of the reader.

Theorem 2.2. *If $\Omega \in \mathcal{O}$ then for every $f \in \mathcal{B}_b(\partial\Omega)$ there exists a unique bounded continuous function $U_\Omega^M f$ on Ω such that*

$$U_\Omega^M f + T_\Omega^M U_\Omega^M f = H_\Omega f. \tag{2.2}$$

Proof. In virtue of the previous proposition there exists at most one bounded continuous function on Ω which satisfies (2.2).

Let $f \in \mathcal{B}_b(\partial\Omega)$ and choose $a > 0$ such that $H_\Omega|f| \leq a$. Considering

$$\Lambda : u \in \mathcal{B}_b(\Omega) \to H_\Omega f - T_\Omega^M \xi(u)$$

where $\xi(t) = \operatorname{sgn}(t)\min(|t|, a), t \in \mathbb{R}$, and remarking that K_Ω^M is a compact operator on $\mathcal{B}_b(\Omega)$ (see, [10, Proposition 3.1]), it follows from the Schauder's fixed point theorem that $\Lambda(v) = v$ for some $v \in \mathcal{B}_b(\Omega)$. Thus,

$$v + T_\Omega^M \xi(v) = H_\Omega f.$$

Similarly, there exists $w \in \mathcal{B}_b(\Omega)$ satisfying

$$w + T_\Omega^M \xi(w) = H_\Omega |f|.$$

Now, replacing Ψ by the function $\Psi_\xi : (x,t) \to \Psi(x, \xi(t))$ in the proof of the previous proposition, we obtain that $|v| \leq w$. Therefore, $|v| \leq a$ and the proof is finished. □

If $D \in \mathcal{O}$ and $f \in \mathcal{B}(X)$ such that f is bounded on ∂D, we shall denote again by $U_D^M f$ the function on X given by

$$U_D^M f = U_D^M (f_{/\partial D}) \text{ on } D \quad \text{and} \quad U_D^M f = f \text{ on } X \backslash D.$$

For every open subset Ω of X we define ${}^M\mathcal{U}^*(\Omega)$ to be the set of all lower semi-continuous (l.s.c) locally bounded functions u on Ω such that

$$U_D^M u \leq u$$

for all $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$. We also define ${}^M\mathcal{U}_*(\Omega) := -{}^M\mathcal{U}^*(\Omega)$ and

$${}^M\mathcal{U}(\Omega) := \{u \in \mathcal{C}(\Omega) : U_D^M u = u, \text{ for all } D \in \mathcal{O} \text{ with } \overline{D} \subset \Omega\}.$$

Theorem 2.3. *If $\Omega \in \mathcal{O}$ and $u \in \mathcal{B}_b(\Omega)$ then $u \in {}^M\mathcal{U}(\Omega)$ (${}^M\mathcal{U}^*(\Omega)$ resp.) if and only if $u + T_\Omega^M u \in \mathcal{H}(\Omega)$ ($\mathcal{S}(\Omega)$ resp.).*

In particular, if u is a Borel measurable locally bounded function on Ω then $u \in {}^M\mathcal{U}(\Omega)$ (${}^M\mathcal{U}^*(\Omega)$ resp.) if and only if $u + T_D^M u \in \mathcal{H}(D)$ ($\mathcal{S}(D)$ resp.) for every $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$.

Proof. By (2.1) and (2.2), for every $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$ we have:

$$\begin{aligned} u + T_\Omega^M u - H_D T_\Omega^M u &= u + T_D^M u, \\ H_D(u + T_\Omega^M u) - H_D T_\Omega^M u &= U_D^M u + T_D^M U_D^M u. \end{aligned}$$

Then, Proposition 2.1 completes the proof. □

The above Theorem assures that ${}^M\mathcal{U}(\Omega)$ is closed under uniform convergence on compact subsets of Ω, and that all functions in ${}^M\mathcal{U}^+(\Omega)$ are subharmonic on Ω. The following proposition follows immediately from Theorem 2.3 and Proposition 2.1.

Proposition 2.4. *Let $\Omega \in \mathcal{O}$ and let $u \in {}^M\mathcal{U}_b^*(\Omega)$, $v \in {}^M\mathcal{U}_{*,b}(\Omega)$ such that $\liminf_{x\to z}[u(x) - v(x)] \geq 0$ for all $z \in \partial\Omega$. Then $u \geq v$ on Ω.*

Theorem 2.5. *Let $\Omega \subset X$ be an open subset and let (u_n) be a sequence in ${}^M\mathcal{U}(\Omega)$ which is uniformly bounded on every compact subset of Ω. The following holds:*

(a) *If (u_n) increases to a function u then $u \in {}^M\mathcal{U}(\Omega)$.*

(b) *There exits a subsequence of (u_n) which converges locally uniformly on Ω.*

(c) *If (u_n) converges pointwise to a function u then $u \in {}^M\mathcal{U}(\Omega)$ and (u_n) converges uniformly to u on every compact subset of Ω.*

Proof. Take $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$ and let $h_n = u_n + T_D^M u_n$ for every $n \geq 1$.

(a): Since (h_n) is an increasing sequence of harmonic functions on D and it is uniformly bounded on D, we get by the convergence property of Bauer that

$$h := \sup_{n\geq 1} h_n$$

is harmonic on D. Then, passing to the limit in the formula $u_n + T_D^M u_n = h_n$ we obtain that $u + T_D^M u = h \in \mathcal{H}(D)$. So, (a) is proved in view of Theorem 2.3.

(b): Let $K \subset D$ be a compact subset and let (h_{n_k}) be a subsequence of (h_n) which converges uniformly on K. Since the family

$$\{T_D^M u_{n_k}, k \geq 1\}$$

is equicontinuous [10, Proposition 3.1], by Ascoli theorem there exists a subsequence (v_k) of (u_{n_k}) such that $(T_D^M v_k)$ converges uniformly on K. Consequently, (v_k) is uniformly convergent on K. To finish the proof of (b) it will be enough to cover Ω by a sequence $(\Omega_n) \in \mathcal{O}$ such that $\overline{\Omega}_n \subset \Omega_{n+1}$ and apply the diagonal procedure.

(c) follows easily from statement (b). □

3 A Liouville property

In the following (Ω_n) will always denote a sequence in $\mathcal{O}$ such that $\overline{\Omega}_n \subset \Omega_{n+1}$ and $X = \cup_{n\geq 1}\Omega_n$. For every function $f \in \mathcal{B}^+(X)$ we define

$$K^M f := \lim_{n\to\infty} K^M_{\Omega_n} f,$$

and for $f \in \mathcal{B}(X)$ such that $K^M|f| < \infty$, $K^M f := K^M(f^+) - K^M(f^-)$. Note that these definitions do not depend on the particular choice of (Ω_n). Furthermore, if $f \in \mathcal{B}^+_b(X)$ with compact support then

$$K^M f \in \mathcal{P}(X) \cap \mathcal{C}(X)$$

This follows from [2, Satz 5.3.6] (for more details see [11, Section 2]).

For any $f \in \mathcal{B}(X)$ such that $K^M\Psi(\cdot, f)$ possesses a sense we define

$$T^M f := K^M \Psi(\cdot, f).$$

Clearly $T^M f = \lim_{n\to\infty} T^M_{\Omega_n} f$ if $f \geq 0$, and from (2.1) we deduce that

$$T^M = T^M_D + H_D T^M \ \text{ for every } D \in \mathcal{O}.$$

Remark. By the same proof we may extend Proposition 2.1 ($(a) \Leftrightarrow (b)$) to the case $\Omega = X$. More precisely, suppose that f, g are real Borel measurable functions on X such that $T^M|f|, T^M|g|$ are finite potentials on X, and $f - g + T^M f - T^M g$ is superharmonic on X. Then

$$f \geq g \ \text{ if and only if } \ f + T^M f \geq g + T^M g.$$

In the sequel, by *comparison principle* we shall mean Proposition 2.1 or the assertion given by this remark.

Let $D, \Omega \in \mathcal{O}$ such that $\overline{D} \subset \Omega$. By Theorem 2.3 and formula (2.2) we see that the operator U^M_Ω maps $\mathcal{B}_b(\partial\Omega)$ into ${}^M\mathcal{U}(\Omega)$ and therefore

$$U^M_D \circ U^M_\Omega = U^M_\Omega.$$

This yields that $U^M_\Omega s \leq U^M_D s$ for every locally bounded function $s \in \mathcal{S}^+(X)$. In particular, if $h \in \mathcal{H}^+(X)$ then the sequence $(U^M_{\Omega_n} h)$ is monotone decreasing. So, defining

$$L^M h := \inf_{n\geq 1} U^M_{\Omega_n} h$$

we obtain, in accordance to Theorem 2.5, an operator $L^M : \mathcal{H}^+(X) \to {}^M\mathcal{U}^+_\mathcal{H}(X)$ where ${}^M\mathcal{U}_\mathcal{H}(X)$ denotes the set of all $u \in {}^M\mathcal{U}(X)$ such that $|u| \leq g$ for some function $g \in \mathcal{H}^+(X)$. It is easy to see that $L^M h$ does not depend on (Ω_n), in fact for every $x \in X$

$$L^M h(x) = \inf_{\Omega \in \mathcal{O}} U_\Omega^M h(x).$$

Since $L^M h$ is a positive subharmonic function on X which is dominated by h, the sequence $(H_{\Omega_n} L^M h)$ increases to a positive harmonic function on X. We define

$$Q^M h := \sup_{n \geq 1} H_{\Omega_n} L^M h,$$

and we denote by ${}^M\mathcal{Q}^+(X)$ the set of all $h \in \mathcal{H}^+(X)$ such that $Q^M h = h$.

Theorem 3.1. *For every $h, g \in \mathcal{H}^+(X)$ the following holds:*
(a) *$L^M h \leq Q^M h \leq h$.*
(b) *If $h \leq g$ then $L^M h \leq L^M g$ and $Q^M h \leq Q^M g$.*
(c) *$T^M L^M h$ is a continuous potential on X and $L^M h + T^M L^M h = Q^M h$.*
(d) *$h \in {}^M\mathcal{Q}^+(X)$ if and only if there exists $u \in {}^M\mathcal{U}_{\mathcal{H}}^+(X)$ such that $u + T^M u = h$. In this case we have $u = L^M h$.*
(e) *$Q^M \circ Q^M = Q^M$ and $L^M \circ Q^M = L^M$.*
(f) *$L^M h = \max\{u \in {}^M\mathcal{U}^+(X) : u \leq h\} = \max\{u \in {}^M\mathcal{U}^+(X) : u \leq Q^M h\}$.*
(g) *$Q^M h = \max\{f \in {}^M\mathcal{Q}^+(X) : f \leq h\} = \min\{f \in \mathcal{H}^+(X) : f \geq L^M h\}$.*

Proof. Assertions (a) and (b) are obvious.

Let $u \in {}^M\mathcal{U}_{\mathcal{H}}^+(X)$. The fact that $H_{\Omega_n} u = u + T_{\Omega_n}^M u$ shows that u is a positive subharmonic function on X and $(H_{\Omega_n} u)$ is increasing. Moreover $H_{\Omega_n} u$ is bounded above by every harmonic function dominating u. Therefore

$$u + T^M u = \sup_{n \geq 1} H_{\Omega_n} u \in \mathcal{H}^+(X)$$

which yields in particular the continuity of the potential $T^M u$. This proves (c) and the “only if” part of (d). Suppose now that $h = u + T^M u$, then

$$U_{\Omega_n}^M h \geq U_{\Omega_n}^M u = u$$

and thereby $L^M h \geq u$ which implies, in accordance to (a) and (c), that $Q^M h = h$. This means that $h \in {}^M\mathcal{Q}^+(X)$. To see that $u = L^M h$ it suffices to apply the comparison principle (see Remark above). Whence (d) is proved.

Assertion (e) is an immediate consequence of (d) and the formula $L^M h + T^M L^M h = Q^M h$. Recalling that U_Ω^M $(\Omega \in \mathcal{O})$ is increasing we obtain (f) in view of (a). Finally the first (second resp.) part of (g) follows from (a) and the monotonicity of Q^M (H_Ω resp.). □

Let $u \in {}^M\mathcal{U}(X)$ and $h \in \mathcal{H}^+(X)$ such that $|u| \leq h$. For every $n \geq 1$ we have

$$-U_{\Omega_n}^M h \leq u \leq U_{\Omega_n}^M h$$

which proves that $|u| \leq L^M h$ and therefore $L^M h = \max\{u \in {}^M\mathcal{U}(X) : |u| \leq h\}$.

The relation on $\mathcal{M}^+(\mathcal{H}) \times \mathcal{M}^+(\mathcal{H})$ given by:

$$N \prec M \ \Leftrightarrow\ M = N + P \text{ for some } P \in \mathcal{M}^+(\mathcal{H}),$$

makes $\mathcal{M}^+(\mathcal{H})$ a partially ordered set. It can be shown that $\mathcal{M}^+(\mathcal{H})$ equipped with "$\prec$" is a lattice.

Proposition 3.2. *Let $\Omega \in \mathcal{O}$ and let $f \in \mathcal{B}_b^+(\partial\Omega)$. For every $M, N \in \mathcal{M}^+(\mathcal{H})$ the following holds:*
(a) $U_\Omega^M f \le U_\Omega^N f$ *if* $N \prec M$.
(b) $U_\Omega^M f + U_\Omega^N f \le H_\Omega f + U_\Omega^{M+N} f$.

Proof. Applying (2.2) we obtain that

$$U_\Omega^N f + T_\Omega^M U_\Omega^N f = H_\Omega f + T_\Omega^M U_\Omega^N f - T_\Omega^N U_\Omega^N f.$$

(a): Since $\Psi(\cdot, U_\Omega^N f) \in \mathcal{B}_b^+(\Omega)$ we conclude that $T_\Omega^M U_\Omega^N f - T_\Omega^N U_\Omega^N f \in \mathcal{P}_b(\Omega)$. Then $U_\Omega^N f \ge U_\Omega^M f$ by the comparison principle.

(b): Let $u = U_\Omega^{M+N} f$. By (a) we have $u \le U_\Omega^M f$ and $u \le U_\Omega^N f$. Then

$$\begin{aligned} H_\Omega f + u - U_\Omega^M f - U_\Omega^N f &= T_\Omega^M U_\Omega^M f + T_\Omega^N U_\Omega^N f - T_\Omega^{M+N} u \\ &\ge K_\Omega^M \Psi(\cdot, u) - K_\Omega^N \Psi(\cdot, u) - K_\Omega^{M+N} \Psi(\cdot, u) = 0. \end{aligned}$$

□

For every $h \in \mathcal{H}^+(X)$ and $M, N \in \mathcal{M}^+(\mathcal{H})$ the above proposition yields that:

$$L^N h \le L^M h \ \text{ and } \ Q^N h \le Q^M h, \text{ if } M \prec N. \tag{3.1}$$

$$L^M h + L^N h \le h + L^{M+N} h \ \text{ and } \ Q^M h + Q^N h \le h + Q^{M+N} h. \tag{3.2}$$

Definitions. 1.- Given $h \in \mathcal{H}^+(X)$ we shall say that a subset $A \subset X$ is *h-thin* if $\hat{R}^A h \in \mathcal{P}(X)$, and we say that A is *h-thick* if $\hat{R}^A h = h$. Notice that h is minimal if and only if every subset of X is either h-thin or h-thick (see, [9] or [12]).

2.- For every $\varphi \in \mathcal{B}_b^+(X)$ and every $M \in \mathcal{M}^+(\mathcal{H})$ we define φM to be the element N of $\mathcal{M}^+(\mathcal{H})$ given by

$$K_\Omega^N f = K_\Omega^M(\varphi f), \quad \Omega \in \mathcal{O},\ f \in \mathcal{B}_b(\Omega).$$

We shall say that M is supported by the Borel subset $A \subset X$ if $1_{A^c} M = 0$. Obviously, $T^M 1_{A^c} = 0$ (or equivalently, $T_\Omega^M 1_{A^c} = 0$ for all $\Omega \in \mathcal{O}$) if M is supported by A. Here, 1_A denotes the characteristic function of A.

Proposition 3.3. *Let $h \in \mathcal{H}^+(X)$ and let $M \in \mathcal{M}^+(\mathcal{H})$.*
(a) *If M is supported by a non-h-thick subset, then there exists $u \in {}^M\mathcal{U}^+(X)$ which is bounded above by h and does not vanish identically on X.*
(b) *Each of the following conditions implies that $h \in {}^M\mathcal{Q}^+(X)$.*
(b.1) *$T^M h$ is a potential on X.*
(b.2) *M is supported by an h-thin subset of X.*

Proof. Let us first recall that for $A \subset X$, the function $\hat{R}^A h \in \mathcal{S}^+(X)$ and the set

$$P = \{x \in A : \hat{R}^A h(x) \neq h(x)\}$$

is semipolar (see Corollary 2.4 and Proposition 5.11 in [3, Chap. VI]). Therefore, $K_\Omega^M 1_P = 0$ for all $\Omega \in \mathcal{O}$ (see Lemma 5.15 in [3, p. 288]) and consequently $T^M f = 0$ for every $f \in \mathcal{B}^+(X)$ which vanishes on the complement of P.

Suppose that M is supported by A. Then M is supported by $A\backslash P$ which yields that $T^M(h - \hat{R}^A h) = 0$. So, for every $\Omega \in \mathcal{O}$ it is obvious that

$$h - \hat{R}^A h + T_\Omega^M (h - \hat{R}^A h) = h - \hat{R}^A h.$$

Recalling that $U_\Omega^M h + T_\Omega^M U_\Omega^M h = h$ and applying Proposition 2.1, we obtain that $h - \hat{R}^A h \leq U_\Omega^M h$ for all $\Omega \in \mathcal{O}$. This implies that

$$0 \leq h - \hat{R}^A h \leq L^M h, \tag{3.3}$$

and thereby

$$0 \leq h - Q^M h \leq \hat{R}^A h. \tag{3.4}$$

Now, if A is not h-thick then (3.3) assures that $L^M h \not\equiv 0$ on X which proves statement (a). If A is h-thin then from (3.4) we deduce that $Q^M h = h$. This finishes the proof of (b.2). To prove (b.1), we remark that on Ω_n we have

$$h = U_{\Omega_n}^M h + T_{\Omega_n}^M U_{\Omega_n}^M h \leq U_{\Omega_n}^M h + T^M h.$$

By letting n tend to infinity we obtain that $h - Q^M h \leq h - L^M h \leq T^M h$. So, if $T^M h \in \mathcal{P}(X)$ then $h = Q^M h$ and (b.1) is shown. □

In the last section we shall give a counterexample proving that the converse statement in (b.1) does not hold even if h is minimal.

Theorem 3.4. *For every $M \in \mathcal{M}^+(\mathcal{H})$ and $h \in \mathcal{H}^+(X)$ the following statements are equivalent:*

(a) $h \in {}^M\mathcal{Q}^+(X)$.

(b) *There exists an h-thin Borel subset (which can be chosen closed or open) $A \subset X$ such that $T^{1_{A^c} M} h$ is a continuous potential on X.*

(c) *M can be decomposed into a sum of $N, P \in \mathcal{M}^+(\mathcal{H})$ such that $T^N h \in \mathcal{P}(X)$ and P is supported by a h-thin subset.*

Proof. (c) follows trivially from (b) (take $N = 1_{A^c} M$ and $P = 1_A M$). The implication (c)⇒(a) is an immediate consequence of Proposition 3.3.(b) and the second inequality in (3.2). Assume now that $h \in {}^M\mathcal{Q}^+(X)$ then from Theorem 3.1.(c) we know that

$$T^M L^M h \in \mathcal{P}(X) \cap \mathcal{C}(X) \;\text{ and }\; h = L^M h + T^M L^M h.$$

Taking $A := \{2L^M h \le h\}$ (we replace the weak inequality by the strong one if we want to get an open subset) we have

$$2T^M L^M h = 2(h - L^M h) \ge h \ \text{ on } A.$$

Then $\hat{R}^A h \le 2T^M L^M h$ and thereby $\hat{R}^A h \in \mathcal{P}(X)$ which means that A is h-thin. Finally to show that $T^{1_{A^c}M} h \in \mathcal{P}(X) \cap \mathcal{C}(X)$ we remark that the inequality $\Psi(\cdot, h)1_{A^c} \le \Psi(\cdot, 2L^M h)$ implies, in view of property (D) imposed on the function Ψ, that

$$\Psi(\cdot, h)1_{A^c} \le \kappa\Psi(\cdot, L^M h).$$

Therefore, $\kappa T^M L^M h - T^{1_{A^c}M} h \in \mathcal{P}(X)$ which yields that $T^{1_{A^c}M} h$ is a continuous potential on X because $\kappa T^M L^M h$ is. □

By analogous arguments as in the proof of the previous theorem (using moreover Proposition 3.3.(a)) we obtain the following result which is an extension of Theorem 7.19 in [9].

Theorem 3.5. *Let $M \in \mathcal{M}^+(\mathcal{H})$ and let $h \in \mathcal{H}^+(X)$. The following statements are equivalent:*

(a) $Q^M h = 0$ *(i.e., $u \in {}^M\mathcal{U}(X)$ and $|u| \le h \Rightarrow u \equiv 0$ on X).*

(b) *Every Borel set $A \subset X$ such that $T^{1_{A^c}M} h \in \mathcal{P}(X)$ is h-thick.*

(c) *Every $P \in \mathcal{M}^+(\mathcal{H})$ such that $P \prec M$ and $T^{M-P} h \in \mathcal{P}(X)$ is supported by an h-thick subset of X.*

4 Further properties of ${}^M\mathcal{Q}^+(X)$

In this section $M \in \mathcal{M}^+(\mathcal{H})$ is fixed. Since ${}^M\mathcal{U}^+(X)$ has not to be stable under additions, the operator L^M is in general non-additive. However, we shall prove that Q^M is linear on $\mathcal{H}^+(X)$.

Proposition 4.1. $U^M_\Omega(f + g) \le U^M_\Omega f + U^M_\Omega g$ *for every $\Omega \in \mathcal{O}$, $f, g \in \mathcal{B}^+_b(\partial\Omega)$. In particular, L^M and Q^M are subadditive on $\mathcal{H}^+(X)$.*

Proof. Let $\Omega \in \mathcal{O}$. By (C) we see that $T^M_\Omega(\phi + \psi) - T^M_\Omega\phi - T^M_\Omega\psi \in \mathcal{P}_b(\Omega)$ for every $\phi, \psi \in \mathcal{B}^+_b(\Omega)$. In particular, if $f, g \in \mathcal{B}^+_b(\partial\Omega)$ then

$$T^M_\Omega(U^M_\Omega f + U^M_\Omega g) - T^M_\Omega U^M_\Omega f - T^M_\Omega U^M_\Omega g := p \in \mathcal{P}_b(\Omega).$$

From (2.2) we have

$$\begin{aligned} U^M_\Omega f + U^M_\Omega g + T^M_\Omega(U^M_\Omega f + U^M_\Omega g) &= H_\Omega(f+g) + p, \\ U^M_\Omega(f+g) + T^M_\Omega U^M_\Omega(f+g) &= H_\Omega(f+g). \end{aligned}$$

Now, it suffices to apply the comparison principle to obtain the first part of the proposition. The second part is then obvious. □

As immediate consequence we get that

$$\sup_{n\geq 1} L^M h_n = L^M h \text{ and } \sup_{n\geq 1} Q^M h_n = Q^M h \tag{4.1}$$

for every increasing sequence $(h_n) \in \mathcal{H}^+(X)$ such that $\sup_{n\geq 1} h_n = h \in \mathcal{H}^+(X)$.

Using the fact that $K^M_{\Omega_n}$ are kernels for all $n \geq 1$, it is easy to show the following convergence lemma.

Lemma 4.2. *Let $f, g, (f_n), (g_n) \in \mathcal{B}(X)$. Assume that $K^M g < \infty$ and that the sequences $(f_n), (g_n), (K^M_{\Omega_n} g_n)$ converge pointwise to $f, g, K^M g$, respectively. If $|f_n| \leq g_n$ for every $n \geq 1$, then $(K^M_{\Omega_n} f_n)$ converges to $K^M f$.*

Theorem 4.3. *Let $h, g \in \mathcal{H}^+(X)$ and let $\alpha \geq 0$. The following holds:*
(a) *Q^M is linear, i.e, $Q^M(\alpha h + g) = \alpha Q^M h + Q^M g$.*
(b) *${}^M\mathcal{Q}^+(X)$ is a convex cone.*
(c) *If $h \leq g$ then ${}^M\mathcal{Q}^+(X)$ contains h if it contains g.*
(d) *If (h_n) is a sequence in ${}^M\mathcal{Q}^+(X)$ which increases to h then $h \in {}^M\mathcal{Q}^+(X)$.*

Proof. Let us first assume that (a) is proved. Then (b) and (d) become obvious. Suppose that $h \leq g$ then from (a) it follows that

$$Q^M h = Q^M g - Q^M(g - h) \geq h + Q^M g - g.$$

If moreover $g \in {}^M\mathcal{Q}^+(X)$ then we get that $Q^M h \geq h$. Thus $Q^M h = h$ which means that $h \in {}^M\mathcal{Q}^+(X)$. So, the proof of the Theorem will be complete if we show statement (a).

Put $\tilde{h} = Q^M h$ and $\tilde{g} = Q^M g$; we claim that

$$\lim_{n\to\infty} T^M_{\Omega_n} U^M_{\Omega_n}(\tilde{h} + \tilde{g}) = T^M L^M(\tilde{h} + \tilde{g}). \tag{4.2}$$

Indeed, by the previous proposition we have $U^M_{\Omega_n}(\tilde{h} + \tilde{g}) \leq U^M_{\Omega_n}\tilde{h} + U^M_{\Omega_n}\tilde{g}$. Then

$$0 \leq \Psi(\cdot, U^M_{\Omega_n}(\tilde{h} + \tilde{g})) \leq \kappa(\Psi(\cdot, U^M_{\Omega_n}\tilde{h}) + \Psi(\cdot, U^M_{\Omega_n}\tilde{g})) := f_n,$$

thanks to the property (D) and the monotonicity of $\Psi(x, \cdot)$. On the other hand, by the continuity of $\Psi(x, \cdot)$ and since $\tilde{h}, \tilde{g} \in {}^M\mathcal{Q}^+(X)$, we see that

$$\lim_{n\to\infty} f_n = \kappa(\Psi(\cdot, L^M h) + \Psi(\cdot, L^M h)) := f$$

and

$$\lim_{n\to\infty} K^M_{\Omega_n} f_n = K^M f = \kappa(T^M L^M h + T^M L^M g) < \infty.$$

Recall that $L^M h = L^M \tilde{h}$ and $L^M g = L^M \tilde{g}$. So, Lemma 4.2 yields (4.2).

Now, letting n tend to infinity in the formula

$$U^M_{\Omega_n}(\tilde{h} + \tilde{g}) + T^M_{\Omega_n} U^M_{\Omega_n}(\tilde{h} + \tilde{g}) = \tilde{h} + \tilde{g}$$

we obtain that $L^M(\tilde{h}+\tilde{g}) + T^M L^M(\tilde{h}+\tilde{g}) = \tilde{h}+\tilde{g}$. This means that

$$Q^M(Q^M h + Q^M g) = Q^M h + Q^M g$$

and consequently $Q^M h + Q^M g \leq Q^M(h+g)$ by monotonicity of Q^M. This and Proposition 4.1 yield the additivity of Q^M. Finally, the fact that

$$Q^M(\alpha h) = \alpha Q^M h$$

follows from the additivity of Q^M, (4.1), and the density of $\mathbb{Q}_+$ in $\mathbb{R}_+$. □

Defining ${}^M\overline{\mathcal{Q}}^+(X)$ to be the set of all $h \in \mathcal{H}^+(X)$ such that $Q^M h = 0$, we obtain a convex cone which satisfies (c) and (d) of Theorem 4.3. It is not difficult to see that ${}^M\mathcal{Q}^+(X) \perp {}^M\overline{\mathcal{Q}}^+(X)$, in the sense that $\inf(h,g) \in \mathcal{P}(X)$ if $h \in {}^M\mathcal{Q}^+(X)$ and $g \in {}^M\overline{\mathcal{Q}}^+(X)$. Moreover,

$$\mathcal{H}^+(X) = {}^M\mathcal{Q}^+(X) \oplus {}^M\overline{\mathcal{Q}}^+(X),$$

i.e, every $h \in \mathcal{H}^+(X)$ has unique decomposition into a sum $h = h_1 + h_2$ where $h_1 \in {}^M\mathcal{Q}^+(X)$ and $h_2 \in {}^M\overline{\mathcal{Q}}^+(X)$.

5 On a boundary value problem

In the last years, serval papers investigated boundary value problems of the type

$$\begin{aligned} \Delta u &= u^\alpha \quad \text{in } \Omega, \\ u &= \nu \quad \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

where α is a real > 1, Ω is a bounded smooth domain of $\mathbb{R}^d$, and ν is a finite positive measure on $\partial\Omega$ (see [17, 7, 18, 19]). Following [7], a solution u of (5.1) has to be understood as the solution of the integral equation

$$u(x) + \int_\Omega G_\Omega(x,y) u^\alpha(y)\, dy = \int_{\partial\Omega} P(x,z)\, d\nu(z) \quad (x \in \Omega) \tag{5.2}$$

where G_Ω denotes the Green function of Ω and P is the Martin kernel on Ω.

If the problem (5.1) is solvable for some measure ν, we call ν the *trace* (see [19]) of u on $\partial\Omega$ and we write $\nu = tr(u)$. Let $\mathcal{M}(\partial\Omega)$ denote the space of all Radon measures on $\partial\Omega$ and define $\mathcal{M}_\alpha^+(\partial\Omega)$ to be the set of all $\nu \in \mathcal{M}^+(\partial\Omega)$ such that $\nu = tr(u)$ for a positive solution u of $\Delta u = u^\alpha$ which is *moderate*, in the sense that $u \leq h$ for some harmonic function h on Ω (i.e., $\Delta h = 0$ in Ω).

If $\nu \in \mathcal{M}^+(\partial\Omega)$ and *cap* is a set function defined on compact subsets of $\partial\Omega$, we shall write $\nu \ll cap$ to mean that $\nu(K) = 0$ for every compact subset $K \subset \partial\Omega$ such that $cap(K) = 0$.

In the case of $1 < \alpha \le 2$, E. B. Dynkin and S. E. Kuznetsov used probabilistic techniques and obtained the following characterizations of $\mathcal{M}_\alpha^+(\partial\Omega)$:

$$\nu \in \mathcal{M}_\alpha^+(\partial\Omega) \Leftrightarrow \nu \ll CM_\alpha \Leftrightarrow \nu \ll CB_{2/\alpha,\alpha'},$$

where CM_α ($CB_{2/\alpha,\alpha'}$ resp.) is defined to be the Martin (Bessel resp.) capacity, and $\alpha' = \alpha/(\alpha-1)$. For more details we refer the reader to [7, 8]. By purely analytic methods M. Marcus and L. Véron investigated in [18, 19] problems of the type (5.1), they proved for $\alpha > 2$ that $\nu \in \mathcal{M}_\alpha^+(\partial\Omega)$ if and only if $\nu \ll CB_{2/\alpha,\alpha'}$.

Let us now fix a Greenian domain $X \subset \mathbb{R}^d$, that is, X has a Green function G_X ($\Delta G_X(\cdot,y) = -\varepsilon_y$ for every $y \in X$), and consider the continuous solutions (in the distributional sense) to the semilinear equation

$$\Delta u \;=\; \Psi(\cdot,u)\mu \quad \text{in } X \tag{5.3}$$

with the boundary data given by

$$u \;=\; \nu \quad \text{on } \partial_1^M X \tag{5.4}$$

where ν is a finite positive measure on the Martin boundary $\partial^M X$ supported by the minimal part of $\partial^M X$ which is denoted by $\partial_1^M X$ following J. L. Doob [6]. The measure μ is assumed to be positive and in the (local) Kato class, in other words $M = (K_\Omega^\mu)_{\Omega\in\mathcal{O}}$ is a family of potential kernels on X, where K_Ω^μ is given by

$$K_\Omega^\mu f = \int_\Omega G_\Omega(\cdot,y) f(y)\, d\mu(y), \quad \Omega \in \mathcal{O}, f \in \mathcal{B}_b(\Omega).$$

Analogously to (5.1), the problem (5.3)-(5.4) is equivalent to the equation

$$u + T^\mu u = h \tag{5.5}$$

where $T^\mu := T^M$ (i.e., $T^\mu u = \int_X G_X(\cdot,y)\Psi(y,u(y))\, d\mu(y)$) and

$$h = P\nu := \int_{\partial^M X} P(\cdot,z)\, d\nu(z). \tag{5.6}$$

For every $A \subset X$, we define Σ_A to be the set of all $z \in \partial^M X$ such that A is *minimal thin* relative to z, that is, $\hat{R}^A P_z \neq P_z$ where P_z denotes the Martin function with pole at z. Recall that Σ_A is a Borel subset of $\partial^M X$ (see, e.g., [21]).

Denote by $\mathcal{M}$ the set of all finite-valued signed measures on $\partial^M X$ which are supported by $\partial_1^M X$, and let $\mathcal{M}_\mu^+$ be the set of all $\nu \in \mathcal{M}^+$ such that $\nu = tr(u)$ for some positive solution u to (5.3) (i.e., u is the solution of (5.5) where $h = P\nu$).

Applying Theorem 3.1.(d) and Theorem 3.4 we obtain the following characterizations of $\mathcal{M}_\mu^+$.

Theorem 5.1. *Let $\nu \in \mathcal{M}^+$ and let $h = P\nu$. The following statements are equivalent:*

(a) $\nu \in \mathcal{M}^+_\mu$.

(b) *There exists a Borel set $A \subset X$ such that ν is supported by Σ_A and such that $T^{1_{A^c}\mu}h$ is a continuous potential on X.*

(c) *μ can be decomposed into a sum of $\mu_1, \mu_2 \in \mathcal{M}^+$ such that $T^{\mu_1}h \in \mathcal{P}(X)$ and μ_2 is supported a subset A of X satisfying $\nu(\Sigma_A) = \nu(\partial^M X)$.*

Proof. It suffices to recall that a subset A of X is h-thin if and only if ν is supported by the set Σ_A (see Corollaire in [21, p. 234]). To see this, let $\hat{\varepsilon}_x^A$ be the balayage of the Dirac measure ε_x on A. Then

$$\hat{R}^A h(x) = \int_{\partial^M X} \int_X P(y,z) d\hat{\varepsilon}_x^A(y)\, d\nu(z) = \int_{\partial^M X} \hat{R}^A P_z(x)\, d\nu(z).$$

In particular,

$$g : x \to \int_{(\partial^M X)\setminus \Sigma_A} P(x,z)\, d\nu(z)$$

is the greatest harmonic minorant of $\hat{R}^A h$. Consequently, $\hat{R}^A h \in \mathcal{P}(X)$ if and only if ν is supported by Σ_A.

Notice that in order to apply Theorem 3.1.(d) and Theorem 3.4 we have to remark that for every open subset Ω of X, ${}^M\mathcal{U}(\Omega)$ coincide with the set of all continuous solutions to the equation (5.3) replacing X by Ω ($M = (K^\mu_\Omega)$). This can be shown by easy computations. □

Let a be a fixed point in X and define for every compact subset $K \subset \partial^M X$

$$cap_\mu(K) := \sup\{\nu(K) : \nu \in \mathcal{M}^+;\ \nu(\partial^M X\setminus K) = 0;\ T^\mu(P\nu)(a) \le 1\}. \tag{5.7}$$

In the case when $\mu = \lambda_d$ is the Lebesgue measure on $\mathbb{R}^d$ and $\Psi(x,t) = t|t|^{\alpha-1}$ for $1 < \alpha \le 2$, $cap_{\lambda_d}(K) = 0$ means that $CM_\alpha(K) = 0$ where CM_α is the Martin capacity. Hence, under some assumptions on the smoothness of X we get that $\nu \in \mathcal{M}^+_{\lambda_d}$ if and only if $\nu \ll cap_{\lambda_d}$ (see [8, 7]). In the following we give a counterexample showing that for Kato measures μ, the condition $\nu \ll cap_\mu$ may not more necessary for ν to be in $\mathcal{M}^+_\mu$.

Counterexample. Consider the function Ψ on $\mathbb{R}^3 \times \mathbb{R}$ defined by $\Psi(x,t) = t|t|^{\alpha-1}$, where α is a real > 1. Let $x_0 = (1,0,0)$, $z = (2,0,0)$, and denote by B (B_1 resp.) the ball of center $0 = (0,0,0)$ (x_0 resp.) and radius 2 (1 resp.). By elementary computations we get that

$$\limsup_{y\in B_1, y\to z} \frac{G_{B_1}(x_0,y)}{G_B(x_0,y)} > 0$$

which implies that $A = B \backslash B_1$ is minimal thin at z (see [21, Théorème 11]). Let $h = P\varepsilon_z$ and choose a Kato measure μ on B which is supported by A and such that

$$\int_B G_B(0,y) h^\alpha(y)\, d\mu(y) = \infty. \tag{5.8}$$

From the preceding theorem we conclude that the Dirac measure ε_z is in the class $\mathcal{M}_\mu^+$. However, we clearly see that $cap_\mu(\{z\}) = 0$. Here, the point a in (5.7) is the origin 0.

Notice that a such Kato measure μ exists. In fact, a way to construct it is the following: Choose a sequence (V_n) of open subsets such that $\overline{V}_n \subset A$ and for any compact set $K \subset A$ there exists $n_K \geq 1$ such that $K \cap V_n = \emptyset$ for all $n \geq n_K$. Then the measure

$$\mu := \sum_{n=1}^{\infty} a_n^{-1} \lambda_{V_n},$$

where $a_n := \int_{V_n} G_B(0,y) h^\alpha(y)\, dy$ and λ_{V_n} is the restriction of the Lebesgue measure on $\mathbb{R}^3$ to V_n, belongs to the Kato class of B and (5.8) is satisfied.

Remarks. 1.- Under additional assumptions on the pair (μ, Ψ) (in particular we need the fact that $\Psi(x,\cdot)$ is convex on $\mathbb{R}_+$) we find that a measure $\nu \in \mathcal{M}$ is a trace of some moderate solution to (5.3) whenever $|\nu| \ll cap_\mu$. This will be treated in a subsequent paper.

2.- The analogous parabolic problem to (5.1) was investigated in [15] (see also [16]) for $\Omega = \mathbb{R}_+ \times D$ a smooth cylinder in $\mathbb{R} \times \mathbb{R}^d$ and for positive measures ν on the lateral boundary of Ω. As in the elliptic case, characterizations of $\mathcal{M}_\alpha^+(\mathbb{R}_+ \times \partial D)$ in terms of exceptional sets were established. For measures supported by the initial boundary of Ω, results in the same direction were given in [20]. Since Theorem 3.4 is true in every $\mathcal{P}$-harmonic Bauer space, similar results as in Theorem 5.1 can be obtained if we consider, instead of (5.3)-(5.4), the following parabolic problem:

$$\Delta u - \frac{\partial u}{\partial t} = \Psi(\cdot, u)\mu \ \text{ in } X \ \text{ and } \ u = \nu \ \text{ on } \partial_1^M X,$$

where X is an arbitrary domain of $\mathbb{R} \times \mathbb{R}^d$. We refer the reader to [6, Chapter XIX] (or [13]) for details on the Martin representation of positive harmonic functions in the parabolic setting.

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