Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$

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Abstract

Given a locally bounded real function g, we examine the existence of a 4-covariation $[g(B^H), B^H, B^H, B^H]$, where B^H is a fractional Brownian motion with a Hurst index $H \geq \frac{1}{4}$. We provide two essential applications. First, we relate the mentioned covariation to one expression involving the derivative of local time, in the case $H = \frac{1}{4}$, generalizing an identity of Bouleau-Yor type, well-known for the classical Brownian motion. A second application is an Itô's formula of Stratonovich type for $f(B^H)$. The main difficulty comes from the fact B^H has only a finite 4-variation.

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Fractional Brownian motion, fourth variation, Itô's formula, local time.
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1 Introduction

The present paper is devoted to generalized covariation processes and an Itô's formula related to the fractional Brownian motion. Classical Itô's formula and classical covariations constitute the kernel of stochastic calculus with respect to semimartingales. Fractional Brownian motion, which in general, is not a semimartingale, is knowing a very intensive research activity in stochastic analysis and it is considered in many applications as hydrology, telecommunications, economics and finance. Finance is the most recent one in spite of the fact, that, according to [31] the general assumption of no arbitrage opportunity is violated. Interesting remarks have been recently done by [7] and [40]. We recall that a mean zero Gaussian process $X = B^H$ is a fractional Brownian motion with Hurst index $H \in]0,1[$ if its covariance function is given by

$$K_H(s,t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}),\ (s,t) \in \mathbb{R}^2 \tag{1.1}$$

An easy consequence of that property is that

$$\mathrm{E}(B_t^H - B_s^H)^2 = (t-s)^{2H} \tag{1.2}$$

Before coming back to this self-similar Gaussian process, we would like to make some general observations. Calculus with respect to integrands which are not semimartingales is now twenty years old. A huge amount of papers have been produced, and it is impossible to list them here; however we are still not so close from having a truely efficient approach for applications.
The techniques for studying non-semimartingales integrators are essentially three.

- Pathwise and related techniques.
- Dirichlet forms.
- Anticipating techniques (Malliavin calculus, Skorohod integration and so on).

Pathwise type integrals are defined very often using discretization, as limit of Riemann sums: an interesting survey on the subject is a book of R.M. Dudley and R. Norvaisa ([14]). They emphasize a big historical literature in the deterministic case. The first contribution in the stochastic framework has been provided by H. Föllmer ([18]) in 1981; through this significant and simply written contribution, the author wished to discuss integration with respect to a Dirichlet process X, that is to say a local martingale plus a zero quadratic variation (or sometimes zero energy) process. In the sequel this approach has been continued and performed by J. Bertoin ([4]).
Since 1991, F. Russo and P. Vallois [35] have developed a regularization procedure, whose philosophy is similar to the discretization. They introduced a forward (generalizing Itô), backward, symmetric (generalizing Stratonovich) stochastic integrals and a generalized quadratic variation. Their techniques are of pathwise nature, but they are not truely pathwise. They make large use of ucp (uniform convergence in probability) related topology. More recently, several papers have followed that strategy, see for instance [36], [37], [38], [41], [16].
In fact the terminology "Dirichlet processes" is inspired by the theory of Dirichlet forms. Tools from that theory have been developed to understand such processes as integrators, see for instance [27], [28]. Dirichlet processes belong to the class of finite quadratic variation processes.
Even though Dirichlet processes generalize semimartingales, fractional Brownian motion is a finite quadratic variation process (even Dirichlet) if and only if the Hurst index is greater or equal to $\frac{1}{2}$. When $H = \frac{1}{2}$, one obtains the classical standard Brownian motion. If $H > \frac{1}{2}$ it is even a zero quadratic variation process. Indeed fractional Brownian motion is a semimartingale if

and only if it is a classical Brownian motion.

The regularization, or discretization technique, for those and related processes have been performed by [15], [17], [22], [39], [43] and [44] in the case of zero quadratic variation, so $H > \frac{1}{2}$. Young [42] integral can be often used under this circumstance. This integral coincides with the forward (but also with the backward or symmetric) integral since the covariation between integrand and integrator is always zero.

When the integrator has paths with finite p-variation for $p > 2$, there is no hope to make use of forward and backward integrals and the reference integral will be for us the symmetric integral which is a generalization of Stratonovich.

The next step was done by T. J. Lyons and coauthors, see [25, 26], who have considered, through an absolutely pathwise approach based on Lévy stochastic area, integrators having $p - variation$ for any $p > 1$, provided one could construct a canonical geometric rough path associated with the process.

This construction has been done in [8] when the integrator is a fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$; in that case, paths are almost surely of finite p- variation for $p < 4$.

Using Russo-Vallois regularization techniques, [16] has considered a stochastic calculus and some ordinary SDEs with respect to integrators with finite p- variation when $p \leq 3$. This applies directly to the fractional Brownian motion case for $H \geq \frac{1}{3}$. A significant object introduced in [16] was the concept of n-covariation $[Y_1, \ldots, Y_n]$ of n processes $Y_1, \ldots, Y_n$.

Since fractional Brownian motion is a Gaussian process, it was natural to use Skorohod-Malliavin apprach, which as we said, constitutes a powerful tool for the analysis of integrators which are not semimartingales.

Using this approach, integration with respect to fractional Brownian motion, has been attacked by by L. Decreusefonds and A. S. Ustunel [11] for $H > \frac{1}{2}$ and it has known then a very intensive study, see [6], [1] and [2], even when the integrator is a more general Gaussian process. Malliavin-Skorohod techniques

allow to treat integration with respect to processes, in several situations where the variation is larger than 2. In particular [2] includes the case of a fractional Brownian motion B^H such that $H > \frac{1}{4}$. The key tool there, is the Skorohod integral which can be related to the symmetric-Stratonovich integral, up to a trace term of some Malliavin derivative of the integrand. In the case of fractional Brownian motion, [2] have discussed a Itô's formula for the Stratonovich integral when the Hurst index H is greater than $\frac{1}{4}$.
Other significant and interesting references about stochastic calculus with fractional Brownian motion, especially for $H > \frac{1}{2}$, are [12, 13, 24, 29, 30]. Some activity is also going on with stochastic PDE's driven by fractional sheets, see [21].
As we said, our paper follows "almost pathwise calculus techniques" developed by Russo and Vallois, and it reaches the $H = \frac{1}{4}$ barrier, developing very detailed Gaussian calculations.

As we said, one motivation of this paper, was to prove a Itô-Stratonovich formula for a fractional Brownian motion $X = B^H$ for $H \geq \frac{1}{4}$. Such a process has a finite 4-variation in the sense of [16] and a finite pathwise p-variation for $p > 4$, if one refers for instance to [14, 25]. We even prove that the cubic variation in the sense of [16] is zero even when the Hurst index is strictly bigger than $\frac{1}{6}$, see Proposition 3.8.
If one wants to remain in the framework of "pathwise" calculus, Itô's formula has to be of Stratonovich type. In fact, if $H < \frac{1}{2}$, such a formula cannot not make use of the forward integral $\int_0^{\cdot} g(B^H)d^-B^H$ considered for instance in [36] because that integral, as well as the bracket $[g(B^H), B^H]$, is not defined because an explosion occurs in the regularization. It is for this enough to analyse the case of $g(x) = x$; for instance, as [2] points out, the forward integral $\int_0^T B_s^H d^- B_s^H$ does not exist. The use of Stratonovich-symmetric integral is natural and it provides cancellation of the term involving the second derivative.

Our Itô's formula is of the following type.

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H) d^\circ B_u^H$$

We recall that the case $H > \frac{1}{4}$ has already been treated using Malliavin calculus techniques by [2].

The natural way to prove a Itô formula for an integrator having a finite 4-variation is to realize a fourth order Taylor expansion:

$$\begin{aligned} f(X_{t+\varepsilon}) &= f(X_t) + f'(X_t)(X_{t+\varepsilon} - X_t) + \frac{f''(X_t)}{2}(X_{t+\varepsilon} - X_t)^2 \\ &+ \frac{f^{(3)}(X_t)}{6}(X_{t+\varepsilon} - X_t)^3 + \frac{f^{(4)}(X_t)}{24}(X_{t+\varepsilon} - X_t)^4 \end{aligned}$$

plus a rest which can be neglected. The second and third order terms can be essentially controlled because one will prove the existence of suitable covariations and the fourth order term provides a finite contribution because X has a finite fourth variation. Significantly in the case $H = \frac{1}{4}$, the third order term can be expressed in terms of a 4-covariation term $[f^{(3)}(X), X, X, X]$; it compensates then with the fourth order term.

At our point of view, the main achievement of this paper is the proof of the existence of the 4- covariation $[g(B^H), B^H, B^H, B^H]$ for $H \geq \frac{1}{4}$, g being locally bounded: this is done in Theorem 3.9. Moreover, we prove that it is Hölder continuous with parameter strictly smaller than $\frac{1}{4}$. The local boundedness assumption on g can be of course relaxed, making a more careful analysis on the density of fractional Brownian motion at each instant. For the moment, we have not investigated that generality.

That result provides, as an application the Itô-Stratonovich formula for $f(B^H)$, f being of class C^4, see Theorem 4.1.

A second application is a generalized Bouleau-Yor formula for fractional Brownian motion. Fractional Brownian motion B^H has a local time (λ_t^a) which has a continuous version in (a, t), for any $0 < H < 1$, as density of the occupation measure, see for instance [3, 20]. In particular, one has

$$g \to \int_0^t g(B_s^H) ds = \int_{\mathbb{R}} g(a) \lambda_t^a da.$$

We recall first the result concerning the classical Brownian motion $B = B^H$ where $H = \frac{1}{2}$. A direct consequence of [19, 38] and [5] is the following.
If f is a locally bounded function, we have

$$[f(B), B]_t = -\int_{\mathbb{R}} f(a)\ell_t(da)$$

where the right member is well-defined, for instance because $(\ell_t(a))_{a\in\mathbb{R}}$ defines a semimartingale.
Our generalization of Bouleau-Yor identity is the following.

$$[f(B^H), B^H, B^H, B^H]_t = -3\int_{\mathbb{R}} f(a)\lambda_t'(a)da.$$

This is done at Corollary 3.10.
We recall also that, for $H > \frac{1}{3}$, a Tanaka type formula has been obtained by [9] involving Skorohod integral.
The technique used here is a "pedestrian" but accurate exploitation of the Gaussian feature of fractional Brownian motion. Other recent papers where similar techniques have been used are for instance by [23] and [32]. Some of the computations are made with a Maple procedure.
The paper is organised as follows. At section 2, we recall some basic definitions and results, in section 3 we state the theorems, we make some basic remarks and we prove part of the results. Section 4 is devoted to Itô's formula. Section 5 will be devoted to the technical proofs and the Appendix will indicate the Maple procedure.

2 Notations and recalls of preliminary results

We start recalling some definitions and results established on some previous papers, see [36, 37, 38, 39]. In this paper X will be a continuous processes and Y will be a process with locally bounded paths. The space of continuous processes will be a metrizable Fréchet space $\mathcal{C}$, if it is equipped with the topology of the *uniform convergence in probability on each compact interval* (ucp). The space of random variables is also a metrizable Fréchet space, denoted by $L^\circ \equiv L^\circ(\Omega)$ and it is equipped with the topology of the convergence in probability.

We define the forward integral

$$\int_0^t Y_u d^- X_u := \lim_{\varepsilon \downarrow 0} \text{ucp} \frac{1}{\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_u) du \tag{2.1}$$

and the covariation

$$[X, Y]_t := \lim_{\varepsilon \downarrow 0} \text{ucp} \frac{1}{\varepsilon} \int_0^t (X_{u+\varepsilon} - X_u)(Y_{u+\varepsilon} - Y_u) du. \tag{2.2}$$

The symmetric-Stratonovich integral is defined as

$$\int_0^t Y_u d^\circ X_u := \lim_{\varepsilon \downarrow 0} \text{ucp} \frac{1}{2\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_{u-\varepsilon}) du. \tag{2.3}$$

The following fundamental equality is valid

$$\int_0^t Y d^\circ X = \int^t Y d^- X + \frac{1}{2}[X, Y]_t, \tag{2.4}$$

provided that the right member is well defined. However, as we will see in the next section, the left member may exist even if the covariation $[X, Y]$ does not exist.

On the other hand the symmetric-Stratonovich integral can also be written as

$$\int_0^t f(X_u) d^\circ X_u = \lim_{\varepsilon \downarrow 0} \text{ucp} \int_0^t (f(X_{u+\varepsilon}) + f(X_u)) \frac{X_{u+\varepsilon} - X_u}{2\varepsilon} du. \tag{2.5}$$

Such a definition will be somehow relaxed later. If X is such that $[X, X]$ exists, X is called finite quadratic variation processes. If $[X, X] = 0$ then X will be called *zero quadratic variation process*. In particular a Dirichlet process (local maringale plus a zero quadratic variation process) is a finite quadratic variation process. If $f \in C^2$ then following Itô formula holds:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^- X_u + \frac{1}{2}[f'(X), X]_t. \tag{2.6}$$

We recall that finite quadratic variation processes are stable by C^1 transformations. In particular, if $f, g \in C^1$ and the vector (X, Y) is such that all mutual covariation exist, then $[f(X), g(Y)]_t = \int_0^t f'(X_s) g'(X_s) d[X, Y]_s$. Then formulas (2.5) and (2.6) give

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^\circ X_u. \tag{2.7}$$

Remark 2.1 *1. If X is a continuous semimartingale and Y is a suitable previsible process, then $\int_0^\cdot Y d^- X$ is the classical Itô's integral, for details see [36].*

2. If X and Y are (continuous) semimartingales then $\int_0^\cdot Y d^\circ X$ is the Fisk-Stratonovich integral and $[X, Y]$ is the ordinary square bracket.

3. If $X = B^H$, then its paths are a.s. Hölder continuous with parameter strictly bigger than H. Therefore, it is easy to see that, if $H > \frac{1}{2}$ then X is a zero quadratic variation process. When $H = \frac{1}{2}$, B^H is the classical Brownian motion and so $[B^H, B^H]_t = t$. In particular Itô formula (2.6) holds for $H \geq \frac{1}{2}$.

4. If X is a classical Brownian motion B, then formula (2.6) holds even for $f \in W^{1,2}_{\mathrm{loc}}$, see [19, 38]. On the other hand, if $(\ell_t(a))$ is the local time associated with B, then [5] has shown that

$$f(B_t) = f(B_0) + \int_0^t f'(B_u) dB_u - \frac{1}{2} \int_{\mathbb{R}} f'(a) \ell_t(da). \tag{2.8}$$

The integral involving local time in the right member of (2.8) was defined directly by Bouleau and Yor, for B being a general semimartingale. However, in the case of fractional Brownian motion, Corollary 1.13 of [5] states that for fixed $t > 0$, $(\ell_t(a))$ is a classical semimartingale with respect to a; indeed that integral has a meaning as a deterministic Itô integral. Thus, for $g \in L^2_{\mathrm{loc}}$, setting f such that $f' = g$ and using (2.6) and (2.8), we obtain what we will call the **Bouleau-Yor identity***:*

$$\int_{\mathbb{R}} g(a) \ell_t(da) = -[g(B), B]. \tag{2.9}$$

Our Corollary 3.10 will generalize this result to the case of fractional Brownian motion B^H with $H = \frac{1}{4}$.

5. *An accurate study of "pathwise stochastic calculus" for finite quadratic variation processes has been done in [39]. That paper provides necessary and sufficient conditions on the covariance of a Gaussian process X so that X is a finite quadratic variation process and that X has a deterministic quadratic variation.*

Since the quadratic variation is not defined for B^H when $H < \frac{1}{2}$, we have to find a substitution tool. Already in [39], a concept of α-variation, that here will be called **strong** α-variation, is the following increasing continuous process:

$$[X]_t^{(\alpha)} := \lim_{\varepsilon \downarrow 0} \text{ucp} \int_0^t \frac{|X_{u+\varepsilon} - X_u|^\alpha}{\varepsilon} du. \tag{2.10}$$

A real attempt to adapt previous approach to integrators X not being of finite quadratic variation has been done in [16]. For a positive integer n, [16] defines the *n-covariation* $[X^1, \ldots, X^n]$ of a vector $(X_1, \ldots, X_n)$ of real processes if one of them at least is continuous, in the following way.

$$[X_1, \ldots, X_n]_t := \lim_{\varepsilon \downarrow 0} \text{ucp} \int_0^t \frac{(X^1_{u+\varepsilon} - X^1_u) \ldots (X^n_{u+\varepsilon} - X^n_u)}{\varepsilon} du. \tag{2.11}$$

In particular, if all the processes X_i are equal to X than the definition gives

$$\underbrace{[X, X, \ldots, X]}_{n \text{ times}}(t) := \lim_{\varepsilon \downarrow 0} \text{ucp} \int_0^t \frac{(X_{u+\varepsilon} - X_u)^n}{\varepsilon} du. \tag{2.12}$$

Previous quantity is called the **n-variation** of process X. Clearly,

$$[X]^{(n)} = \underbrace{[X, X, \ldots, X]}_{n \text{ times}}, \text{ for even integer } n.$$

Remark 2.2 *1. If $n = 2$, the 2-variation $[X_1, X_2]$ extends the covariation defined for instance in [37].*

2. *If the strong n-variation of X exists, then for all $m > n$, $\underbrace{[X, X, \dots, X]}_{m \text{ times}} = 0$ (see [16], Remark 2.6.3, p. 7).*

3. *If $\underbrace{[X, X, \dots, X]}_{n \text{ times}}$ and $[X]^{(n)}$ exist then*

$$\lim_{\varepsilon \downarrow 0} \operatorname{ucp} \int_0^t g(X_u) \frac{(X_{u+\varepsilon} - X_u)^n}{\varepsilon} du = \int_0^t g(X_u) d[X, X, \dots, X]_u, \tag{2.13}$$

where $g \in \mathrm{C}(\mathbb{R})$, see [16], Remark 2.6.6, p. 8 and Remark 2.1, p. 5). Taking in particular $g \equiv 1$, one obtains that the existence of the strong n-variation of X implies the existence of the n-variation of X.

In reality, the result of [16] was a bit stronger, since it did not exactly require the existence of $[X]^{(n)}$, but only the fact that the approximating r. v. are bounded in probability.

4. *Let $f_1, \dots, f_n$ of class $C^1(\mathbb{R})$ and X being a strong n-variation continuous process. Then $[f_1(X), \dots, f_n(X)] = \int_0^\cdot f_1'(X) \dots f_n'(X) d[X_1, \cdots, X_n]_s$.*

5. *In [16], Proposition 3.4, if X is a continuous strong 3-variation process then, for $f \in \mathrm{C}^3(\mathbb{R})$ one writes a Itô's type formula:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^\circ X_u - \frac{1}{12} \int_0^t f^{(3)}(X_u) d[X, X, X]_u. \tag{2.14}$$

In particular previous point implies that the it can also be expressed as

$$f(X_0) + \int_0^t f'(X_u) d^\circ X_u - \frac{1}{12} [f''(X), X, X]_t.$$

The authors also consider extensions to the n-dimensional case.

6. *Let us consider again $X = B^H$. In [16], Proposition 3.1, it is proved that the strong 3-variation exists if $H \geq \frac{1}{3}$ but, even for the limiting case $H = \frac{1}{3}$, we have that $[B^H, B^H, B^H] = 0$.*

7. *In [39], Proposition 3.14, p. 22, it is proved that the strong* $1/H$*-variation exists and equals* $\rho_H t$*, where* $\rho_H = \mathrm{E}[|G|^{1/H}]$*, with* G *a standard normal random variable.*
   Consequently, if $H = \frac{1}{4}$*, then* $[B^H]^{(4)}_t = [B^H, B^H, B^H, B^H]_t = 3t$*; if* $H > \frac{1}{4}$ *then* $[B^H]^{(4)} \equiv 0$*.*

In section 5, we will be able to write a Itô's formula for the fractional Brownian motion with index $H < \frac{1}{3}$, more precisely for $H \in [\frac{1}{4}, \frac{1}{3}[$. We observe that, in that case, B^H admits a (strong) 4-variation but not a strong 3-variation.

3 The third order type integrals and 4-covariations

In order to understand the case of fractional Brownian motion for $H \geq \frac{1}{4}$, besides the family of integrals introduced until now, we need to introduce a new class of integrals of third order type. All along the paper, T will stay for a positive number.

Let again $(X_t)_{t\geq 0}$ be a continuous process and $(Y_t)_{t\geq 0}$ be a process with locally bounded paths. We define the following **third order integrals** as the limit in probability of

$$\begin{aligned} \int_0^T Y_u d^{-3} X_u &:= \lim_{\varepsilon \downarrow 0+} \frac{1}{\varepsilon} \int_0^T Y_u (X_{u+\varepsilon} - X_u)^3 du; \\ \int_0^T Y_u d^{+3} X_u &:= \lim_{\varepsilon \downarrow 0+} \frac{1}{\varepsilon} \int_0^T Y_u (X_u - X_{(u-\varepsilon \vee 0)})^3 du; \\ \int_0^T Y_u d^{\circ 3} X_u &:= \lim_{\varepsilon \downarrow 0+} \frac{1}{2\varepsilon} \int_0^T (Y_u + Y_{u+\varepsilon}) (X_{u+\varepsilon} - X_u)^3 du. \end{aligned} \tag{3.1}$$

We will call them respectively **forward, backward** and **symmetric** third order integral. If the above L°- valued function, $t \to \int_0^t Y_s d^{-3} X_s$ (resp. $t \to \int_0^t Y_s d^{\circ 3} X_s$, $t \to \int_0^t Y_s d^{+3} X_s$) exists for any $t > 0$, and it admits a continuous version, then such a version will be called **indefinite third order integral** and it will be denoted again by $(\int_0^t Y_s d^{-3} X_s)_t$ (resp. $(\int_0^t Y_s d^{+3} X_s)_t$, $(\int_0^t Y_s d^{\circ 3} X_s)_t$).

Remark 3.1 *1. If X is a strong 3-variation process, then $[X,X,X]$ will be a finite variation process and*

$$\int_0^{\cdot} Y d^{-3}X = \int_0^{\cdot} Y d^{+3}X = \int_0^{\cdot} Y d[X,X,X]. \tag{3.2}$$

2. If Y is a semimartingale and X is a fractional Brownian motion, $B^H, H \geq \frac{1}{3}$, an obvious integration by parts shows that all the quantities in (3.2) are zero.

The following results relate previously defined integrals with the notion of n-covariation. Its proof is elementary.

Proposition 3.2 *1. $\int_0^{\cdot} Y d^{\circ 3}X = \frac{1}{2}(\int_0^{\cdot} Y d^{-3}X + \int_0^{\cdot} Y d^{+3}X)$ provided two of the three previous quantities exist.*

2.

$$\int_0^{\cdot} Y d^{+3}X - \int_0^{\cdot} Y d^{-3}X = [Y,X,X,X]$$

if two of the above terms exist.

Corollary 3.3 *Let X be a continuous process having a 4- variation. Let $f \in \mathrm{C}^1$.*

1. If $\int_0^{\cdot} f(X) d^{-3}X$ exists then

$$\int_0^{\cdot} f(X) d^{+3}X = \int_0^{\cdot} f(X) d^{-3}X + \int_0^{\cdot} f'(X) d[X,X,X,X].$$

2. Let $f \in \mathrm{C}^2$. If $\int_0^{\cdot} f'(X) d^{-3}X$ exists then

$$[f(X),X,X] = \int_0^{\cdot} f'(X) d^{-3}X + \frac{1}{2}\int_0^{\cdot} f''(X) d[X,X,X,X]$$

Proof.

Point 1) follows immediately from Proposition 3.2 and Remark 2.2 point 4). Concerning point 2), a second order Taylor expansion, for $s, \varepsilon > 0$, gives

$$f(X_{s+\varepsilon}) - f(X_s) = f'(X_s)(X_{s+\varepsilon} - X_s) + \frac{f''(X_s)}{2}(X_{s+\varepsilon} - X_s)^2 + R(f,\varepsilon,s)(X_{s+\varepsilon} - X_s)^2$$

where $R(f, \varepsilon, s)$ converges ucp to zero in s when ε goes to zero, because of the uniform continuity of f and X paths on each compact interval.
Multiplying by $(X_{s+\varepsilon} - X_s)^2$, integrating previous expression from 0 to t, dividing by ε and using Remark 2.2 point 3) we obtain the result. ■

From now on we will concentrate on the case when X is a fractional Brownian motion B^H with Hurst index H.
In some situations, the third order integrals will be shown to exist even if $[X]^{(3)}$ does not exist; however $[X, X, X]$ will exist and it will be zero. On the other hand, they will not need to be finite variation processes.
In spite of the now classical definition of the symmetric integral given in (2.5), next result will be a bit weaker than expected. From now on, we will say that the symmetric integral of a process Y with respect to an integrator X exists if

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_{u-\varepsilon}) du. \tag{3.3}$$

in probability and whenever, the limiting L°- valued function has a continuous version. We will still denote that process (unique up to indistinguishability) by $\int_0^t Y_u d^\circ X_u$.
At the same way, in this paper the concept of 4-covariation will be understood in a weaker sense with respect to [16]. We will say that the 4-variation $[X_1, X_2, X_3, X_4]$ exists if for each $t > 0$,

$$\lim_{\varepsilon \downarrow 0} \int_0^t \frac{(X^1_{u+\varepsilon} - X^1_u) \dots (X^4_{u+\varepsilon} - X^4_u)}{\varepsilon} du.$$

exists in probability and if that the limiting L° valued function has a continuous version.
Clearly if $\int_0^\cdot Y d^\circ X$ in the classical sense of Russo and Vallois, see (2.5), then it exists also in this relaxed meaning; similarly if $[X_1, X_2, X_3, X_4]$ exists in the (2.11) sense, that it will exist in the relaxed sense. We remark that when all the processes are equal, then a Dini type lemma, as in [39] allows to show that the two definitions of 4-covariations are equivalent. We remark that Proposition 3.2 and Corollary 3.3 are still valid with these conventions.

The fundamental, even though technical result, of this section is the following. We will say that a real function g fulfills the **subexponential inequality** if

$$|g(x)| \leq Le^{\ell|x|}, \ \textit{for } \ell, L \textit{ positive constants.} \tag{3.4}$$

Proposition 3.4 *Let $H \in [\frac{1}{4}, \frac{1}{3}[$ and g be a real locally bounded function. The following properties hold.*

a) The third order forward integral $\int_0^T g(B_u^H) d^{-3} B_u^H$ exists.

a') The third order backward integral $\int_0^T g(B_u^H) d^{+3} B_u^H$ exists.

b) For $\frac{1}{4} < H < \frac{1}{3}$

$$\int_0^T g(B_u^H) d^{\pm 3} B_u^H = 0. \tag{3.5}$$

c) Suppose that g fulfills the subexponential inequality (3.4). For $H = \frac{1}{4}$ the expectation and the second moment of $\int_0^T g(B_u^H) d^{-3} B_u^H$ are given by:

$$\mathrm{E}\left\{\int_0^T g(B_u^H) d^{-3} B_u^H\right\} = -\frac{3}{2}\int_0^T \frac{du}{\sqrt{u}} \mathrm{E}[g(B_u^H) B_u^H], \tag{3.6}$$

and,

$$\mathrm{E}\left\{\left(\int_0^T g(B_u^H) d^{-3} B_u^H\right)^2\right\} = \frac{9}{2} \iint\limits_{0<u<v<T} du\, dv\, \mathrm{E}\left[g(B_u^H) g(B_v^H)\right. \tag{3.7}$$

$$\left. \times \left(\lambda_{11}\lambda_{12}(B_u^H)^2 + (\lambda_{11}\lambda_{22} + \lambda_{12}^2) B_u^H B_v^H + \lambda_{12}\lambda_{22}(B_v^H)^2 - \lambda_{12}\right)\right],$$

where the right hand sides of (3.6) and (3.7) are absolute convergent integrals. Here

$$\begin{aligned} \lambda_{11} &= \frac{\sqrt{v}}{\sqrt{uv - K_H(u,v)^2}}, \ \lambda_{22} = \frac{\sqrt{u}}{\sqrt{uv - K_H(u,v)^2}}, \\ \lambda_{12} &= -\frac{K_H(u,v)}{\sqrt{uv - K_H(u,v)^2}}. \end{aligned} \tag{3.8}$$

c') Suppose again that g fulfills the subexponential inequality (3.4). For $H=\frac{1}{4}$ and with the conventions above,

$$\mathrm{E}\left\{\int_0^T g(B_u^H)d^{+3}B_u^H\right\} = -\mathrm{E}\left\{\int_0^T g(B_u^H)d^{-3}B_u^H\right\}, \tag{3.9}$$

and,

$$\mathrm{E}\left\{\left(\int_0^T g(B_u^H)d^{+3}B_u^H\right)^2\right\} = \mathrm{E}\left\{\left(\int_0^T g(B_u^H)d^{-3}B_u^H\right)^2\right\}. \tag{3.10}$$

d) Let $H=\frac{1}{4}$ and $g,h\in L^\infty_{\mathrm{loc}}(\mathbb{R})$ fulfilling the subexponential inequality (3.4). Then

$$\begin{aligned} &\mathrm{E}\left\{\int_0^T g(B_u^H)d^{-3}B_u^H \int_0^T h(B_u^H)du\right\} \\ &= -\frac{3}{2}\mathrm{E}\left\{\int_0^T dv\int_0^T du g(B_u^H)h(B_v^H)(\lambda_{11}B_u^H+\lambda_{12}B_v^H)\right\}\end{aligned}$$

e) If $H=\frac{1}{4}$, $g\in \mathrm{C}^1(\mathbb{R})$ with g,g' bounded, then the following holds:

$$\begin{aligned} &\mathrm{E}\left\{\left(\int_0^T g(B_u^H)d^{-3}B_u^H\right)^2\right\} \\ &= \frac{9}{4}\mathrm{E}\left\{\left(\int_0^T g'(B_u^H)du\right)^2\right\}.\end{aligned} \tag{3.11}$$

e') If $H=\frac{1}{4}$, $g\in \mathrm{C}^1(\mathbb{R})$, $h\in \mathrm{C}^\circ(\mathbb{R})$ with g,g',h bounded, we have:

$$\begin{aligned} &\mathrm{E}\left\{\left(\int_0^T g(B_u^H)d^{-3}B_u^H\right)\left(\int_0^T h(B_u^H)du\right)\right\} \\ &= -\frac{3}{2}\mathrm{E}\left\{\left(\int_0^T g'(B_u^H)du\right)\left(\int_0^T h(B_u^H)du\right)\right\}.\end{aligned} \tag{3.12}$$

f) The indefinite third order integrals $(\int_0^t g(B_u^H)d^{\pm 3}B_u^H)_t$, exist and they are Hölder continuous with parameter less than $\frac{1}{4}$.

Proof.

It will be postponed to the last section. ■

Corollary 3.5 *The maps $g \to \int_0^T g(B^H)d^{\pm}B^H$ and $g \to \int_0^T g(B^H)d^{\circ}B^H$ are continuous from $L^{\infty}_{\rm loc}$ to $L^{\circ}(\Omega)$.*

Proof.

Let $(g_n)_n$ be a sequence in $L^{\infty}_{\rm loc}$ converging to zero, which means that for every $M > 0$, $\sup_{|x|\le M} |g_n(x)| \to 0$.

We have to show that the sequence of random variables $(\int_0^T g_n(B^H)d^{\pm}B^H)_n$ converges to zero in probability.

Let $M > 0$. Since the paths of B^H are continuous, so locally bounded, setting

$$\Omega_M = \{\omega \in \Omega | \sup_{t\in[0,T]} |B_t^H|\},$$

we have

$$\int_0^T g_n(B_u^H)d^{\pm 3}B_u^H \quad 1_{\Omega_{M+1}} = \int_0^T g_n^M(B_u^H)d^{\pm 3}B_u^H \quad 1_{\Omega_{M+1}},$$

where $g^M = g1_{[-M,M]}$. Therefore, the sequence (g_n) can be considered converging in L^{∞} to zero. An obvious argument of localization allows to conclude. But now, point c) and c') of Proposition 3.4 show that the convergence holds in L^2.

The case of the symmetric third order integral follows from point 1) of Proposition 3.2.

■

By point e) of Proposition 3.4, we observe that, when g is smooth, the third order forward integral has the same expectation and the same second moment as $-\frac{3}{2}\int_0^T g'(B_u^H)du$. The following will show that those integrals are in fact the same.

Theorem 3.6 *Let* $H \geq \frac{1}{4}$, $g \in \mathrm{C}^1(\mathbb{R})$.

1.

$$\int_0^t g(B_u^H) d^{+3} B_u^H = \frac{1}{2} \int_0^t g'(B_u^H) d[B^H]_u^{(4)} = - \int_0^t g(B_u^H) d^{-3} B_u^H \tag{3.13}$$

2. $\int_0^t g(B_u^H) d^{03} B_u^H = 0$

Proof.

Point 2) follows immediately from point 1) and Proposition 3.2 1).
If $H > \frac{1}{4}$, point 1) is a consequence of $[B_t^H]^{(4)} \equiv 0$ and of Proposition 3.4 b). It remains to show the validity of point 1) under the assumption $H = \frac{1}{4}$. Using Corollary 3.5, and an obvious approximation argument, it is enough to suppose that g and g' are bounded. Since all the integrals in (3.13) are continuous, we only need to verify that for fixed $t > 0$,

$$\mathrm{E} \left(\int_0^t g(B_u^H) d^{-3} B_u^H + \frac{3}{2} \int_0^t g'(B_v^H) dv \right)^2 = 0. \tag{3.14}$$

The left member of (3.14) gives

$$J_{11} + \frac{9}{4} J_{22} + 3 J_{12} \tag{3.15}$$

where

$$\begin{aligned} J_{11} &= \mathrm{E} \left(\int_0^t g(B_u^H) d^{-3} B_u^H \right)^2 \\ J_{22} &= \mathrm{E} \left(\int_0^t g'(B_v^H) dv \right)^2 \\ J_{12} &= \mathrm{E} \left(\int_0^t g(B_u^H) d^{-3} B_u^H \int_0^t g'(B_v^H) dv \right) \end{aligned}$$

Proposition 3.4 e) implies that

$$\begin{aligned} J_{11} &= \frac{9}{4} \mathrm{E} \left(\int_0^t g'(B_u^H) du \right)^2 \\ J_{12} &= -\frac{1}{2} \mathrm{E} \left(\int_0^t g'(B_u^H) du \right)^2 \end{aligned}$$

Finally (3.15) allows to conclude (3.13). ■

Remark 3.7 *Let $H \geq \frac{1}{4}$. Suppose again $g \in C^1(\mathbb{R})$. Remark 2.2 4) says that*

$$[g(B^H), B^H, B^H, B^H] = \int_0^{\cdot} g'(B^H) d[B^H]^{(4)};$$

hence, from previous theorem it follows that

$$\int_0^t g(B_u^H) d^{+3} B_u^H = \frac{1}{2}[g(B^H), B^H, B^H, B^H]_t = -\int_0^t g(B_u^H) d^{-3} B_u^H.$$

This property will be generalized by theorem 3.9, to any Borel locally bounded g.

As it follows from above, the 3-variation of a fractional Brownian motion B^H is zero when $H \geq \frac{1}{4}$. In fact it is the third order integral of the constant function 1. This result can be extended to the case of lower Hurst index.

Proposition 3.8 *If $H > \frac{1}{6}$, then*

$$[B^H, B^H, B^H]_T = \int_0^T d^{-3} B_u^H = 0. \tag{3.16}$$

Proof.
It will be postponed to the last section. ■

Next result states the existence of a significant fourth order covariation related to the fractional Brownian motion B^H with Hurst index $H \geq \frac{1}{4}$.

Theorem 3.9 *Let $g \in L^\infty_{\text{loc}}, T > 0$. The process $([g(B^H), B^H, B^H, B^H]_t)_{t \geq 0}$ is well defined. Moreover, the following properties hold.*

1. *The map $g \to [g(B^H), B^H, B^H, B^H]_T$ is continuous from L^∞_{loc} to $L^\circ(\Omega)$.*

2. *$[g(B^H), B^H, B^H, B^H]$ has Hölder continuous paths of parameter strictly less than $\frac{1}{4}$.*

3.

$$[g(B^H), B^H, B^H, B^H] = 2\int_0^{\cdot} g(B_u^H) d^{+3} B_u^H = -2\int_0^{\cdot} g(B_u^H) d^{-3} B_u^H$$

4. $\int_0^{\cdot} g(B^H_u) d^{\circ 3} B^H_u \equiv 0$.

Proof.

The result is trivial if $H > \frac{1}{4}$. Therefore we will suppose $H = \frac{1}{4}$.

According to Proposition 3.4 points a), a'), the third order integrals

$$
\begin{aligned}
FI &= \int_0^{\cdot} g(B^H_u) d^{-3} B^H_u \\
BI &= \int_0^{\cdot} g(B^H_u) d^{+3} B^H_u .
\end{aligned}
$$

are well-defined. Proposition 3.2 implies

$$[g(B^H), B^H, B^H, B^H] = BI - FI \tag{3.17}$$

exists. According to Corollary 3.5, the applications $g \to \int_0^T g(B^H_u) d^{\pm 3} B^H_u$ are continuous from L^∞_{loc} to $L^\circ(\Omega)$. Therefore, point 1) follows from (3.17). Point 2) follows from point f) of Proposition 3.4. Point 4) is a consequence of Point 3) of this Proposition and of the definition of symmetric third order integral. It remains to prove Point 3); Remark 3.7 states the result if $g \in C^1$; point 1) and Corollary 3.5 say that the 4-covariation and the third order integrals are continuous in probability at fixed time $t > 0$; the result follows then by continuous extension.

■

One consequence of previous theorem concerns local time. Let $(\lambda_t(a)$ be the local time as the occupation measure density, see [3, 20]. It exists for any $0 < H < 1$. Moreover, if $H < \frac{1}{3}$, it is absolutely continuous with respect to a. We denote by $\lambda'_t(a)$ the corresponding derivative.

Corollary 3.10 *If* $H = \frac{1}{4}$, $g \in L^\infty_{\mathrm{loc}}$ *then*

$$[g(B^H), B^H, B^H, B^H]_t = -3 \left(\int g(a) \lambda'_t(a) da \right) . \tag{3.18}$$

Proof.
We recall that $[B^H, B^H, B^H, B^H]_t = 3t$ and so

$$[g(B^H), B^H, B^H, B^H]_t = 3\int_0^t g'(B_s^H)ds,$$

whenever $g \in C^1$. By density occupation formula, previous expression becomes $3\int g'(a)\lambda_t(a)da$. Integrating by parts, we obtain the right member of (3.18). This shows the equality for smooth g. To obtain the final statement, we regularize $g \in L^\infty_{\text{loc}}$, we apply the equality for g being smooth and we take the limit. For the limit of left members, we use the continuity of the considered 4-covariation and for the right members, we use the Lebesgue dominated convergence theorem. ∎

Remark 3.11 *Our theorem extends the Bouleau-Yor type equality (2.9), discussed at Remark 2.1 for the case of the classical Brownian motion, to the fractional Brownian motion with* $H = \frac{1}{4}$.

4 Itô's formula

Let B^H be again a fractional Brownian motion with Hurst index H. The following Itô's formula holds.

Theorem 4.1 *Let* $H \geq \frac{1}{4}$ *and* $f \in \mathrm{C}^4(\mathbb{R})$. *Then the symmetric integral* $\int_0^t f'(B_u^H)d^\circ B_u^H$ *exists and a Itô's type formula can be written:*

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H)d^\circ B_u^H. \tag{4.1}$$

Remark 4.2 *The most interesting case concerns the critical limiting case* $H = \frac{1}{4}$, *since when,* $H > \frac{1}{4}$ *the result has been established by [2].*

Proof.

Theorem 4.1 will be in fact a consequence of Theorem 3.6 2). Let fix $T > 0$. Technically speaking, we will prove that

$$f(B_T^H) = f(B_0^H) + \int_0^T f'(B_u^H) d^\circ B_u^H - \frac{1}{12} \int_0^T f^{(3)}(B_u^H) d^{\circ 3} B_u^H. \qquad (4.2)$$

This will imply the final result because Theorem 3.6 2) says that $\int_0^T g(B_s^H) d^{\circ 3} B_s^H$ vanishes for any $g \in C^1$.

Therefore, it remains to show (4.2) for any $f \in C^4$.

We start with Taylor formula; for $a, b \in \mathbb{R}$ we have

$$\begin{aligned} f(b) - f(a) &= f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} \\ &+ f^{(3)}(a)\frac{(b-a)^3}{6} + \frac{(b-a)^4}{6} \int_0^1 \lambda^3 f^{(4)}\left(\lambda a + (1-\lambda) b\right) d\lambda \end{aligned} \qquad (4.3)$$

and also

$$\begin{aligned} f(a) - f(b) &= f'(b)(a-b) + f''(b)\frac{(a-b)^2}{2} \\ &+ f^{(3)}(b)\frac{(a-b)^3}{6} + \frac{(a-b)^4}{6} \int_0^1 \lambda^3 f^{(4)}\left(\lambda b + (1-\lambda) a\right) d\lambda \end{aligned}$$

which also equals

$$\begin{aligned} &- f'(b)(b-a) + f''(b)\frac{(b-a)^2}{2} - f^{(3)}(b)\frac{(b-a)^3}{6} \\ &+ \frac{(b-a)^4}{6} \int_0^1 (1-\lambda)^3 \left[f^{(4)}(\lambda a + (1-\lambda) b)\right] d\lambda \end{aligned}$$

Since

$$\begin{aligned} f''(b) &= f''(a) + f^{(3)}(a)(b-a) + \\ &+ (b-a)^2 \int_0^1 \lambda \left[f^{(4)}(\lambda a + (1-\lambda) b)\right] d\lambda \end{aligned}$$

and

$$f^{(3)}(b) = f^{(3)}(a) + (b-a)\int_0^1 \left[f^{(4)}(\lambda a + (1-\lambda)b)\right] d\lambda,$$

we can write

$$\begin{aligned} f(a) - f(b) &= -f'(b)(a-b) + f''(a)\frac{(b-a)^2}{2} + f^{(3)}(a)\frac{(b-a)^3}{3} \\ &+ (b-a)^4 \int_0^1 \left[\frac{\lambda^2}{2} - \frac{\lambda^3}{6}\right] f^{(4)}(\lambda a + (1-\lambda)b) d\lambda \end{aligned} \tag{4.4}$$

Taking the difference between (4.3) and (4.4) and dividing by 2, we get

$$\begin{aligned} f(b) - f(a) &= \frac{f'(a) + f'(b)}{2}(b-a) - \frac{1}{12} f^{(3)}(a)(b-a)^3 \\ &+ (b-a)^4 \int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4}\right) f^{(4)}(\lambda a + (1-\lambda)b) d\lambda. \end{aligned} \tag{4.5}$$

On the other hand, exchanging the role of a and b, we get

$$\begin{aligned} f(a) - f(b) &= -\frac{f'(a) + f'(b)}{2}(b-a) + \frac{1}{12} f^{(3)}(a)(b-a)^3 \\ &+ (b-a)^4 \int_0^1 \left(\frac{(1-\lambda)^3}{6} - \frac{(1-\lambda)^2}{4}\right) f^{(4)}(\lambda a + (1-\lambda)b) d\lambda. \end{aligned} \tag{4.6}$$

Taking this time the difference between (4.5) and (4.6) and dividing by 2, we get

$$\begin{aligned} f(b) - f(a) &= \frac{f'(a) + f'(b)}{2}(b-a) - \frac{f^{(3)}(a) + f^{(3)}(b)}{12}(b-a)^3 \\ &+ (b-a)^4 J(a,b) \end{aligned} \tag{4.7}$$

where

$$J(a,b) = \int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4} + \frac{1}{24}\right) f^{(4)}(\lambda a + (1-\lambda)b) d\lambda$$

Since

$$\int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4} + \frac{1}{24} \right) d\lambda = 0,$$

$J(a, b)$ can be written as

$$\int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4} + \frac{1}{24} \right) \left[f^{(4)}(\lambda a + (1-\lambda)b) - f^{(4)}(a) \right] d\lambda.$$

Let $T > 0$. For simplicity, we shall write X for B^H. Setting in (4.7) $a = X_u$ and $b = X_{u+\varepsilon}$, we get

$$\begin{aligned} f(X_{u+\varepsilon}) - f(X_u) &= (f'(X_u) + f'(X_{u+\varepsilon})) \frac{X_{u+\varepsilon} - X_u}{2} \\ &- \left(f^{(3)}(X_u) + f^{(3)}(X_{u+\varepsilon}) \right) \frac{(X_{u+\varepsilon} - X_u)^3}{12} \\ &+ J(X_u, X_{u+\varepsilon})(X_{u+\varepsilon} - X_u)^4. \end{aligned} \tag{4.8}$$

Using the uniform continuity on each compact real interval of $f^{(4)}$ and of X, we observe that $\sup_u J(X_u, X_{u+\varepsilon}) \to 0$, in probability when $\varepsilon \downarrow 0$. We integrate (4.8) in u on $[0, T]$ and we divide by ε:

$$\frac{1}{\varepsilon} \int_0^T (f(X_{u+\varepsilon}) - f(X_u)) du = \int_0^T (f'(X_{u+\varepsilon}) + f'(X_u)) \frac{X_{u+\varepsilon} - X_u}{2\varepsilon} du$$

$$- \int_0^T \left(f^{(3)}(X_u) + f^{(3)}(X_{u+\varepsilon}) \right) \frac{(X_{u+\varepsilon} - X_u)^3}{12\varepsilon} du + \int_0^T J(X_u, X_{u+\varepsilon}) \frac{(X_{u+\varepsilon} - X_u)^4}{\varepsilon} du$$

By a simple change of variable we can transform the left-hand side and we finally obtain

$$\begin{aligned} \frac{1}{\varepsilon} \int_T^{T+\varepsilon} f(X_u) du &- \frac{1}{\varepsilon} \int_0^{\varepsilon} f(X_u) du \\ &= \int_0^T (f'(X_{u+\varepsilon}) + f'(X_u)) \frac{X_{u+\varepsilon} - X_u}{2\varepsilon} du \\ &- \int_0^T \left(f^{(3)}(X_u) + f^{(3)}(X_{u+\varepsilon}) \right) \frac{(X_{u+\varepsilon} - X_u)^3}{12\varepsilon} du \\ &+ \int_0^T J(X_u, X_{u+\varepsilon}) \frac{(X_{u+\varepsilon} - X_u)^4}{\varepsilon} du. \end{aligned} \tag{4.9}$$

The left-hand side of (4.9) tends, as $\varepsilon \downarrow 0$, toward $f(X_T) - f(X_0)$. The last term on the right-hand side of (4.9) tends to zero because of the convergence $\sup_{u\in[0,T]} J(X_u, X_{u+\varepsilon}) \to 0$ and because the strong 4-variation of the process exists. The second term in the right-hand side converges to $\int_0^T f^{(3)}(X_u) d^{\circ 3} X_u$, which exists because of Theorem 3.6. Therefore, the second term on the right-hand side of (4.9) is also forced to have a limit in probability. According to point *e)* of Proposition 3.4, the symmetric third order integral has a continuous version in T; therefore the second term must have a continuous version and it will be of course the symmetric integral $\int_0^T f'(X_u) d^\circ X_u$.

■

5 Proof of the most technical results

The main topic of this section is the proof of Proposition 3.4 which will be articulated from step *I)* to step *XI)*.

Let us denote

$$I_\varepsilon(g)_T := \int_0^T g(B_u^H) \frac{(B_{u+\varepsilon}^H - B_u^H)^3}{\varepsilon} du. \tag{5.1}$$

Index T will be omitted in the sequel, when it will be considered as fixed.

In steps *I-IV)* we assume the existence of the L^2-limit of $I_\varepsilon(g)$, and we prove points *c), c')* of Proposition 3.4. *V)* concerns the proof of point b), which will be a consequence of an intermediate step by the proof of c). The existence of the third type forward integral for fixed T, (point a)) will be discussed at step *VI)*; the proof of the existence of a continuous version in t of $(I_\varepsilon(g)_t)$ (point f)), is treaten only at step *XI)*: this means that the indefinite third order forward integral exists. As a side effect of point a), we prove the existence of the L^2-limit of $I_\varepsilon(g)$, when g fulfills the subexponential inequality (3.4). The discussion concerning the backward third type integral is similar, see Step *VII)*. At step *VIII)*, we establish point d) and at step *IX) and X)*, we will prove points e) and e').

After having proved Proposition 3.4, we establish Proposition 3.8. The end of this section is devoted to the proof of Lemma 5.2 which is stated and used in the proof of point c).

For simplicity of notations, from now on, we will fix $T = 1$.
Before starting the proof, we start with a Remark which justifies the fact that we will compose Borel functions and fractional Brownian motion related variables without any worry.

Remark 5.1 *If g is a, Lebesgue a.e. defined, locally bounded Borel function then the composition $g(B_t^H), t > 0$ is a well defined, up to an a.s. equivalence, random variable. In fact, if g_1, g_2 are two Lebesgue a.e. modifications of g than $g_1(B_t^H) = g_2(B_t^H)$ a.s.; the reason is that B_t^H has a law density.*
Moreover, if (g_n) is a sequence of r.v. converging to zero in L^∞, it is very easy to see that $(g_n(B_t^H))_n$ converges to zero in $L^2(\Omega)$.

I) Proof of (3.6) as the first part of c) of Proposition 3.4.
Let $H \in [\frac{1}{4}, \frac{1}{3}[$. To compute the expectation of $I_\varepsilon(g)$ we shall use the linear regression for $X_{u+\varepsilon} - X_u$, which is a centered Gaussian random variable with variance ε^{2H}. It can be written as

$$X_{u+\varepsilon} - X_u = \frac{K_H(u, u+\varepsilon) - K_H(u,u)}{K_H(u,u)} X_u + Z_\varepsilon, \tag{5.2}$$

where Z_ε is a Gaussian mean-zero random variable, independent from X_u with variance $\varepsilon^{2H} - \frac{1}{4u^{2H}}((u+\varepsilon)^{2H} - u^{2H} - \varepsilon^{2H})^2$. Therefore,

$$X_{u+\varepsilon} - X_u = \alpha_\varepsilon(u) X_u + \beta_\varepsilon(u) N, \tag{5.3}$$

where N is a standard normal random variable independent from X_u such that for $u > 0$ fixed,

$$\alpha_\varepsilon(u) = \frac{1}{2u^{2H}} \left((u+\varepsilon)^{2H} - u^{2H} - \varepsilon^{2H}\right) = \frac{1}{2}\left(\frac{\varepsilon}{u}\right)^{2H} \phi_0\left(\frac{\varepsilon}{u}\right) \tag{5.4}$$

and

$$\beta_\varepsilon(u)^2 = \varepsilon^{2H} - \alpha_\varepsilon^2(u) u^{2H} = \varepsilon^{2H} \phi_1\left(\frac{\varepsilon}{u}\right), \tag{5.5}$$

where $x^{2H}\phi_0(x) = (1+x)^{2H} - 1 - x^{2H}$, $\phi_1(x) = \left(1 - \frac{1}{4}x^{2H}\phi_0(x)\right)_+$, with ϕ_0 being a continuous bounded function, ϕ_1 a bounded function with the property $\lim_{x\downarrow 0}\phi_0(x) = -1$, $\lim_{x\downarrow 0}\phi_1(x) = 1$. Since $2H < 1$ we can also write

$$\alpha_\varepsilon(u) = -\frac{\varepsilon^{2H}}{2u^{2H}}\left(1 - 2Hu^{2H-1}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \text{ as } \varepsilon \downarrow 0. \tag{5.6}$$

Moreover

$$\beta_\varepsilon^2(u) = \varepsilon^{2H}\left(1 - \frac{\varepsilon^{2H}}{4u^{2H}}\right) + o(\varepsilon^{2H}) \text{ as } \varepsilon \downarrow 0. \tag{5.7}$$

We can now compute the first moment of $I_\varepsilon(g)$. Injecting (5.3) in the expression of $I_\varepsilon(g)$ and because of the independence of N and X_u, we get

$$\mathrm{E}\left[I_\varepsilon(g)\right] = \int_0^1 \frac{\alpha_\varepsilon^3(u)}{\varepsilon}\mathrm{E}\left[g(X_u)X_u^3\right]du + \int_0^1 \frac{3\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon}\mathrm{E}\left[g(X_u)X_u\right]du.$$

Cauchy-Schwarz inequality and the hypothesis on g imply that, for $0 < u < 1$,

$$\mathrm{E}\left[|g(X_u)X_u|\right] \le L\mathrm{E}\left[e^{\ell|X_u|}|X_u|\right] \le L'\mathrm{E}\left[e^{\ell X_u}|X_u|\right] \le \text{cst.}\sqrt{\mathrm{E}(X_u^2)} \le \text{cst.}u^H < \infty.$$

In a similar way, it follows

$$\mathrm{E}\left[|g(X_u)X_u^3|\right] \le \text{cst.}\sqrt{\mathrm{E}(X_u^6)} = \text{cst. } u^{3H}.$$

Hence, since $\frac{1}{4} \le H < \frac{1}{3}$, as $\varepsilon \downarrow 0$,

$$\frac{\alpha_\varepsilon^3(u)}{\varepsilon}u^{3H} = \frac{1}{8}\frac{\varepsilon^{6H-1}}{u^{3H}}\phi_0^3\left(\frac{\varepsilon}{u}\right), \text{ with } \int_0^1 \frac{\mathrm{du}}{\mathrm{u}^{3\mathrm{H}}} < \infty.$$

Since $\frac{1}{4} \le H < \frac{1}{3}$, letting ε go to 0 we get

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}\left[I_\varepsilon(g)\right] = \int_0^1 \left(\lim_{\varepsilon\downarrow 0}\frac{3\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon}\right)\mathrm{E}\left[g(X_u)X_u\right]du$$

and (3.6) is obtained using (5.4) and (5.5). Indeed, since $\frac{1}{4} \le H < \frac{1}{3}$, we have

$$\frac{\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon}u^H = \frac{1}{2}\frac{\varepsilon^{4H-1}}{u^H}(\phi_0\phi_1)\left(\frac{\varepsilon}{u}\right), \text{ with } \int_0^1 \frac{du}{u^H} < \infty.$$

- If $H = \frac{1}{4}$, Lebesgue dominated convergence implies that

$$\lim_{\varepsilon \downarrow 0} \mathrm{E}\left[I_\varepsilon(g)\right] = -\frac{3}{2}\int_0^1 \frac{1}{\sqrt{u}} \mathrm{E}\left[g(X_u)X_u\right] du$$

  and then (3.6) follows using part *a)* of Propositon 3.4.

- If $\frac{1}{4} < H < \frac{1}{3}$ then clearly

$$\lim_{\varepsilon \downarrow 0} \mathrm{E}\left[I_\varepsilon(g)\right] = 0. \tag{5.8}$$

□

II) Proof of (3.7) as the second part of c) of Proposition 3.4.
The computation of the second moment of $I_\varepsilon(g)$ is done using again the Gaussian feature of the process. We express the linear regression for the random vector $(X_{u+\varepsilon} - X_u, X_{v+\varepsilon} - X_v)$. Let us denote $G = (G_1, G_2, G_3^\varepsilon, G_4^\varepsilon)$ the Gaussian mean-zero random vector $(X_u, X_v, X_{u+\varepsilon} - X_u, X_{v+\varepsilon} - X_v)$ and we use a similar idea as in *I)*. For instance (5.2) will be replaced by

$$\begin{pmatrix} G_3^\varepsilon \\ G_4^\varepsilon \end{pmatrix} = A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + \begin{pmatrix} Z_1^\varepsilon \\ Z_2^\varepsilon \end{pmatrix}, \tag{5.9}$$

where the Gaussian mean-zero random vector $Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon)$ is independent from (G_1, G_2).
Clearly,

$$I_\varepsilon(g)^2 = 2 \iint\limits_{0<u<v<1} g(X_u)g(X_v) \frac{(X_{u+\varepsilon} - X_u)^3}{\varepsilon} \frac{(X_{v+\varepsilon} - X_v)^3}{\varepsilon} dudv,$$

hence

$$\mathrm{E}\left[I_\varepsilon(g)^2\right] = 2\mathrm{E}\left\{\iint_{0<u<v<1} g(G_1)g(G_2)\,\mathrm{E}\left(\frac{(G_3^\varepsilon)^3 (G_4^\varepsilon)^3}{\varepsilon^2} \mid G_1, G_2\right) dudv\right\}. \tag{5.10}$$

Therefore we need to compute the conditional expectation in (5.10). For that reason, we need the following lemma which will be useful again at step *VI)* of the proof where we prove the existence of the L^2-limit of I_ε. For random variables $\xi, \zeta\ \phi_\varepsilon$, we will denote

$$\xi \stackrel{\text{(law)}}{=} \zeta + o(\varepsilon) \text{ as } \varepsilon \downarrow 0, \quad \text{if } \xi \stackrel{\text{(law)}}{=} \zeta + \varepsilon \phi_\varepsilon, \text{ with } \mathrm{E}\left[\sup_{0<\varepsilon<1} |\phi_\varepsilon|^p\right] < \infty, \forall p.$$

Lemma 5.2 *Consider the Gaussian mean-zero random vector*

$$G = (G_1, G_2, G_3^\varepsilon, G_4^\varepsilon) = (X_u, X_v, X_{u+\varepsilon} - X_u, X_{v+\varepsilon} - X_v), \tag{5.11}$$

and denote

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} := \begin{pmatrix} u^{2H} & K_H(u,v) \\ K_H(v,u) & v^{2H} \end{pmatrix}^{-1} = K^{-1}_{(G_1,G_2)}, \tag{5.12}$$

$$Q_1 := -\frac{1}{2}\left(\lambda_{11} G_1 + \lambda_{12} G_2\right), \ Q_2 := -\frac{1}{2}\left(\lambda_{12} G_1 + \lambda_{22} G_2\right). \tag{5.13}$$

a) For $\frac{1}{4} \le H < \frac{1}{3}$, as $\varepsilon \downarrow 0$,

$$\mathrm{E}\left(\frac{(G_3^\varepsilon)^3 (G_4^\varepsilon)^3}{\varepsilon^2} \mid G_1, G_2\right) \stackrel{\textit{(law)}}{=} \varepsilon^{8H-2}\left(9Q_1Q_2 - \frac{9}{4}\lambda_{12} + o(1)\right). \tag{5.14}$$

a1) For $\frac{1}{4} \le H < \frac{1}{3}$, as $\varepsilon \downarrow 0$,

$$\mathrm{E}\left(\frac{(G_3^\varepsilon)^3}{\varepsilon} \mid G_1, G_2\right) \stackrel{\textit{(law)}}{=} \varepsilon^{4H-1}\left(3Q_1 + o(1)\right). \tag{5.15}$$

a2) For $\frac{1}{4} \le H < \frac{1}{3}$, as $\varepsilon \downarrow 0$,

$$\mathrm{E}\left(\frac{(G_4^\varepsilon)^3}{\varepsilon} \mid G_1, G_2\right) \stackrel{\textit{(law)}}{=} \varepsilon^{4H-1}\left(3Q_2 + o(1)\right). \tag{5.16}$$

b) Denote $G_4^\delta = X_{v+\delta} - X_v$ and $G_1, G_2, G_3^\varepsilon$ as previously. Then, for $H = \frac{1}{4}$, as $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$\mathrm{E}\left(\frac{(G_3^\varepsilon)^3 (G_4^\delta)^3}{\varepsilon\delta} \mid G_1, G_2\right) \stackrel{\textit{(law)}}{=} 9Q_1Q_2 - \frac{9}{4}\lambda_{12} + o(1). \tag{5.17}$$

c) Equivalents in (5.14), (5.15), (5.16) and (5.17) are uniform on $\{1 < u, 1 < v-u\}$.

d) For $\kappa > 0$,

$$(G_1(\kappa u), G_2(\kappa v), G_3^{\kappa\varepsilon}(\kappa u), G_4^{\kappa\varepsilon}(\kappa v)) \overset{(law)}{=} \kappa^H (G_1(u), G_2(v), G_3^\varepsilon(u), G_4^\varepsilon(v)) \tag{5.18}$$

and

$$\left(G_1(\kappa u), G_2(\kappa v), Q_1(\kappa u, \kappa v)Q_2(\kappa u, \kappa v) - \frac{1}{4}\lambda_{12}(\kappa u, \kappa v)\right)$$

$$\overset{(law)}{=} \left(\kappa^H G_1(u), \kappa^H G_2(v), \kappa^{-2H}(Q_1(u,v)Q_2(u,v) - \frac{1}{4}\lambda_{12}(u,v))\right). \tag{5.19}$$

Remark 5.3 *The computation of limits when* ε *or* (ε, δ) *go to zero requires asymptotic equivalent expressions of the conditional expectations (parts a) and b) of Lemma 5.2). However, since we have to integrate on the domain* $\{0 < u < v < 1\}$, *we need to check uniform that those are uniform on* u, v, *see part c) of Lemma 5.2).*

We postpone the proof of Lemma 5.2 and we finish the proof of (3.7).
Let $0 < \rho < 1$. The second moment of $I_\varepsilon(g)$ can be written as

$$\mathrm{E}\left[\frac{1}{2}I_\varepsilon^2(g)\right] = \iint_{0<u<\varepsilon^{1-\rho}, u<v<1} \mathrm{E}\left\{g(G_1)g(G_2)\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\} dudv$$

$$+ \iint_{0<v-u<\varepsilon^{1-\rho}, 0<u,v<1} \mathrm{E}\left\{g(G_1)g(G_2)\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\} dudv$$

$$+ \iint_{\varepsilon^{1-\rho}<u<1, \varepsilon^{1-\rho}<v-u<1, v<1} \mathrm{E}\left\{g(G_1)g(G_2)\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\} dudv$$

Using assumptions on g we can bound the first term by

$$\mathrm{cst.} \iint_{0<u<\varepsilon^{1-\rho}, u<v<1} \frac{\varepsilon^{3H}\varepsilon^{3H}}{\varepsilon^2} dudv = \mathrm{cst.}\, \varepsilon^{6H-2+1-\rho}.$$

In the sequel of this step, we will use in a significant way point d) of Lemma 5.2.

Choosing $0 < \rho < 6H - 1$, we can see that the first term converges to 0, as $\varepsilon \downarrow 0$. A similar reasoning implies that the second term converges also to 0. Let us denote $\varepsilon^{1-\rho} = \kappa$ and $\varepsilon^{\rho} = \tilde{\varepsilon}$ (hence $\varepsilon = \kappa\tilde{\varepsilon}$). In the third term we operate the change of variables $u = \kappa\tilde{u}$ and $v = \kappa\tilde{v}$. Hence, as $\varepsilon \downarrow 0$,

$$
\begin{aligned}
&\iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{ g(G_1(u))g(G_2(v)) \frac{(G_3^{\varepsilon}(u))^3 (G_4^{\varepsilon}(v))^3}{\varepsilon^2} \right\} dudv \\
&= \iint_{1<\tilde{u}<\frac{1}{\kappa},1<\tilde{v}-\tilde{u}<\frac{1}{\kappa},\tilde{v}<\frac{1}{\kappa}} \mathrm{E}\left\{ g(G_1(\kappa\tilde{u}))g(G_2(\kappa\tilde{v})) \frac{(G_3^{\kappa\tilde{\varepsilon}}(\kappa\tilde{u}))^3 (G_4^{\kappa\tilde{\varepsilon}}(\kappa\tilde{v}))^3}{\kappa^2\tilde{\varepsilon}^2} \right\} \kappa^2 d\tilde{u}d\tilde{v} \\
&\overset{(5.18)}{=} \iint_{1<\tilde{u}<\frac{1}{\kappa},1<\tilde{v}-\tilde{u}<\frac{1}{\kappa},\tilde{v}<\frac{1}{\kappa}} \mathrm{E}\left\{ g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v})) \frac{\kappa^{6H}(G_3^{\tilde{\varepsilon}}(u))^3 (G_4^{\tilde{\varepsilon}}(v))^3}{\tilde{\varepsilon}^2} \right\} d\tilde{u}d\tilde{v} \\
&\quad = \iint_{1<\tilde{u}<\frac{1}{\kappa},1<\tilde{v}-\tilde{u}<\frac{1}{\kappa},\tilde{v}<\frac{1}{\kappa}} \mathrm{E}\left\{ g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v}))\kappa^{6H} \right. \\
&\qquad \left. \times \mathrm{E}\left(\frac{(G_3^{\tilde{\varepsilon}}(u))^3 (G_4^{\tilde{\varepsilon}}(v))^3}{\tilde{\varepsilon}^2} \mid G_1(\tilde{u}), G_2(\tilde{v}) \right) \right\} d\tilde{u}d\tilde{v} \\
&\quad \overset{(5.14)}{\sim} \iint_{1<\tilde{u}<\frac{1}{\kappa},1<\tilde{v}-\tilde{u}<\frac{1}{\kappa},\tilde{v}<\frac{1}{\kappa}} \mathrm{E}\left\{ g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v}))\kappa^{6H}\tilde{\varepsilon}^{8H-2} \right. \\
&\qquad \left. \times \left(9Q_1(\tilde{u},\tilde{v})Q_2(\tilde{u},\tilde{v}) - \frac{9}{4}\lambda_{12}(\tilde{u},\tilde{v}) \right) \right\} d\tilde{u}d\tilde{v} \\
&\quad = \iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{ g(\kappa^H G_1(\frac{u}{\kappa}))g(\kappa^H G_2(\frac{v}{\kappa}))(\kappa\tilde{\varepsilon})^{6H}\tilde{\varepsilon}^{2H-2} \right. \\
&\qquad \left. \times \left(9Q_1(\frac{u}{\kappa},\frac{v}{\kappa})Q_2(\frac{u}{\kappa},\frac{v}{\kappa}) - \frac{9}{4}\lambda_{12}(\frac{u}{\kappa},\frac{v}{\kappa}) \right) \right\} \frac{dudv}{\kappa^2} \\
&\quad \overset{(5.19)}{=} \iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{ g(G_1(u))g(G_2(v))(\kappa\tilde{\varepsilon})^{6H}\tilde{\varepsilon}^{2H-2} \right. \\
&\qquad \left. \times\, \kappa^{2H-2} \left(9Q_1(u,v)Q_2(u,v) - \frac{9}{4}\lambda_{12}(u,v) + o(1) \right) \right\} dudv \\
&\quad = \varepsilon^{8H-2} \iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{ g(G_1(u))g(G_2(v)) \right. \\
&\qquad \left. \times \left(9Q_1(u,v)Q_2(u,v) - \frac{9}{4}\lambda_{12}(u,v) + o(1) \right) \right\} dudv
\end{aligned}
$$

where we have also used point *c)* of Lemma 5.2 to replace the conditional expectation by the uniform equivalent asymptotics in (5.14) on $\{1 < \tilde{u}, 1 < \tilde{v} - \tilde{u}\}$. Therefore, as $\varepsilon \downarrow 0$,

$$\mathrm{E}\left[I_\varepsilon(g)^2\right] \sim \varepsilon^{8H-2}\mathrm{E}\{\frac{9}{2}\iint du\,dv\, g(G_1)g(G_2)((\lambda_{11}G_1+\lambda_{12}G_2)(\lambda_{12}G_1+\lambda_{22}G_2)-\lambda_{12})\}.$$

From the expression above (3.7) can follow. Moreover

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}\left[I_\varepsilon(g)^2\right] = 0, \text{ if } \frac{1}{4} < H < \frac{1}{3}. \tag{5.20}$$

□ *III) Proof of the absolute convergence of the integrals in (3.6) and (3.7) as the third part of c) in Proposition 3.4*

The absolute convergence of the integral on the right hand side of (3.6) is already explained by the reasoning operated in *I)*. We need however to justify the absolute convergence of the integral on the right hand side of (3.7), which means

$$J := \iint_{0<u<v<1} du\,dv\,\mathrm{E}|g(B_u^H)g(B_v^H)$$
$$\times(\lambda_{11}\lambda_{12}(B_u^H)^2 + (\lambda_{11}\lambda_{22}+\lambda_{12}^2)B_u^H B_v^H + \lambda_{12}\lambda_{22}(B_v^H)^2 - \lambda_{12})| < \infty.$$

We can write $J = J_1 + J_2 + J_3 + J_4$, where

$$J_i = \iint_{0<u<v<1} \mathrm{E}(|\mathcal{E}_i(u,v)|)du\,dv,\ i = 1,2,3,$$

$$J_4 = \iint_{0<u<v<1} \mathrm{E}(|g(B_u^H)g(B_v^H)\lambda_{12}|)du\,dv.$$

where

$$\mathcal{E}_1(u,v) = g(B_u^H)g(B_v^H)\left(\lambda_{11}\lambda_{12}+\lambda_{11}\lambda_{22}+\lambda_{12}^2+\lambda_{12}\lambda_{22}\right)(B_u^H)^2, \tag{5.21}$$

$$\mathcal{E}_2(u,v) = g(B_u^H)g(B_v^H)(\lambda_{11}\lambda_{22}+\lambda_{12}^2)B_u^H(B_v^H - B_u^H) \tag{5.22}$$

$$\mathcal{E}_3(u,v) = g(B_u^H)g(B_v^H)\lambda_{12}\lambda_{22}(B_v^H + B_u^H)(B_v^H - B_u^H). \tag{5.23}$$

We set $v = u(1+\eta)$ so that

$$J_i = \iint_{0<u<1, 0<\eta<\frac{1}{u}-1} \mathrm{E}(|\mathcal{E}_i(u,\eta)|)u\,du\,d\eta,\ i = 1,2,3,$$

$$J_4 = \iint_{0<u<1, 0<\eta<\frac{1}{u}-1} \mathrm{E}(|g(B_u^H)g(B_{u(1+\eta)}^H)\lambda_{12}(u,\eta)|)u\,du\,d\eta.$$

Moreover,

$$K_{1/2}(u, u(1+\eta)) = \sqrt{u}\hat{K}(\eta), \text{ with } \hat{K}(\eta) = \frac{1}{2}(1+\sqrt{1+\eta}-\sqrt{\eta})$$

and

$$\Delta := \sqrt{u \cdot u(1+\eta)} - K_{1/2}^2(u, u(1+\eta)) = u\hat{\Delta}(\eta), \text{ with } \hat{\Delta}(\eta) = \sqrt{1+\eta} - \hat{K}^2(\eta).$$

We remark that

$$\hat{K}(\eta) \sim 1, \text{ as } \eta \downarrow 0 \text{ and } \hat{K}(\eta) \sim \frac{1}{2}, \text{ as } \eta \uparrow \infty,$$

whereas

$$\hat{\Delta}(\eta) \sim \sqrt{\eta}, \text{ as } \eta \downarrow 0 \text{ or as } \eta \uparrow \infty.$$

Using (3.8) we can write

$$\lambda_{11} = \frac{1}{\sqrt{u}}\frac{\sqrt{1+\eta}}{\hat{\Delta}(\eta)},\ \lambda_{22} = \frac{1}{\sqrt{u}}\frac{1}{\hat{\Delta}(\eta)},\ \lambda_{12} = -\frac{1}{\sqrt{u}}\frac{\hat{K}(\eta)}{\hat{\Delta}(\eta)}.$$

We can now prove that each each J_i is a convergent double integral. To illustrate this fact, we prove the convergence of J_2; the computation will be similar for the other J_i. We recall that

$$J_2 = \iint_{0<u<1, 0<\eta<\frac{1}{u}-1} \mathrm{E}(|\lambda_{11}\lambda_{22}+\lambda_{12}^2||g(B_u^H)g(B_{u(1+\eta)}^H)B_u^H(B_{u(1+\eta)}^H - B_u^H)|)u\,du\,d\eta$$

$$= \iint_{0<u<1, 0<\eta<\frac{1}{u}-1} \frac{\sqrt{1+\eta}+\hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \mathrm{E}(|g(B_u^H)g(B_{u(1+\eta)}^H)B_u^H(B_{u(1+\eta)}^H - B_u^H)|)du\,d\eta.$$

By Cauchy-Schwarz inequality and taking in account the assumption on g we can write

$$\mathrm{E}|g(B_u^H)g(B_{u(1+\eta)}^H)B_u^H(B_{u(1+\eta)}^H - B_u^H)| \leq \mathrm{cst}.u^{1/2}\eta^{1/4}.$$

On the other hand

$$\frac{\sqrt{1+\eta}+\hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \sim \frac{2}{\eta}, \text{ as } \eta \downarrow 0 \text{ and } \frac{\sqrt{1+\eta}+\hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \sim \frac{1}{\sqrt{\eta}}, \text{ as } \eta \uparrow \infty.$$

Hence, we need now to study respectively the integrals

$$\iint_{0<u<1, 0<\eta<1} \frac{u^{1/2}}{\eta^{3/4}}\, du\, d\eta < \infty$$

and

$$\iint_{0<u<1, 1<\eta<\frac{1}{u}-1} \frac{u^{1/2}}{\eta^{1/4}}\, du\, d\eta = \int_1^{\infty} \frac{d\eta}{\eta^{1/4}} \int_0^{\frac{1}{\eta+1}} u^{1/2} du = \frac{2}{3} \int_1^{\infty} \frac{d\eta}{\eta^{1/4}(\eta+1)^{3/2}} < \infty.$$

This concludes the proof of point c) of the Proposition 3.4. □

IV) Proof of c') Propositon 3.4

We omit it because, its proof follows the same scheme and it develops the same argument as point c).

V) Proof of b) of Propositon 3.4.

Immediate from (5.8) and (5.20). □

VI) Proof of a) of Propositon 3.4. Suppose for a moment that we know the result when g is bounded. Since the paths of B^H are continuous, we can operate by localization to treat the case when g is only locally bounded.

Suppose that g is locally bounded. Let $\alpha > 0$. We will show that $(I_\varepsilon(g))$ is Cauchy with respect to the convergence in probability, i.e.

$$P\{|I_\varepsilon(g) - I_\delta(g)| \geq \alpha\} \to 0, \qquad \varepsilon, \delta \to 0.$$

Let $K > 0$, $\Omega_K = \{|B_u| \leq K; \forall u \in [0, T+1]\}$. On Ω_K, we have $I_\varepsilon(g) = I_\varepsilon(g_K)$ and $I_\delta(g) = I_\delta(g_K)$ where g_K is a function with compact support, which coincides on g on the compact interval $[-K, K]$.

Therefore,

$$P\{|I_\varepsilon(g) - I_\delta(g)| \geq \alpha, \Omega_K^c\} \leq P(\Omega_K^c).$$

We choose K large enough, so that $P(\Omega_K^c)$ is uniformly small with respect to ε and δ. Then

$$P\{|I_\varepsilon(g) - I_\delta(g)| \geq \alpha, \Omega_K\} = P\{|I_\varepsilon(g_K) - I_\delta(g_K)| \geq \alpha, \Omega_K\}$$
$$\leq P\{|I_\varepsilon(g_K) - I_\delta(g_K)| \geq \alpha\}$$

Since g_K has compact support, $I_\varepsilon(g_K)$ converges in probability. Thus, it remains to prove that the sequence $(I_\varepsilon(g))$ converges in probability, when g is bounded. For this purpose, we will even show that, in that case, it is even Cauchy in L^2.

We will have finished if we prove that the sequence $(I_\varepsilon(g))$ is Cauchy in L^2 when g is bounded.

We have only to verify the point *a)* for $H = \frac{1}{4}$, since for $\frac{1}{4} < H < \frac{1}{3}$, we have already proved that the limit equals 0. We will prove the Cauchy criterium for $\{I_\varepsilon(g)\}_{\varepsilon > 0}$:

$$\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \mathrm{E}\left(|I_\varepsilon(g) - I_\delta(g)|^2\right) = 0.$$

The expectation above equals

$$\mathrm{E}\left[I_\varepsilon(g)^2\right] + \mathrm{E}\left[I_\delta(g)^2\right] - 2\mathrm{E}\left[I_\varepsilon(g) I_\delta(g)\right];$$

moreover the first two terms converge to the same limit given in (3.7) as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$. It remains to show that $\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \mathrm{E}\left[I_\varepsilon(g) I_\delta(g)\right]$ equals to the right hand-side of (3.7), which is the same limit and then the Cauchy criterium will be fulfilled. A simple change of variable gives,

$$I_\varepsilon(g) I_\delta(g) = \iint_{0<u<v<1} g(X_u) g(X_v) \frac{(X_{u+\varepsilon} - X_u)^3}{\varepsilon} \frac{(X_{v+\delta} - X_v)^3}{\delta}\, dudv$$
$$+ \iint_{0<u<v<1} g(X_u) g(X_v) \frac{(X_{u+\delta} - X_u)^3}{\delta} \frac{(X_{v+\varepsilon} - X_v)^3}{\varepsilon}\, dudv.$$

Taking the expectation of the expression above gives

$$\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \mathrm{E}\left[I_\varepsilon(g) I_\delta(g)\right] = 2 \lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \mathrm{E}\left\{\iint_{0<u<v<1} g(G_1) g(G_2) \mathrm{E}\left(\frac{(G_3^\varepsilon)^3 (G_4^\delta)}{\varepsilon\delta} \mid G_1, G_2\right) dudv\right\}$$

so that the result will be a consequence of (5.17). □

VII) Proof of a') of Propositon 3.4.

It follows the same arguments as point a).

VIII) Proof of point d) of the Proposition 3.4.

It concerns the evaluation of the covariance of the third order integral $\int_0^1 g(B_u^H)d^{-3}B_u^H$ with a r.v. of the form $\int_0^1 h(B_v^H)dv$. We follow for this a similar scheme as for the evaluation of the second moment of the third order integral, see point c).

Since $\int_0^1 g(B_u^H)d^{-3}B_u^H$ is the limit in $L^2(\Omega)$ of $I_\varepsilon(g)$, then

$$\mathrm{E}\left(\int_0^1 g(B_u^H)d^{-3}B_u^H \int_0^1 h(B_v^H)dv\right) \tag{5.24}$$

is the limit of

$$\frac{J_\varepsilon^1 + J_\varepsilon^2}{\varepsilon} \tag{5.25}$$

where

$$\begin{aligned} J_\varepsilon^1 &= \int_0^1 dv \int_0^v du \mathrm{E}\left(g(B_u^H)(B_{u+\varepsilon}^H - B_u^H)^3 h(B_v^H)\right) \\ J_\varepsilon^2 &= \int_0^1 dv \int_0^v du \mathrm{E}\left(g(B_v^H)(B_{v+\varepsilon}^H - B_v^H)^3 h(B_u^H)\right). \end{aligned}$$

Using the same notations as for the evaluation of the second moment at point c), we can write

$$\begin{aligned} J_\varepsilon^1 &= \int_0^1 \int_0^v du \mathrm{E}\left(g(G_1)h(G_2)(G_3^\varepsilon)^3\right) \\ J_\varepsilon^2 &= \int_0^1 \int_0^v du \mathrm{E}\left(g(G_2)h(G_1)(G_4^\varepsilon)^3\right). \end{aligned}$$

J_ε^1 equals

$$\int_0^1 \int_0^v du \mathrm{E}\left(g(G_1)h(G_2)\mathrm{E}((G_3^\varepsilon)^3 | G_1, G_2)\right)$$

Point a1) of the lemma says that the expectation in the above integral gives

$$3\varepsilon^{4H-1}\left\{\mathrm{E}\left(g(G_1)h(G_2)\left(-\frac{\lambda_{11}}{2}G_1 - \frac{\lambda_{12}}{2}G_2\right)\right) + o(1)\right\}. \tag{5.26}$$

Point c) of lemma 5.2 implies that the estimates in (5.26) are uniform in u and v. Therefore Lebesgue dominated convergence theorem says that

$$\lim_{\varepsilon\to 0}\frac{J_\varepsilon^1}{\varepsilon} = -\frac{3}{2}\int_0^1 dv\int_0^v du\mathrm{E}\left(g(B_u^H)h(B_v^H)(\lambda_{11}B_u^H+\lambda_{12}B_v^H)\right)$$

Proceeding similarly for J_ε^2, using this time point a2) of the lemma, we obtain

$$\lim_{\varepsilon\to 0}\frac{J_\varepsilon^2}{\varepsilon} = -\frac{3}{2}\int_0^T dv\int_0^v du\mathrm{E}\left(g(B_v^H)h(B_u^H)(\lambda_{12}B_u^H+\lambda_{22}B_v^H)\right).$$

This equals

$$-\frac{3}{2}\int_0^1 dv\int_v^1 du\mathrm{E}\left(g(B_u^H)h(B_v^H)(\lambda_{12}B_v^H+\lambda_{11}B_u^H)\right).$$

Finally (5.25) gives

$$-\frac{3}{2}\int_0^1 dv\int_0^1 du\mathrm{E}\left(g(B_u^H)h(B_v^H)(\lambda_{11}B_u^H+\lambda_{12}B_v^H)\right),$$

which is the desired quantity. This proves point d) of the Proposition. □

Since it is not easy to recognize the positivity from the right-hand side of the second moment of the third order integrals, we try to obtain expressions of the second moment of the third order integrals, see points c), c'). We also try to better understand the covariance type expressions given at point d). This will be possible when g is smooth.

Step IX) Proof of (3.11) as point e) of Proposition 3.4

To simplify notations, we write K for $K_H(u,v)$ and Δ for $\sqrt{uv}-K^2$. Hence

$$\lambda_{11} = \frac{\sqrt{v}}{\Delta},\ \lambda_{22} = \frac{\sqrt{u}}{\Delta},\ \lambda_{12} = -\frac{K}{\Delta}.$$

Let us introduce the matrix

$$M = \begin{pmatrix} u^H & 0 \\ \frac{K}{u^H} & \frac{\sqrt{\Delta}}{u^H}\end{pmatrix},\ \text{ with } M^{-1} = \begin{pmatrix} u^H & 0 \\ -u^{-H}\frac{K}{\sqrt{\Delta}} & \frac{u^H}{\sqrt{\Delta}}\end{pmatrix}$$

and observe that, by (5.12), $MM^* = K_{(B_u^H,B_v^H)}$. Furthermore, if N_1, N_2 are two independent standard normal random variables, then

$$\begin{pmatrix} B_u^H \\ B_v^H\end{pmatrix} = M\begin{pmatrix} N_1 \\ N_2\end{pmatrix};$$

after some algebraic computations, we obtain

$$\lambda_{11}\lambda_{12}(B_u^H)^2 + (\lambda_{11}\lambda_{22} + \lambda_{12}^2)B_u^H B_v^H + \lambda_{12}\lambda_{22}(B_v^H)^2 - \lambda_{12}$$

$$= \left((M^{-1})^* \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}\right)_1 \cdot \left((M^{-1})^* \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}\right)_2 - \left((M^{-1})^* M^{-1}\right)_{12}$$

$$= \frac{N_1 N_2}{\sqrt{\Delta}} - \frac{K\,N_2^2}{\Delta} + \frac{K}{\Delta}.$$

Therefore, by (3.7),

$$\mathrm{E}\left\{\left(\int_0^1 g(B_u^H)d^{-3}B_u^H\right)^2\right\}$$

$$= \frac{9}{2}\iint_{0<u<v<1} du\,dv \mathrm{E}\left[g(u^H N_1) g\left(\frac{K}{u^H}N_1 + \frac{\sqrt{\Delta}}{u^H}N_2\right)\left(\frac{N_1 N_2}{\sqrt{\Delta}} - \frac{K\,N_2^2}{\Delta} + \frac{K}{\Delta}\right)\right]$$

$$= \frac{9}{2}\iint_{0<u<v<1} du\,dv \mathrm{E}\left[g'(u^H N_1) g'\left(\frac{K}{u^H}N_1 + \frac{\sqrt{\Delta}}{u^H}N_2\right)\right]$$

$$= \frac{9}{4}\,\mathrm{E}\left\{\left(\int_0^T g'(B_u^H)du\right)^2\right\}.$$

The second equality is given by the following identity

$$\begin{aligned} &\mathrm{E}\left[g(aN_1) g\left(\frac{b}{a}N_1 + \frac{c}{a}N_2\right)\left(\frac{1}{c}N_1 N_2 - \frac{b}{c^2}\left(N_2^2 - 1\right)\right)\right] \\ = \;&\mathrm{E}\left[g'(aN_1) g'\left(\frac{b}{a}N_1 + \frac{c}{a}N_2\right)\right]. \end{aligned} \tag{5.27}$$

Indeed, using the assumption on g and integration by parts,

$$2\pi\,\mathrm{E}\left[g(aN_1) g\left(\frac{b}{a}N_1 + \frac{c}{a}N_2\right)\left(\frac{1}{c}N_1 N_2 - \frac{b}{c^2}\left(N_2^2 - 1\right)\right)\right]$$

$$= \frac{1}{c}\int_{\mathbb{R}} y e^{-y^2/2} dy \int_{\mathbb{R}} g(ax) g\left(\frac{b}{a}x + \frac{c}{a}y\right) d(-e^{-x^2/2})$$

$$+ \frac{b}{c^2}\int_{\mathbb{R}} g(ax) e^{-x^2/2} dx \int_{\mathbb{R}} g(\frac{b}{a}x + \frac{c}{a}y) d(y e^{-y^2/2})$$

$$= \frac{a}{c}\int_{\mathbb{R}} y e^{-y^2/2} dy \int_{\mathbb{R}} g'(ax) g\left(\frac{b}{a}x + \frac{c}{a}y\right) e^{-x^2/2} dx$$

$$+\frac{b}{ac}\int_{\mathbb{R}} ye^{-y^2/2}dy \int_{\mathbb{R}} g(ax)g'\left(\frac{b}{a}x+\frac{c}{a}y\right)e^{-x^2/2}dx$$
$$-\frac{b}{ac}\int_{\mathbb{R}} g(ax)e^{-x^2/2}dx \int_{\mathbb{R}} g'\left(\frac{b}{a}x+\frac{c}{a}y\right)ye^{-y^2/2}dy$$
$$=\frac{a}{c}\int_{\mathbb{R}} g'(ax)e^{-x^2/2}dx \int_{\mathbb{R}} g\left(\frac{b}{a}x+\frac{c}{a}y\right)d(-e^{-y^2/2})$$
$$=\int_{\mathbb{R}} g'(ax)e^{-x^2/2}dx \int_{\mathbb{R}} g'\left(\frac{b}{a}x+\frac{c}{a}y\right)e^{-y^2/2}dy = 2\pi\,\mathrm{E}\left[g'(aN_1)g'\left(\frac{b}{a}N_1+\frac{c}{a}N_2\right)\right].$$

This concludes the proof of (3.11).

Step X) Proof of (3.12) as point e') of Proposition 3.4

Point d) of Proposition 3.4 implies that the left member of (3.12) equals

$$-\frac{3}{2}\int_0^1 dv \int_0^1 dug(B_u^H)h(B_V^H)\left(\frac{\sqrt{v}}{\Delta}B_u^H-\frac{K}{\Delta}B_v^H\right) \tag{5.28}$$

where

$$\Delta=\sqrt{uv}-K_H^2(u,v), \quad K=K_H^2(u,v).$$

Making the same conventions as in the proof of point a), we can write

$$B_u^H=u^H N_1, \quad B_v^H=\frac{K}{u^H}N_1+\frac{\sqrt{\Delta}}{u^H}N_2$$

where N_1, N_2 are again independent $N(0,1)$ random variables. Therefore (5.28) gives

$$\begin{aligned}&-\frac{3}{2}\int_0^1 dv\int_0^1 du\mathrm{E}\left\{g(u^H N_1)h\left(\frac{K}{u_H}N_1+\frac{\sqrt{\Delta}}{u^H}N_2\right)\right.\\ &\left.\left[\frac{N^1}{u^H}-\frac{K}{\sqrt{\Delta}u^H}\right]\right\}.\end{aligned} \tag{5.29}$$

Similarly to identity (5.27), we can establish the following for $a,b,c\in\mathbb{R}$, $a>0$:

$$\begin{aligned}&\mathrm{E}\left(g(aN_1)h\left(\frac{b}{a}N_1+\frac{c}{a}N_2\right)\left(\frac{N_1}{a}-\frac{b}{ac}N_2\right)\right)\\ =\ &\mathrm{E}\left(g'(aN_1)h\left(\frac{b}{a}N_1+\frac{c}{a}N_2\right)\right)\end{aligned} \tag{5.30}$$

The proof follows easily again, through integration by parts.

We apply (5.30) with

$$a = u^H, \quad b = K, \quad c = \sqrt{\Delta}$$

Hence, (5.29) gives

$$-\frac{3}{2}\int_0^1 dv \int_0^1 du \mathrm{E}\left\{g'(u^H N_1) h\left(\frac{K}{u_H}N_1 + \frac{\sqrt{\Delta}}{u^H}N_2\right)\right)$$

and finally the desired quantity

$$-\frac{3}{2}\int_0^1 dv \int_0^1 du \mathrm{E}\left\{g'(B_u^H)h(B_v^H)\right\}$$

□

XI) Proof of the existence of a continuous version of $\int_0^\cdot g(B_u^H)d^{-3}B_u^H$ as point f) of Proposition 3.4.

We make the reasoning for the case of the forward third order integral, the other case being analogous.

It is enough to show the existence of a continuous version for $t \in [0, T]$, for any $T > 0$.

Suppose for a moment that for every g bounded we can show the existence of a (Hölder) continuous version for $(\int_0^t g(B^H)d^- B^H)_{t\in[0,T]}$. We denote it by $(\tilde{I}(g)_t)$. Then, we can define the associated version for a general $g \in L^\infty_{\mathrm{loc}}$, by

$$\tilde{I}(g)(\omega) = \tilde{I}(g^M)(\omega),$$

where $g^M = g1_{[-M,M]}$, if $\omega \in \{\sup_{t\in[0,T]} |B_t^H| \leq M\}$.

Therefore, it remains to prove that the forward third order integral has a Hölder continuous version, with parameter strictly less than $\frac{1}{4}$, when g is bounded and continuous.

We prove that, for $H = \frac{1}{4}$, the L^2- valued function

$$t \mapsto I(g)(t) := \int_0^t g(B_u^H)d^{-3}B_u^H$$

has a Hölder continuous version on $[0,T]$. We need to control, for $s<t$, s,t in compact intervals,

$$\mathrm{E}\left[(I(g)(t)-I(g)(s))^2\right]=\mathrm{E}\left[\left(\int_s^t g(B_u^H)d^{-3}[B^H]_u\right)^2\right]$$

$$\leq \iint_{s\leq u<v\leq t} du\,dv \mathrm{E}[|g(B_u^H)g(B_v^H)||\mathcal{E}_1(u,v)+\mathcal{E}_2(u,v)+\mathcal{E}_3(u,v)-\lambda_{12}|],$$

where $\mathcal{E}_i(u,v)$, $i=1,2,3$, are given by (5.21) and (5.22). Let us denote

$$\mathcal{E}_1(u,v)=\tilde{\mathcal{E}}_1(u,v)(B_u^H)^2,\ \mathcal{E}_2(u,v)=\tilde{\mathcal{E}}_2(u,v)B_u^H(B_v^H-B_u^H),$$

$$\mathcal{E}_3(u,v)=\tilde{\mathcal{E}}_3(u,v)(B_v^H+B_u^H)(B_v^H-B_u^H).$$

We denote again $\eta=v-u$. Therefore

$$\tilde{\mathcal{E}}_1(u,u+\eta)=\lambda_{11}\lambda_{12}+\lambda_{11}\lambda_{22}+\lambda_{12}^2+\lambda_{12}\lambda_{22}=\frac{1}{2\Delta^2}\eta\frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{u}}$$

$$=\frac{1}{2\Delta^2}\eta\frac{\sqrt{u/\eta}}{\sqrt{1+u/\eta}+\sqrt{u/\eta}},$$

$$\tilde{\mathcal{E}}_2(u,u+\eta)=\lambda_{11}\lambda_{22}+\lambda_{12}^2=\frac{1}{2\Delta^2}\left(u+\eta+3\sqrt{u}\sqrt{u+\eta}-\sqrt{u}\sqrt{\eta}-\sqrt{\eta}\sqrt{u+\eta}\right)$$

$$\tilde{\mathcal{E}}_3(u,u+\eta)=\lambda_{12}\lambda_{22}=-\frac{1}{2\Delta^2}u\left(1+\frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{\eta}}\right)=-\frac{1}{2\Delta^2}u\left(1+\frac{\sqrt{u/\eta}}{1+\sqrt{1+u/\eta}}\right)$$

and

$$-\lambda_{12}=\frac{1}{2\Delta}\sqrt{u}\left(1+\frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{\eta}}\right)=\frac{1}{2\Delta}\sqrt{u}\left(1+\frac{\sqrt{u/\eta}}{1+\sqrt{1+u/\eta}}\right),$$

where

$$\Delta=\sqrt{u(u+\eta)}-K_H^2(u,u+\eta)=$$
$$\frac{1}{2}\sqrt{u}\sqrt{\eta}\left(1+\frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{\eta}}+\frac{\sqrt{\eta}}{\sqrt{u+\eta}+\sqrt{u}}\right)\geq\frac{1}{2}\sqrt{u}\sqrt{\eta}$$

The functions $\psi_1(x)=\frac{\sqrt{x}}{\sqrt{x}+\sqrt{1+x}}$ and respectively $\psi_2(x)=\frac{\sqrt{x}}{1+\sqrt{1+x}}$ are positive increasing on $[0,+\infty[$ with limit $\frac{1}{2}$, respectively 1 as $x\uparrow\infty$. Moreover, we see that $\sqrt{u+\eta}\leq\sqrt{u}+\sqrt{\eta}$. Therefore

$$0\leq\tilde{\mathcal{E}}_1(u,u+\eta)\leq\frac{1}{u},\ |\tilde{\mathcal{E}}_2(u,u+\eta)|\leq\frac{8}{\eta}+\frac{4}{u}+\frac{10}{\sqrt{u}\sqrt{\eta}}$$

$$|\tilde{\mathcal{E}}_2(u,u+\eta)| \leq \frac{4}{\eta},\ 0 \leq -\lambda_{12} \leq \frac{2}{\sqrt{\eta}}.$$

Hence

$$\iint_{s\leq u<v\leq t} \mathrm{E}\left[|g(B_u^H)g(B_v^H)||\tilde{\mathcal{E}}_1(u,v)|(B_u^H)^2\right] du\, dv$$

$$\leq \text{cst.} \iint_{s\leq u\leq t, 0<\eta\leq t-s} \frac{du d\eta}{\sqrt{u}} = \text{cst.}(t-s)^{3/2},$$

$$\iint_{s\leq u<v\leq t} \mathrm{E}\left[|g(B_u^H)g(B_v^H)||\tilde{\mathcal{E}}_2(u,v)|\,|B_u^H(B_v^H - B_u^H)|\right] du\, dv$$

$$\leq \text{cst.} \iint_{s\leq u\leq t, 0<\eta\leq t-s} \left(8\frac{u^{1/4}}{\eta^{3/4}} + 4\frac{\eta^{1/4}}{u^{3/4}} + 10\frac{1}{u^{1/4}\eta^{1/4}}\right) du d\eta$$

$$= \text{cst.}(8(t^{\frac{5}{4}} - s^{\frac{5}{4}})(t-s)^{\frac{1}{4}} + 4(t^{\frac{1}{4}} - s^{\frac{1}{4}})(t-s)^{\frac{5}{4}} + 10(t^{\frac{3}{4}} - s^{\frac{3}{4}})(t-s)^{\frac{3}{4}})$$

$$\leq \text{cst.}(t-s)^{\frac{3}{2}-\rho}, \ \text{ with } \ \rho > 0,$$

$$\iint_{s\leq u<v\leq t} \mathrm{E}\left[|g(B_u^H)g(B_v^H)||\mathcal{E}_3(u,v)|\,|(B_v^H + B_u^H)(B_v^H - B_u^H)|\right] du\, dv$$

$$\leq \text{cst.} \iint_{s\leq u\leq t, 0<\eta\leq t-s} \frac{du d\eta}{\sqrt{u}\,\eta^{3/4}} = \text{cst.}(t-s)^{5/4}$$

and

$$\iint_{s\leq u<v\leq t} \mathrm{E}\left[|g(B_u^H)g(B_v^H)||\lambda_{12}|\right] du\, dv \leq \text{cst.}(t-s)^{3/2}.$$

Therefore

$$\mathrm{E}\left[(I(g)(t) - I(g)(s))^2\right] \leq \text{cst.}(t-s)^{1+\frac{1}{2}-\rho}, \ \text{ with } \ \rho > 0.$$

The classical Kolmogorov criterion allows then to conclude. □

This achieves the proof of Proposition 3.4.

We can now go on with the proof Proposition 3.8.

Proof of Proposition 3.8.

We need to prove that the limit when ε goes to zero of

$$\mathrm{E}\left(\int_0^u \frac{(B_{u+\varepsilon}^H - B_u^H)^3}{\varepsilon}\right)^2,$$

is zero, for a fractional Brownian motion with index $H > \frac{1}{6}$. We will prove in fact that the limit, when $\varepsilon \downarrow 0$ of the following integral

$$\mathcal{J}_\varepsilon := 2 \iint_{0<u<v<1} \mathrm{E}\left(\frac{(G_3^\varepsilon)^3 (G_4^\varepsilon)^3}{\varepsilon^2} \right) dudv$$

gives zero.

We can see that

$$\mathrm{E}\left((G_3^\varepsilon)^3 (G_4^\varepsilon)^3 \right) = 6\mathrm{Cov}^3(G_3^\varepsilon, G_4^\varepsilon) + 9\mathrm{Cov}(G_3^\varepsilon, G_4^\varepsilon)\mathrm{Var}(G_3^\varepsilon)\mathrm{Var}(G_4^\varepsilon).$$

Indeed, it is enough to write

$$\mathrm{E}\left((G_3^\varepsilon)^3 (G_4^\varepsilon)^3 \right) = \mathrm{E}\left[(G_3^\varepsilon)^3 \mathrm{E}\left((G_4^\varepsilon)^3 \mid G_3^\varepsilon \right) \right]$$

and to use again linear regression (see the proof of Lemma 3.7, p. 15 [39] for a similar computation).

Therefore, previous integral $\mathcal{J}_\varepsilon$ can be written as

$$\mathcal{J}_\varepsilon = 12 \iint_{0<u<v<1} \frac{(\Theta^\varepsilon(u,v))^3}{\varepsilon^2} dudv + 9 \cdot 2^{4H+1} \varepsilon^{4H-2} \iint_{0<u<v<1} \Theta^\varepsilon(u,v) dudv =: \mathcal{J}_\varepsilon^1 + \mathcal{J}_\varepsilon^2,$$

where $\Theta^\varepsilon(u,v) = \mathrm{Cov}(G_3^\varepsilon, G_4^\varepsilon)$. Clearly,

$$\Theta^\varepsilon(u,v) = \frac{1}{2}\left(|v-u+2\varepsilon|^{2H} + |v-u-2\varepsilon|^{2H} - |v-u|^{2H} \right)$$

A direct computation shows that

$$\int_0^v \Theta^\varepsilon(u,v) du = \frac{1}{4H} \begin{cases} (v+2\varepsilon)^{2H+1} + (v-2\varepsilon)^{2H+1} - 2v^{2H+1}, & \text{if } v \geq 2\varepsilon \\ (v+2\varepsilon)^{2H+1} - (2\varepsilon - v)^{2H+1} - 2v^{2H+1}, & \text{if } 0 \leq v \leq 2\varepsilon \end{cases}$$

and then, when $\varepsilon \downarrow 0$,

$$\iint_{0<u<v<1} \Theta^\varepsilon(u,v) dudv = \int_0^{2\varepsilon} dv \int_0^v \Theta^\varepsilon(u,v) du + \int_{2\varepsilon}^1 dv \int_0^v \Theta^\varepsilon(u,v) du$$

$$\sim \frac{2H+2}{H}\varepsilon^2 + \frac{4^{2H+2} - 2 \cdot 2^{2H+2}}{4H(2H+1)} \varepsilon^{2H+2} \sim \frac{2H+2}{H}\varepsilon^2.$$

Hence, for any $H > 0$, when $\varepsilon \downarrow 0$,

$$\mathcal{J}_\varepsilon^2 \sim 9 \cdot 2^{4H+2} \frac{H+1}{H} \varepsilon^{4H}$$

and $\lim_{\varepsilon\downarrow 0}\mathcal{J}^2_\varepsilon = 0$ for any $H>0$. To compute $\mathcal{J}^1_\varepsilon$ we set $\eta = v-u$. Then

$$\mathcal{J}^1_\varepsilon = \frac{3}{2\varepsilon^2}\int_0^1 \left((\eta+2\varepsilon)^{2H} + |\eta-2\varepsilon|^{2H} - 2\eta^{2H}\right)^3 (1-\eta)d\eta$$
$$= 3\cdot 2^{6H}\varepsilon^{6H-1}\int_0^{1/2\varepsilon} \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 (1-2\varepsilon\theta)d\theta$$
$$=: 3\cdot 2^{6H}\varepsilon^{6H-1}\mathcal{J}^{11}_\varepsilon - 3\cdot 2^{6H+1}\varepsilon^{6H}\mathcal{J}^{12}_\varepsilon$$

Clearly,

$$\lim_{\varepsilon\downarrow 0}\mathcal{J}^{11}_\varepsilon = \int_0^\infty \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 d\theta < \infty,$$

if $H<\frac{5}{6}$. A similar calculation shows that the second term tends to a convergent integral under the same condition on H. This yields, as $\varepsilon\downarrow 0$,

$$\mathcal{J}^2_\varepsilon \sim 3\cdot 2^{6H}\varepsilon^{6H-1}\int_0^\infty \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 d\theta.$$

This gives us the conclusion, since $\frac{1}{6}<H$. □

We will proceed now with the proof of the key technical Lemma.

Proof of Lemma 5.2

Proof of point a) We write the covariance matrix of $(G_1, G_2, G_3^\varepsilon, G_4^\varepsilon)$ by blocks:

$$\Lambda_\varepsilon = \begin{pmatrix} \Lambda_{11} & \Lambda^\varepsilon_{12} \\ \Lambda^\varepsilon_{21} & \Lambda^\varepsilon_{22}\end{pmatrix}.$$

By classical Gaussian analysis, we know that the matrix A_ε and the covariance matrix of the vector Z^ε in (IV.1) can be expressed as:

$$A_\varepsilon = \Lambda^\varepsilon_{21}\Lambda^{-1}_{11} \text{ and } K_{Z^\varepsilon} = \Lambda^\varepsilon_{22} - A_\varepsilon(\Lambda^\varepsilon_{21})^*. \tag{5.31}$$

Here,

$$\Lambda_{11} = \begin{pmatrix} u^{2H} & K_H(u,v) \\ K_H(v,u) & v^{2H}\end{pmatrix}$$

and

$$\Lambda^\varepsilon_{21} = \begin{pmatrix} \alpha_\varepsilon(u)u^{2H} & \gamma_\varepsilon(u,v) \\ \gamma_\varepsilon(v,u) & \alpha_\varepsilon(v)v^{2H}\end{pmatrix},\ \Lambda^\varepsilon_{22} = \begin{pmatrix} \varepsilon^{2H} & \eta_\varepsilon(u,v) \\ \eta_\varepsilon(v,u) & \varepsilon^{2H}\end{pmatrix}, \tag{5.32}$$

where α_ε is given by (5.4) and

$$\gamma_\varepsilon(u,v) = \mathrm{Cov}(G_3^\varepsilon, G_2) = \frac{1}{2}\left((u+\varepsilon)^{2H} - u^{2H} - |v-u-\varepsilon|^{2H} + |v-u|^{2H}\right),$$

$$\eta_\varepsilon(u,v) = \mathrm{Cov}(G_3^\varepsilon, G_4^\varepsilon) = \frac{1}{2}\left(|v-u+\varepsilon|^{2H} + |v-u-\varepsilon|^{2H} - 2|v-u|^{2H}\right).$$

We can see that

$$\gamma_\varepsilon(u,v) = H\left(u^{2H-1} + |v-u|^{2H-1}\right)\varepsilon + o(\varepsilon) \text{ as } \varepsilon \downarrow 0, \tag{5.33}$$

and

$$\eta_\varepsilon(u,v) = H(2H-1)|v-u|^{2H-2}\varepsilon^2 + o(\varepsilon^2), \text{ as } \varepsilon \downarrow 0. \tag{5.34}$$

We first proceed expanding in ε the matrix A_ε.

Step 1: expansion of the matrix A_ε.

We express its components by

$$A_\varepsilon := \begin{pmatrix} a_{11}^\varepsilon & a_{12}^\varepsilon \\ a_{21}^\varepsilon & a_{22}^\varepsilon \end{pmatrix}, \tag{5.35}$$

and recall that

$$\Lambda_{11}^{-1} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}.$$

Using (5.6), (5.31) and (5.33), when $\varepsilon \downarrow 0$, gives

$$\begin{aligned} a_{11}^\varepsilon &= \lambda_{11}\alpha_\varepsilon(u)u^{2H} + \lambda_{12}\gamma_\varepsilon(u,v) \\ &= -\frac{\lambda_{11}}{2}\varepsilon^{2H} + H\left((\lambda_{11}+\lambda_{12})u^{2H-1} + \lambda_{12}|v-u|^{2H-1}\right)\varepsilon + o(\varepsilon). \end{aligned} \tag{5.36}$$

The asymptotics of the other coefficients a_{ij}^ε behaves similarly, since

$$\begin{aligned} a_{12}^\varepsilon &= \lambda_{12}\alpha_\varepsilon(u)u^{2H} + \lambda_{22}\gamma_\varepsilon(u,v), \\ a_{21}^\varepsilon &= \lambda_{12}\alpha_\varepsilon(v)v^{2H} + \lambda_{11}\gamma_\varepsilon(v,u),\ a_{22}^\varepsilon = \lambda_{22}\alpha_\varepsilon(v)v^{2H} + \lambda_{12}\gamma_\varepsilon(v,u). \end{aligned} \tag{5.37}$$

The expansion as $\varepsilon \downarrow 0$ for the matrix A_ε becomes

$$A_\varepsilon = \begin{pmatrix} -\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon) & -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon) \\ \\ -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{21}\varepsilon + o(\varepsilon) & -\frac{\lambda_{22}}{2}\varepsilon^{2H} + k_{22}\varepsilon + o(\varepsilon) \end{pmatrix}, \tag{5.38}$$

where $k_{ij} := k_{ij}(u,v)$ $i,j = 1,2$,

$$\begin{pmatrix} k_{11}(u,v) & k_{12}(u,v) \\ k_{21}(u,v) & k_{22}(u,v) \end{pmatrix} =$$

$$H \begin{pmatrix} (\lambda_{11}+\lambda_{12})u^{2H-1} + \lambda_{12}|v-u|^{2H-1} & (\lambda_{12}+\lambda_{22})u^{2H-1} + \lambda_{22}|v-u|^{2H-1} \\ \\ (\lambda_{12}+\lambda_{11})v^{2H-1} + \lambda_{11}|u-v|^{2H-1} & (\lambda_{22}+\lambda_{12})v^{2H-1} + \lambda_{12}|u-v|^{2H-1} \end{pmatrix}.$$

Step 2: expansion of the matrix K_{Z^ε}.

We claim that the expansion of the matrix K_{Z^ε} when $\varepsilon \downarrow 0$, is:

$$K_{Z^\varepsilon} = \begin{pmatrix} K_{Z^\varepsilon}(1,1) & K_{Z^\varepsilon}(1,2) \\ \\ K_{Z^\varepsilon}(1,2) & K_{Z^\varepsilon}(2,2) \end{pmatrix}, \tag{5.39}$$

with

$$\begin{cases} K_{Z^\varepsilon}(1,1) = \varepsilon^{2H} - \frac{\lambda_{11}}{4}\varepsilon^{4H} + k_{11}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}) \\ \\ K_{Z^\varepsilon}(1,2) = -\frac{\lambda_{12}}{4}\varepsilon^{4H} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}) \\ \\ K_{Z^\varepsilon}(2,2) = \varepsilon^{2H} - \frac{\lambda_{22}}{4}\varepsilon^{4H} + k_{22}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}) \end{cases}. \tag{5.40}$$

We compute K_{Z^ε} explicitely. Using (5.31), (5.32), (5.34) and (5.38) for $\varepsilon \downarrow 0$,

$$K_{Z^\varepsilon}(1,1) = \varepsilon^{2H} - a_{11}^\varepsilon \alpha_\varepsilon(u) u^{2H} - a_{12}^\varepsilon \gamma_\varepsilon(u,v)$$

$$= \varepsilon^{2H} - \varepsilon^{4H}\left(-\frac{\lambda_{11}}{2} + k_{11}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)\left(-\frac{1}{2} + Hu^{2H-1}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)$$

$$-\varepsilon^{1+2H}\left(-\frac{\lambda_{12}}{2}+k_{12}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(H\left(u^{2H-1}+|v-u|^{2H-1}\right)+o(1)\right)$$

$$=\varepsilon^{2H}-\varepsilon^{4H}\left(\frac{\lambda_{11}}{4}-(\frac{\lambda_{11}}{2}Hu^{2H-1}+\frac{k_{11}}{2})\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)$$

$$-\varepsilon^{1+2H}\left(-\frac{\lambda_{12}}{2}H\left(u^{2H-1}+|v-u|^{2H-1}\right)+o(1)\right)$$

$$=\varepsilon^{2H}-\frac{\lambda_{11}}{4}\varepsilon^{4H}+k_{11}\varepsilon^{1+2H}+o(\varepsilon^{1+2H}),$$

and

$$K_{Z^\varepsilon}(2,2)=\varepsilon^{2H}-a^\varepsilon_{22}\alpha_\varepsilon(v)v^{2H}-a^\varepsilon_{21}\gamma_\varepsilon(v,u)$$

$$=\varepsilon^{2H}-\varepsilon^{4H}\left(-\frac{\lambda_{22}}{2}+k_{22}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(-\frac{1}{2}+Hv^{2H-1}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)$$

$$-\varepsilon^{1+2H}\left(-\frac{\lambda_{12}}{2}+k_{12}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(H\left(v^{2H-1}+|u-v|^{2H-1}\right)+o(1)\right)$$

$$=\varepsilon^{2H}-\frac{\lambda_{22}}{4}\varepsilon^{4H}+k_{22}\varepsilon^{1+2H}+o(\varepsilon^{1+2H}),$$

whereas

$$K_{Z^\varepsilon}(1,2)=\eta_\varepsilon(u,v)-a^\varepsilon_{12}\alpha_\varepsilon(v)v^{2H}-a^\varepsilon_{11}\gamma_\varepsilon(v,u)$$

$$=\varepsilon^{2}\left(H(2H-1)|v-u|^{2H-2}+o(1)\right)$$

$$-\varepsilon^{4H}\left(-\frac{\lambda_{12}}{2}+k_{12}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(-\frac{1}{2}+Hv^{2H-1}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)$$

$$-\varepsilon^{1+2H}\left(-\frac{\lambda_{11}}{2}+k_{11}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(H\left(v^{2H-1}+|v-u|^{2H-1}\right)+o(1)\right)$$

$$=-\frac{\lambda_{12}}{4}\varepsilon^{4H}+\frac{k_{12}+k_{21}}{2}\varepsilon^{1+2H}+o(\varepsilon^{1+2H}).$$

Step 4: the law of the vector Z^ε.

Using (5.39) and (5.40) we observe that the Gaussian vector Z^ε can be written as

$$\begin{pmatrix} Z_1^\varepsilon \\ Z_2^\varepsilon \end{pmatrix} \overset{\text{(law)}}{=} \begin{pmatrix} \nu(\varepsilon)N_1 \\ \mu(\varepsilon)N_1+\theta(\varepsilon)N_2 \end{pmatrix}, \tag{5.41}$$

where N_1, N_2 are independent standard normal random variables, independent also of G_1, G_2. Moreover, for $\varepsilon \downarrow 0$,

$$\begin{cases} \nu(\varepsilon) = \varepsilon^H \left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon)\right) \\ \\ \mu(\varepsilon) = \varepsilon^{3H} \left(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\ \\ \theta(\varepsilon) = \varepsilon^H \left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + c_3\varepsilon + o(\varepsilon)\right) \end{cases}, \tag{5.42}$$

where $c_i := c_i(u, v)$, $i = 1, 2, 3$,

$$c_1(u, v) := \begin{cases} \frac{k_{11}}{2}, & \text{if } H > \frac{1}{4} \\ \frac{k_{11}}{2} - \frac{\lambda_{11}^2}{128}, & \text{if } H = \frac{1}{4} \end{cases},$$

$$c_2(u, v) := \begin{cases} \frac{k_{12}+k_{21}}{2}, & \text{if } H > \frac{1}{4} \\ \frac{k_{12}+k_{21}}{2} - \frac{\lambda_{11}\lambda_{12}}{32}, & \text{if } H = \frac{1}{4} \end{cases},$$

and

$$c_3(u, v) := \begin{cases} \frac{k_{22}}{2}, & \text{if } H > \frac{1}{4} \\ \frac{k_{22}}{2} + \frac{\lambda_{12}^2}{32} - \frac{\lambda_{22}^2}{128}, & \text{if } H = \frac{1}{4} \end{cases}.$$

Indeed, when $\varepsilon \downarrow 0$,

$$\nu(\varepsilon) = \sqrt{K_{Z^\varepsilon}(1,1)} = \varepsilon^H \left(1 - \frac{\lambda_{11}}{4}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right)^{1/2}$$

$$= \varepsilon^H \left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + \frac{k_{11}}{2}\varepsilon - \frac{\lambda_{11}^2}{128}\varepsilon^{4H} + o(\varepsilon)\right)$$

$$= \begin{cases} \varepsilon^H \left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + \frac{k_{11}}{2}\varepsilon + o(\varepsilon)\right), & \text{if } H > \frac{1}{4} \\ \\ \varepsilon^H \left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + (\frac{k_{11}}{2} - \frac{\lambda_{11}^2}{128})\varepsilon + o(\varepsilon)\right), & \text{if } H = \frac{1}{4} \end{cases},$$

$$\mu(\varepsilon) = \frac{K_{Z^\varepsilon}(1,2)}{\nu(\varepsilon)} = \frac{\varepsilon^{4H}\left(-\frac{\lambda_{12}}{4} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)}{\varepsilon^H\left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon)\right)}$$

$$= \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)\left(1 + \frac{\lambda_{11}}{8}\varepsilon^{2H} - c_1\varepsilon + \frac{\lambda_{11}^2}{64}\varepsilon^{4H} + o(\varepsilon)\right)$$

$$= \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} - \frac{\lambda_{11}\lambda_{12}}{32}\varepsilon^{2H} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)$$

$$= \begin{cases} \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right), & \text{if } H > \frac{1}{4} \\ \\ \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} + (\frac{k_{12}+k_{21}}{2} - \frac{\lambda_{11}\lambda_{12}}{32})\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right), & \text{if } H = \frac{1}{4} \end{cases},$$

and

$$\theta(\varepsilon) = \sqrt{K_{Z^\varepsilon}(2,2) - \mu^2(\varepsilon)}$$

$$= \sqrt{\varepsilon^{2H} - \frac{\lambda_{22}}{4}\varepsilon^{4H} + k_{22}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}) - \varepsilon^{6H}\left(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)^2}$$

$$= \sqrt{\varepsilon^{2H} - \frac{\lambda_{22}}{4}\varepsilon^{4H} + k_{22}\varepsilon^{1+2H} - \frac{\lambda_{12}^2}{16}\varepsilon^{6H} + o(\varepsilon^{1+2H})}$$

$$= \varepsilon^{H}\left(1 - \frac{\lambda_{22}}{4}\varepsilon^{2H} + k_{22}\varepsilon - \frac{\lambda_{12}^2}{16}\varepsilon^{4H} + o(\varepsilon)\right)^{1/2}$$

$$= \varepsilon^{H}\left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + \frac{k_{22}}{2}\varepsilon - (\frac{\lambda_{12}^2}{32} + \frac{\lambda_{22}^2}{128})\varepsilon^{4H} + o(\varepsilon)\right)$$

$$= \begin{cases} \varepsilon^{H}\left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + \frac{k_{22}}{2}\varepsilon + o(\varepsilon)\right), & \text{if } H > \frac{1}{4} \\ \\ \varepsilon^{H}\left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + (\frac{k_{22}}{2} - \frac{\lambda_{12}^2}{32} - \frac{\lambda_{22}^2}{128})\varepsilon + o(\varepsilon)\right), & \text{if } H = \frac{1}{4} \end{cases}$$

Step 5: the law of the vector $(G_3^\varepsilon, G_4^\varepsilon)$.

We claim that, for $\varepsilon \downarrow 0$,

$$\begin{pmatrix} G_3^\varepsilon \\ \\ G_4^\varepsilon \end{pmatrix} \overset{(\text{law})}{=} \begin{pmatrix} N_1\varepsilon^H + Q_1\varepsilon^{2H} - \frac{\lambda_{11}}{8}N_1\varepsilon^{3H} + R_1\varepsilon + o(\varepsilon) \\ \\ N_2\varepsilon^H + Q_2\varepsilon^{2H} - (\frac{\lambda_{12}}{4}N_1 + \frac{\lambda_{22}}{8}N_2)\varepsilon^{3H} + R_2\varepsilon + o(\varepsilon) \end{pmatrix}, \tag{5.43}$$

where

$$R_1 := k_{11}G_1 + k_{12}G_2,\ R_2 := k_{21}G_1 + k_{22}G_2$$

Indeed, using (5.35), (5.38), (5.41) and (5.42), when $\varepsilon \downarrow 0$, we get

$$G_3^\varepsilon = a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2 + Z_1^\varepsilon$$

$$\stackrel{\text{(law)}}{=} \left(-\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right) G_1 + \left(-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)\right) G_2$$

$$+\varepsilon^H \left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon)\right) N_1$$

and

$$G_4^\varepsilon = a_{21}^\varepsilon G_1 + a_{22}^\varepsilon G_2 + Z_2^\varepsilon$$

$$\stackrel{\text{(law)}}{=} \left(-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{21}\varepsilon + o(\varepsilon)\right) G_1 + \left(-\frac{\lambda_{22}}{2}\varepsilon^{2H} + k_{22}\varepsilon + o(\varepsilon)\right) G_2$$

$$+\varepsilon^{3H} \left(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) N_1 + \varepsilon^H \left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + c_3\varepsilon + o(\varepsilon)\right) N_2.$$

Step 6: evaluation of the law of $G_3^\varepsilon G_4^\varepsilon$.

As a consequence of previous step,

$$G_3^\varepsilon G_4^\varepsilon \stackrel{\text{(law)}}{=} \varepsilon^{2H} \left(N_1 + Q_1\varepsilon^H - \frac{\lambda_{11}}{8}N_1\varepsilon^{2H} + R_1\varepsilon^{1-H} + o(\varepsilon^{1-H})\right)$$

$$\times \left(N_2 + Q_2\varepsilon^H - (\frac{\lambda_{12}}{4}N_1 + \frac{\lambda_{22}}{8}N_2)\varepsilon^{2H} + R_2\varepsilon^{1-H} + o(\varepsilon^{1-H})\right)$$

$$\stackrel{\text{(law)}}{=} \varepsilon^{2H}(N_1N_2 + (N_1Q_2 + N_2Q_1)\varepsilon^H + (Q_1Q_2 - \frac{\lambda_{12}}{4}N_1^2 - \frac{\lambda_{11}+\lambda_{22}}{8}N_1N_2)\varepsilon^{2H} + o(\varepsilon^{2H}))$$

$$\stackrel{\text{(law)}}{=} \varepsilon^{2H}\left(N_1N_2 + S_\varepsilon\right),$$

where

$$S_\varepsilon \stackrel{\text{(law)}}{=} \varepsilon^H(N_1Q_2 + N_2Q_1 + (Q_1Q_2 - \frac{\lambda_{12}}{4}N_1^2 - \frac{\lambda_{11}+\lambda_{22}}{8}N_1N_2)\varepsilon^H + o(\varepsilon^H)).$$

Step 7: evaluation of the law of $(G_3^\varepsilon G_4^\varepsilon)^3$.

We observe that, when $\varepsilon \downarrow 0$,

$$S_\varepsilon^2 \stackrel{\text{(law)}}{=} \varepsilon^{2H}((N_1Q_2 + N_2Q_1)^2 + o(1)),$$

and

$$S_\varepsilon^3 \stackrel{\text{(law)}}{=} o(\varepsilon^{3H}).$$

Hence

$$(G_3^\varepsilon G_4^\varepsilon)^3 \stackrel{\text{(law)}}{=} \left\{\varepsilon^{2H}(N_1N_2+S_\varepsilon)\right\}^3 \stackrel{\text{(law)}}{=} \varepsilon^{6H}\left(N_1^3N_2^3+3N_1^2N_2^2S_\varepsilon+3N_1N_2S_\varepsilon^2+S_\varepsilon^3\right)$$

$$\stackrel{\text{(law)}}{=} \varepsilon^{6H}\{N_1^3N_2^3+3N_1^2N_2^2[N_1Q_2+N_2Q_1]\varepsilon^H$$

$$+[9N_1^2N_2^2Q_1Q_2-\frac{3\lambda_{12}}{4}N_1^4N_2^2-3\frac{\lambda_{11}+\lambda_{22}}{8}N_1^3N_2^3+3N_1^3N_2Q_2^2+3N_1N_2^3Q_1^2]\varepsilon^{2H}+o(\varepsilon^{2H})\}.$$

Step 8: computation of the conditional expectation in (5.14).
Consequently, for $\varepsilon \downarrow 0$

$$\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2} \stackrel{\text{(law)}}{=} \varepsilon^{6H-2}\{N_1^3N_2^3+[3N_1^3N_2^2Q_2+3N_1^2N_2^3Q_1]\varepsilon^H \quad (5.44)$$

$$+ \quad [9N_1^2N_2^2Q_1Q_2-\frac{3\lambda_{12}}{4}N_1^4N_2^2-3\frac{\lambda_{11}+\lambda_{22}}{8}N_1^3N_2^3 \quad (5.45)$$

$$+ \quad 3N_1^3N_2Q_2^2+3N_1N_2^3Q_1^2]\varepsilon^{2H}+o(\varepsilon^{2H})\}. \quad (5.46)$$

Since N_1, N_2 are independent standard normal random variables, also independent of G_1, G_2, we obtain the conditional expectation in (5.14). □

Proof of b) of Lemma 5.2. The proof is similar as for *a)*. We will only provide the most significant arguments. Asymptotics for $\varepsilon \downarrow 0$ $\delta \downarrow 0$ of some functions of (ε, δ) in fractional powers are done using a Maple procedure which is given in the Appendix. The equalities involving such a procedure will be indicated by $(\star)$.

Step 1: linear regression
We can write

$$\begin{pmatrix} G_3^\varepsilon \\ G_4^\delta \end{pmatrix} = A_{\varepsilon,\delta}\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + \begin{pmatrix} Z_1^{\varepsilon,\delta} \\ Z_2^{\varepsilon,\delta} \end{pmatrix}, \quad (5.47)$$

with

$$A_{\varepsilon,\delta} = \Lambda_{12}^{\varepsilon,\delta}\Lambda_{11}^{-1} \text{ and } K_{Z^{\varepsilon,\delta}} = \Lambda_{22}^{\varepsilon,\delta} - A_{\varepsilon,\delta}(\Lambda_{12}^{\varepsilon,\delta})^*. \quad (5.48)$$

Here

$$\Lambda_{12}^{\varepsilon,\delta} = \begin{pmatrix} \alpha_\varepsilon(u)\sqrt{u} & \gamma_\varepsilon(u,v) \\ \gamma_\delta(v,u) & \alpha_\delta(v)\sqrt{v} \end{pmatrix}, \ \Lambda_{22}^{\varepsilon,\delta} = \begin{pmatrix} \varepsilon^{\frac{1}{2}} & \eta_{\varepsilon,\delta}(u,v) \\ \eta_{\varepsilon,\delta}(u,v) & \delta^{\frac{1}{2}} \end{pmatrix}, \tag{5.49}$$

with

$$\eta_{\varepsilon,\delta}(u,v) = \mathrm{Cov}(G_3^\varepsilon, G_4^\delta) = \frac{1}{2}\left(|v-u+\delta|^{\frac{1}{2}} + |v-u-\varepsilon|^{\frac{1}{2}} - |v-u|^{\frac{1}{2}} - |v-u+\delta-\varepsilon|^{\frac{1}{2}}\right).$$

Therefore, when $\varepsilon \downarrow 0, \delta \downarrow 0$

$$\eta_{\varepsilon,\delta}(u,v) = -\frac{\varepsilon\delta}{8|v-u|^{\frac{3}{2}}} + o((\varepsilon+\delta)^2). \tag{5.50}$$

Step 2: expansion and computations for the matrix $A_{\varepsilon,\delta}$.

We can write

$$A_{\varepsilon,\delta} := \begin{pmatrix} a_{11}^\varepsilon & a_{12}^\varepsilon \\ a_{21}^\delta & a_{22}^\delta \end{pmatrix},$$

with

$$a_{11}^\varepsilon = \lambda_{11}\alpha_\varepsilon(u)\sqrt{u} + \lambda_{12}\gamma_\varepsilon(u,v), \ a_{12}^\varepsilon = \lambda_{12}\alpha_\varepsilon(u)\sqrt{u} + \lambda_{22}\gamma_\varepsilon(u,v),$$

$$a_{21}^\delta = \lambda_{12}\alpha_\delta(v)\sqrt{v} + \lambda_{11}\gamma_\delta(v,u), \ a_{22}^\delta = \lambda_{22}\alpha_\delta(v)\sqrt{v} + \lambda_{12}\gamma_\delta(v,u).$$

Hence, as in step 1 part *a)*, as $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$A_{\varepsilon,\delta} = \begin{pmatrix} -\frac{\lambda_{11}}{2}\varepsilon^{\frac{1}{2}} + k_{11}\varepsilon + o(\varepsilon) & -\frac{\lambda_{12}}{2}\varepsilon^{\frac{1}{2}} + k_{12}\varepsilon + o(\varepsilon) \\ \\ -\frac{\lambda_{12}}{2}\delta^{\frac{1}{2}} + k_{21}\delta + o(\delta) & -\frac{\lambda_{22}}{2}\delta^{\frac{1}{2}} + k_{22}\delta + o(\delta) \end{pmatrix} \tag{5.51}$$

where,

$$k_{11} = \frac{\lambda_{11}+\lambda_{12}}{4\sqrt{u}} + \frac{\lambda_{12}}{4\sqrt{|v-u|}}, \ k_{12} = \frac{\lambda_{12}+\lambda_{22}}{4\sqrt{u}} + \frac{\lambda_{22}}{4\sqrt{|v-u|}},$$

$$k_{21} = \frac{\lambda_{11}+\lambda_{12}}{4\sqrt{v}} + \frac{\lambda_{11}}{4\sqrt{|v-u|}}, \ k_{22} = \frac{\lambda_{12}+\lambda_{22}}{4\sqrt{v}} + \frac{\lambda_{12}}{4\sqrt{|v-u|}}.$$

Step 3: computations related to matrix $K_{Z^{\varepsilon,\delta}}$.

We write

$$K_{Z^{\varepsilon,\delta}} = \begin{pmatrix} K_{Z^{\varepsilon,\delta}}(1,1) & K_{Z^{\varepsilon,\delta}}(1,2) \\ \\ K_{Z^{\varepsilon,\delta}}(1,2) & K_{Z^{\varepsilon,\delta}}(2,2) \end{pmatrix}, \tag{5.52}$$

Clearly,

$$K_{Z^{\varepsilon,\delta}}(1,1) = \varepsilon^{\frac{1}{2}} - a_{11}^{\varepsilon}\alpha_{\varepsilon}(u)\sqrt{u} - a_{12}^{\varepsilon}\gamma_{\varepsilon}(u,v)$$

and

$$K_{Z^{\varepsilon,\delta}}(2,2) = \delta^{\frac{1}{2}} - a_{22}^{\delta}\alpha_{\delta}(v)\sqrt{v} - a_{21}^{\delta}\gamma_{\delta}(v,u),$$

hence the expansions of those two coefficients are similar as in step 3 part a). The expansion of the remaining element behaves differently. Indeed, for $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$\begin{aligned} K_{Z^{\varepsilon,\delta}}(1,2) &= \eta_{\varepsilon,\delta}(u,v) - a_{12}^{\varepsilon}\alpha_{\delta}(v)\sqrt{v} - a_{11}^{\varepsilon}\gamma_{\delta}(v,u) \\ &= -\frac{\varepsilon\delta}{8|v-u|^{\frac{3}{2}}} + o((\varepsilon+\delta)^2) \\ &-\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}}\left(-\frac{\lambda_{12}}{2} + k_{12}\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}})\right)\left(-\frac{1}{2} + \frac{1}{4\sqrt{v}}\delta^{\frac{1}{2}} + o(\delta^{\frac{1}{2}})\right) \\ &-\varepsilon^{\frac{1}{2}}\delta\left(-\frac{\lambda_{11}}{2} + k_{11}\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}})\right)\left(\frac{1}{4\sqrt{v}} + \frac{1}{4\sqrt{|u-v|}} + o(1)\right) \\ &\overset{(\star)}{=} -\frac{\lambda_{12}}{4}\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}} + \frac{k_{12}}{2}\varepsilon\delta^{\frac{1}{2}} + \frac{k_{21}}{2}\varepsilon^{\frac{1}{2}}\delta + o((\varepsilon+\delta)^2). \end{aligned}$$

We summarize now the obtained results: if $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we have

$$\begin{cases} K_{Z^{\varepsilon,\delta}}(1,1) = \varepsilon^{\frac{1}{2}} - \frac{\lambda_{11}}{4}\varepsilon + k_{11}\varepsilon^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}}) \\ \\ K_{Z^{\varepsilon,\delta}}(1,2) = -\frac{\lambda_{12}}{4}\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}} + \frac{k_{12}}{2}\varepsilon\delta^{\frac{1}{2}} + \frac{k_{21}}{2}\varepsilon^{\frac{1}{2}}\delta + o((\varepsilon+\delta)^2) \\ \\ K_{Z^{\varepsilon,\delta}}(2,2) = \delta^{\frac{1}{2}} - \frac{\lambda_{22}}{4}\delta + k_{22}\delta^{\frac{3}{2}} + o(\delta^{\frac{3}{2}}) \end{cases} . \tag{5.53}$$

Step 4: the law of the vector $(Z_1^{\varepsilon,\delta}, Z_2^{\varepsilon,\delta})$.

At this level we can write

$$\begin{pmatrix} Z_1^{\varepsilon,\delta} \\ Z_2^{\varepsilon,\delta} \end{pmatrix} \overset{\text{(law)}}{=} \begin{pmatrix} \nu(\varepsilon) N_1 \\ \mu(\varepsilon,\delta) N_1 + \theta(\varepsilon,\delta) N_2 \end{pmatrix}, \tag{5.54}$$

where N_1, N_2 are independent standard normal random variable, and also independent of G_1, G_2.

$\nu(\varepsilon)$ is given by the first line of (5.42), when $\varepsilon \downarrow 0$. The other computations, when $\varepsilon \downarrow 0$, $\delta \downarrow 0$, give

$$\mu(\varepsilon,\delta) = \frac{K_{Z^{\varepsilon,\delta}}(1,2)}{\nu(\varepsilon)} = \frac{-\frac{\lambda_{12}}{4}\varepsilon^{\frac12}\delta^{\frac12} + \frac{k_{12}}{2}\varepsilon\delta^{\frac12} + \frac{k_{21}}{2}\varepsilon^{\frac12}\delta + o((\varepsilon+\delta)^2)}{\varepsilon^{\frac14} - \frac{\lambda_{11}}{8}\varepsilon^{\frac34} + o(\varepsilon^{\frac34})}$$

$$= \overset{(\star)}{\cdots} = -\frac{\lambda_{12}}{4}\varepsilon^{\frac14}\delta^{\frac12} + o((\varepsilon^{\frac12} + \delta^{\frac12})^2),$$

whereas

$$\theta(\varepsilon,\delta) = \sqrt{K_{Z^{\varepsilon,\delta}}(2,2) - \mu^2(\varepsilon,\delta)}$$

$$= \sqrt{\delta^{\frac12} - \frac{\lambda_{22}}{4}\delta + k_{22}\delta^{\frac32} + o(\delta^{\frac32}) - \left(-\frac{\lambda_{12}}{4}\varepsilon^{\frac14}\delta^{\frac12} + o((\varepsilon^{\frac12}+\delta^{\frac12})^2)\right)^2}$$

$$= \overset{(\star)}{\cdots} = \delta^{\frac14} - \frac{\lambda_{22}}{8}\delta^{\frac34} + \frac{\lambda_{12}}{8}\varepsilon^{\frac14}\delta^{\frac14} - \frac{\lambda_{12}^2}{128}\varepsilon^{\frac12}\delta^{\frac14}$$

$$+ \left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)\varepsilon^{\frac34}\delta^{\frac14} + \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)\varepsilon^{\frac14}\delta^{\frac34} + o((\varepsilon^{\frac12}+\delta^{\frac12})^2).$$

We have

$$\begin{cases} \nu(\varepsilon) = \varepsilon^{\frac14} - \frac{\lambda_{11}}{8}\varepsilon^{\frac34} + o(\varepsilon^{\frac34}) \\ \mu(\varepsilon,\delta) = -\frac{\lambda_{12}}{4}\varepsilon^{\frac14}\delta^{\frac12} + o((\varepsilon^{\frac12}+\delta^{\frac12})^2) \\ \theta(\varepsilon,\delta) = \delta^{\frac14} + \frac{\lambda_{12}}{8}\varepsilon^{\frac14}\delta^{\frac14} - \frac{\lambda_{22}}{8}\delta^{\frac34} - \frac{\lambda_{12}^2}{128}\varepsilon^{\frac12}\delta^{\frac14} \\ + \left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)\varepsilon^{\frac34}\delta^{\frac14} + \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)\varepsilon^{\frac14}\delta^{\frac34} + o(\varepsilon^{\frac12}+\delta^{\frac12}) \end{cases} \tag{5.55}$$

hence

$$Z_1^{\varepsilon,\delta} \overset{(\mathrm{law})}{=} N_1\varepsilon^{\frac14} - \frac{\lambda_{11}}{8}N_1\varepsilon^{\frac34} + o(\varepsilon^{\frac34})$$

and

$$Z_2^{\varepsilon,\delta} \overset{(\mathrm{law})}{=} N_2\delta^{\frac14} + \frac{\lambda_{12}}{8}N_2\varepsilon^{\frac14}\delta^{\frac14} - \frac{\lambda_{12}^2}{128}N_2\varepsilon^{\frac12}\delta^{\frac14} - \frac{\lambda_{12}}{4}N_1\varepsilon^{\frac14}\delta^{\frac12} - \frac{\lambda_{22}}{8}N_2\delta^{\frac34}$$

$$+\left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)N_2\varepsilon^{\frac34}\delta^{\frac14} + \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)N_2\varepsilon^{\frac14}\delta^{\frac34} + o((\varepsilon^{\frac12}+\delta^{\frac12})^2),$$

Step 6: the law of the vector $(G_3^\varepsilon, G_4^\delta)$.

Using the first line of (5.43), (5.55) and (5.53), when $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we obtain

$$G_3^\varepsilon \overset{(\mathrm{law})}{=} N_1\varepsilon^{\frac14} + Q_1\varepsilon^{\frac12} - \frac{\lambda_{11}}{8}N_1\varepsilon^{\frac34} + o(\varepsilon^{\frac34}) \tag{5.56}$$

$$\begin{aligned} G_4^\delta \quad & \overset{(\mathrm{law})}{=} \underset{\cdots}{(\star)} \overset{(\mathrm{law})}{=} N_2\delta^{\frac14} + Q_2\delta^{\frac12} - \frac{\lambda_{22}}{8}N_2\delta^{\frac34} + R_2\delta \\ + \quad & \frac{\lambda_{12}}{8}N_2\varepsilon^{\frac14}\delta^{\frac14} - \frac{\lambda_{12}^2}{128}N_2\varepsilon^{\frac12}\delta^{\frac14} - \frac{\lambda_{12}}{4}N_1\varepsilon^{\frac14}\delta^{\frac12} \\ + \quad & \left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)N_2\varepsilon^{\frac34}\delta^{\frac14} \\ + \quad & \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)N_2\varepsilon^{\frac14}\delta^{\frac34} + o((\varepsilon^{\frac12}+\delta^{\frac12})^2), \end{aligned} \tag{5.57}$$

with Q_1, Q_2 given by (5.13) and R_2 is as in step 5 part a).

Step 7: computation of the law of $G_3^\varepsilon G_4^\delta$.

From (5.56) and (5.57),when $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we get

$$G_3^\varepsilon G_4^\delta \overset{(\mathrm{law})}{=} \underset{\cdots}{(\star)} \overset{(\mathrm{law})}{=} N_1N_2\varepsilon^{\frac14}\delta^{\frac14} + \left(\frac{\lambda_{12}}{8}N_1N_2 + Q_1N_2\right)\varepsilon^{\frac12}\delta^{\frac14} + N_1Q_2\varepsilon^{\frac14}\delta^{\frac12}$$

$$+\left(-\frac{\lambda_{12}^2}{128}N_1N_2 - \frac{\lambda_{11}}{8}N_1N_2 + \frac{\lambda_{12}}{8}Q_1N_2\right)\varepsilon^{\frac34}\delta^{\frac14} + \left(-\frac{\lambda_{12}}{4}N_1^2 + Q_1Q_2\right)\varepsilon^{\frac12}\delta^{\frac12}$$

$$-\frac{\lambda_{22}}{8}N_1N_2\varepsilon^{\frac14}\delta^{\frac34} + o((\varepsilon^{\frac12}+\delta^{\frac12})^2)$$

Step 8: Computation of the conditional expectation in (5.17).

When $\varepsilon \downarrow 0$, $\delta \downarrow 0$, it follows

$$\frac{(G_3^\varepsilon)^3 (G_4^\delta)^3}{\varepsilon\delta} \overset{\text{(law)}}{=} \overset{(\star)}{\because} \overset{\text{(law)}}{=} N_1^3 N_2^3 \varepsilon^{-\frac{1}{4}} \delta^{-\frac{1}{4}} + 3\left(\frac{\lambda_{12}}{8} N_1^3 N_2^3 + Q_1 N_1^2 N_2^3\right) \delta^{-\frac{1}{4}} + 3 Q_2 N_1^3 N_2^2 \varepsilon^{-\frac{1}{4}}$$

$$+3\left(\left(\frac{\lambda_{12}^2}{128} - \frac{\lambda_{11}}{8}\right) N_1^3 N_2^3 + Q_1^2 N_1 N_2^3 + \frac{3\lambda_{12}}{8} Q_1 N_1^2 N_2^3\right) \varepsilon^{\frac{1}{4}} \delta^{-\frac{1}{4}}$$

$$+3\left(-\frac{\lambda_{22}}{8} N_1^3 N_2^3 + Q_2^2 N_1^3 N_2\right) \varepsilon^{-\frac{1}{4}} \delta^{\frac{1}{4}} + \frac{3\lambda_{12}}{4} Q_2 N_1^3 N_2^2 - \frac{3\lambda_{12}}{4} N_1^4 N_2^2 + 9 Q_1 Q_2 N_1^2 N_2^2 + o(1)$$

Since N_1, N_2 are independent standard normal random variables, independent of G_1, G_2, we deduce finally the conditional expectation in (5.17). □

Proof of a1), a2) of Lemma 5.2.

Using notations (5.9), we recall that

$$\begin{aligned} G_3^\varepsilon &= \left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_1 + Z_1^\varepsilon, \\ G_4^\varepsilon &= \left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_2 + Z_2^\varepsilon \end{aligned}$$

So, the left member of (5.15) gives

$$\phi_\varepsilon^1(G_1, G_2),$$

where

$$\begin{aligned} \phi_\varepsilon^1(G_1, G_2) &= \mathrm{E}\left(\left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_1 + Z_1^\varepsilon\right)^3 \\ &= \left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_1^3 + 3\left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_1 Var Z_1^\varepsilon \\ &= (a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2)^3 + (a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2) K_{Z^\varepsilon}(1,1) \end{aligned}$$

according to the notations in (5.35), (5.36) and (5.39)

We recall from (5.40) that

$$K_{Z^\varepsilon}(1,1) = \varepsilon^{2H} - \frac{\lambda_{11}}{4}\varepsilon^{4H} + o(\varepsilon) \tag{5.58}$$

From (5.38), we recall that

$$a_{11}^\varepsilon = -\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)$$

where

$$k_{11} = H(\lambda_{11} + \lambda_{12})u^{2H-1} - \lambda_{12}H|u-v|^{2H-1}$$

and

$$a_{12}^\varepsilon = -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)$$

with

$$k_{12} = H(\lambda_{12} + \lambda_{22})u^{2H-1} - \lambda_{22}H|u-v|^{2H-1}.$$

Hence, we have

$$\begin{aligned}
& \phi_\varepsilon^1(G_1, G_2) \\
= \ & \left\{\left[-\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right] G_1 + \left[-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)\right] G_2\right\}^3 \\
+ \ & \left\{\left[-\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right] G_1 + \left[-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)\right] G_2\right\} \\
& 3(\varepsilon^{2H} - \frac{\lambda_{11}}{4}\varepsilon^{4H} + o(\varepsilon)) \\
= \ & 3\varepsilon^{4H}\left[-\lambda_{11}\frac{G_1}{2} - \lambda_{12}\frac{G_2}{2}\right] + o(\varepsilon).
\end{aligned}$$

This gives (5.15). Formula (5.16) can be established proceeding in a complete analogous way. □

Proof of c) of Lemma 5.2

We need to show that the asymptotics in (5.14), (5.15), (5.16) and (5.17) are uniform in u and v. We make the job for (5.14), the others behaving similarly. It is enough to analyze the uniformity of the expansions on $\{1 < u, 1 < v-u\}$

of $\alpha_\varepsilon(u)$, $\gamma_\varepsilon(u,v)$ and $\eta_\varepsilon(u,v)$, when $\varepsilon \downarrow 0$, because the other asymptotics are obtained in terms of those ones. When $\varepsilon \downarrow 0$, by (5.4) we have

$$\alpha_\varepsilon(u) = \frac{1}{2u^{2H}}\left((u+\varepsilon)^{2H} - u^{2H} - \varepsilon^{2H}\right)$$

$$= \frac{1}{2}\left(\left(1+\frac{\varepsilon}{u}\right)^{2H} - 1 - \left(\frac{\varepsilon}{u}\right)^{2H}\right) = -\frac{1}{2}\varepsilon^{2H} + H\frac{\varepsilon}{u} + o(\frac{\varepsilon}{u});$$

this provides a uniform expansion on $\{u > 1\}$. Similarly, when $\varepsilon \downarrow 0$, one obtains

$$\gamma_\varepsilon(u,v) = \frac{1}{2}\left((u+\varepsilon)^{2H} - u^{2H} - |v-u-\varepsilon|^{2H} + (v-u)^{2H}\right)$$

$$= \frac{1}{2}\left[u^{2H}\left(\left(1+\frac{\varepsilon}{u}\right)^{2H} - 1\right) - (v-u)^{2H}\left(\left|1-\frac{\varepsilon}{v-u}\right|^{2H} - 1\right)\right]$$

$$= H\left(u^{2H-1} + |v-u|^{2H-1}\right)\varepsilon + o(\varepsilon)$$

uniformly on $\{1 < u, 1 < v-u\}$ and when $\varepsilon \downarrow 0$,

$$\eta_\varepsilon(u,v) = \frac{1}{2}\left((v-u+\varepsilon)^{2H} + |v-u-\varepsilon|^{2H} - 2(v-u)^{2H}\right)$$

$$= \frac{(v-u)^{2H}}{2}\left[\left(1+\frac{\varepsilon}{v-u}\right)^{2H} + \left|1-\frac{\varepsilon}{v-u}\right|^{2H} - 2\right]$$

$$= H(2H-1)|v-u|^{2H-2}\varepsilon^2 + o(\varepsilon^2),$$

uniformly on $\{1 < v-u\}$. □

Proof of d) of Lemma 5.2

We look for the homogeneity degree of all quantities used so far. For a function $f = f(\varepsilon,u,v)$ we shall denote

$$\deg_{\varepsilon,u,v}(f) =: p \Leftrightarrow f(\kappa\varepsilon,\kappa u,\kappa v) = \kappa^p f(\varepsilon,u,v).$$

where we make the convention that

$$\gamma(\varepsilon,u,v) = \gamma_\varepsilon(u,v), K_Z(i,j)(\varepsilon,u,v) = K_{Z^\varepsilon}(i,j)(u,v)$$

We have:

$$\begin{aligned}
&\deg_{\varepsilon,u}(\alpha) = 0, \ \text{ by } (5.4)\ , \\
&\deg_{\varepsilon,u,v}(\lambda_{ij}) = -2H, \ \text{ by } (5.12)\ , \\
&\deg_{\varepsilon,u,v}(\gamma) = 2H, \ \text{ by } (5.33)\ , \\
&\deg_{\varepsilon,u,v}(\eta) = 2H, \ \text{ by } (5.34)\ , \\
&\deg_{\varepsilon,u,v}(a_{ij}) = 0, \ \text{ by } (5.31),\ (5.32) \text{ and } (5.35)\ , \\
&\deg_{\varepsilon,u,v}(K_Z(i,j)) = 2H, \ \text{ by } (5.31),\ (5.32) \text{ and } (5.39)\ , \\
&\deg_{\varepsilon,u,v}(\nu) = \deg_{\varepsilon,u,v}(\mu) = \deg_{\varepsilon,u,v}(\theta) = H, \ \text{ by } (5.41)
\end{aligned}$$

From this, (5.9) and (5.35), recalling that $G_1(u) = B_u^H$, $G_2(v) = B_v^H$, we deduce that

$$\begin{aligned}
G_3^{\kappa\varepsilon}(\kappa u) &= a_{11}^{\kappa\varepsilon}(\kappa u, \kappa v)G_1(\kappa u) + a_{12}^{\kappa\varepsilon}(\kappa u, \kappa v)G_2(\kappa v) + Z_1^{\kappa\varepsilon}(\kappa u, \kappa v) \\
&\overset{\text{(law)}}{=} a_{11}^{\varepsilon}(u,v)\kappa^H G_1(u) + a_{12}^{\varepsilon}(u,v)\kappa^H G_2(v) + \kappa^H Z_1^{\varepsilon}(u,v) \overset{\text{(law)}}{=} \kappa^H G_3^{\varepsilon}(u),
\end{aligned}$$

and in a similar way, $G_4^{\kappa\varepsilon}(\kappa v) = \kappa^H G_4^{\varepsilon}(v)$. Therefore (5.18) is proved. On the other hand, using (3.8) and (5.13), we obtain

$$\begin{aligned}
9Q_1(\kappa u, \kappa v)Q_2(\kappa u, \kappa v) - \frac{9}{4}\lambda_{12}(\kappa u, \kappa v) &= \frac{9}{4}[(\lambda_{11}(\kappa u, \kappa v)G_1(\kappa u) + \lambda_{12}(\kappa u, \kappa v)G_2(\kappa v)) \\
&\times(\lambda_{12}(\kappa u, \kappa v)G_1(\kappa u) + \lambda_{22}(\kappa u, \kappa v)G_2(\kappa v)) - \lambda_{12}(\kappa u, \kappa v)] \\
&\overset{\text{(law)}}{=} \frac{9}{4}[\kappa^{-2H}(\lambda_{11}(u,v)G_1(u) + \lambda_{12}(u,v)G_2(v)) \\
&\times(\lambda_{12}(u,v)G_1(u) + \lambda_{22}(u,v)G_2(v)) - \kappa^{-2H}\lambda_{12}(u,v)]
\end{aligned}$$

and consequently (5.19) is also proved. □

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6 Appendix: Maple Procedure

($\star$) We describe here a simple Maple procedure used to compute the expansions denoted by ($\star$).

Introduce the functions $\alpha1_{\varepsilon}(u)\sqrt{u}$ and $\alpha2_{\delta}(v)\sqrt{v}$ which are obtained from the functions $\alpha_{\varepsilon}(u)$, and $\alpha_{\delta}(v)$ multiplying respectively by $\sqrt{u}$ and $\sqrt{v}$:

```
> alpha1[u] := epsilon ->
  (1/2)*((u+epsilon)^(1/2)-u^(1/2)-epsilon^(1/2));
  alpha2[v] := delta -> (1/2)*((v+delta)^(1/2)-v^(1/2)-delta^(1/2));
```

$$\alpha 1_u := \varepsilon \to \frac{1}{2}\sqrt{u+\varepsilon} - \frac{1}{2}\sqrt{u} - \frac{1}{2}\sqrt{\varepsilon}$$

$$\alpha 2_v := \delta \to \frac{1}{2}\sqrt{v+\delta} - \frac{1}{2}\sqrt{v} - \frac{1}{2}\sqrt{\delta}$$

Introduce the functions $\gamma_{\varepsilon}(u,v)$ and $\gamma_{\delta}(u,v)$:

```
> gamma1[u,v] := epsilon ->
  (1/2)*((u+epsilon)^(1/2)-u^(1/2)-(v-u-epsilon)^(1/2)+(v-u)^(1/2));
  gamma2[u,v] := delta ->
  (1/2)*((v+delta)^(1/2)-v^(1/2)-(v-u-delta)^(1/2)+(v-u)^(1/2));
```

$$\gamma 1_{u,\,v} := \varepsilon \to \frac{1}{2}\sqrt{u+\varepsilon} - \frac{1}{2}\sqrt{u} - \frac{1}{2}\sqrt{v-u-\varepsilon} + \frac{1}{2}\sqrt{v-u}$$

$$\gamma 2_{u,\,v} := \delta \to \frac{1}{2}\sqrt{v+\delta} - \frac{1}{2}\sqrt{v} - \frac{1}{2}\sqrt{v-u-\delta} + \frac{1}{2}\sqrt{v-u}$$

Introduce the function $\eta_{\varepsilon,\delta}(u,v)$:

```
> eta[u,v]:=(epsilon,delta)->(1/2)*((v-u+delta)^(1/2)+(v-u-epsilon)^
  (1/2)-(v-u)^(1/2)-(v-u+delta-epsilon)^(1/2));
```

$$\eta_{u,\,v} := (\varepsilon, \delta) \to \frac{1}{2}\sqrt{v-u+\delta} + \frac{1}{2}\sqrt{v-u-\varepsilon} - \frac{1}{2}\sqrt{v-u} - \frac{1}{2}\sqrt{v-u+\delta-\varepsilon}$$

Compute the elements of the matrix $A_{\varepsilon,\delta}$ as functions of ε and δ:

```
> a[11]:= epsilon
  ->lambda[11]*alpha1[u](epsilon)+lambda[12]*gamma1[u,v](epsilon);
  a[12]:= epsilon
  ->lambda[12]*alpha1[u](epsilon)+lambda[22]*gamma1[u,v](epsilon);
  a[21]:= delta
  ->lambda[12]*alpha2[v](delta)+lambda[11]*gamma2[u,v](delta);
  a[22]:= delta
  ->lambda[22]*alpha2[v](delta)+lambda[12]*gamma2[u,v](delta);
```

$$a_{11} := \varepsilon \to \lambda_{11}\,\alpha 1_u(\varepsilon) + \lambda_{12}\,\gamma 1_{u,\,v}(\varepsilon)$$
$$a_{12} := \varepsilon \to \lambda_{12}\,\alpha 1_u(\varepsilon) + \lambda_{22}\,\gamma 1_{u,\,v}(\varepsilon)$$
$$a_{21} := \delta \to \lambda_{12}\,\alpha 2_v(\delta) + \lambda_{11}\,\gamma 2_{u,\,v}(\delta)$$
$$a_{22} := \delta \to \lambda_{22}\,\alpha 2_v(\delta) + \lambda_{12}\,\gamma 2_{u,\,v}(\delta)$$

Compute the elements of the covariance matrix $K_{Z^{\varepsilon,\delta}}$:

```
> K[12]:=(epsilon,delta)->
  eta[u,v](epsilon,delta)-a[12](epsilon)*alpha2[v](delta)-a[11](epsi
  lon)*gamma2[u,v](delta); K[11]:=epsilon
  ->epsilon^(1/2)-a[11](epsilon)*alpha1[u](epsilon)-a[12](epsilon)*g
  amma1[u,v](epsilon);
  K[22]:=delta ->
  delta^(1/2)-a[22](delta)*alpha2[v](delta)-a[21](delta)*gamma2[u,v]
  (delta);
```

$$K_{12} := (\varepsilon, \delta) \to \eta_{u,v}(\varepsilon, \delta) - a_{12}(\varepsilon)\, \alpha 2_v(\delta) - a_{11}(\varepsilon)\, \gamma 2_{u,v}(\delta)$$

$$K_{11} := \varepsilon \to \sqrt{\varepsilon} - a_{11}(\varepsilon)\, \alpha 1_u(\varepsilon) - a_{12}(\varepsilon)\, \gamma 1_{u,v}(\varepsilon)$$

$$K_{22} := \delta \to \sqrt{\delta} - a_{22}(\delta)\, \alpha 2_v(\delta) - a_{21}(\delta)\, \gamma 2_{u,v}(\delta)$$

Expansion of $K(1,2)$ as a function of $(\varepsilon^{2},\delta^{2})$ up to order 5:

```
> readlib(mtaylor):
  assume(epsilon>0):assume(delta>0):mtaylor(K[12](epsilon^2,delta^2)
  ,[epsilon=0,delta=0],5);
```

$$-\frac{1}{4}\lambda_{12}\, \varepsilon\sim\, \delta\sim + \left(\frac{1}{8}\frac{\lambda_{12}}{\sqrt{v}} + \frac{1}{2}\lambda_{11}\left(\frac{1}{4}\frac{1}{\sqrt{v}} + \frac{1}{4}\frac{1}{\sqrt{v-u}}\right)\right)\delta\sim^2\, \varepsilon\sim$$
$$+ \left(\frac{1}{8}\frac{\lambda_{12}}{\sqrt{u}} + \frac{1}{2}\lambda_{22}\left(\frac{1}{4}\frac{1}{\sqrt{v-u}} + \frac{1}{4}\frac{1}{\sqrt{u}}\right)\right)\delta\sim\, \varepsilon\sim^2 + \left(-\frac{1}{8}\frac{1}{(v-u)^{(3/2)}}\right.$$
$$- \left(\frac{1}{4}\frac{\lambda_{11}}{\sqrt{u}} + \lambda_{12}\left(\frac{1}{4}\frac{1}{\sqrt{v-u}} + \frac{1}{4}\frac{1}{\sqrt{u}}\right)\right)\left(\frac{1}{4}\frac{1}{\sqrt{v}} + \frac{1}{4}\frac{1}{\sqrt{v-u}}\right) - \frac{1}{4}\frac{\frac{1}{4}\frac{\lambda_{12}}{\sqrt{u}} + \lambda_{22}\left(\frac{1}{4}\frac{1}{\sqrt{v-u}} + \frac{1}{4}\frac{1}{\sqrt{u}}\right)}{\sqrt{v}}$$
$$\left.\vphantom{\frac{1}{1}}\right)\delta\sim^2\, \varepsilon\sim^2$$

Compute coefficients $\nu(\varepsilon)$, $\mu(\varepsilon,\delta)$, $\theta(\varepsilon,\delta)$:

```
> nu:=epsilon->sqrt(K[11](epsilon));
  mu:=(epsilon,delta)->(K[12](epsilon,delta))*(nu(epsilon)^(-1));
  theta:=(epsilon,delta)->sqrt(K[22](delta)-mu(epsilon,delta));
```

$$\nu := \varepsilon \to \sqrt{K_{11}(\varepsilon)}$$

$$\mu := (\varepsilon, \delta) \to \frac{K_{12}(\varepsilon, \delta)}{\nu(\varepsilon)}$$

$$\theta := (\varepsilon, \delta) \to \sqrt{K_{22}(\delta) - \mu(\varepsilon, \delta)}$$

Expansion of μ and θ as functions of $(\varepsilon^{4},\delta^{4})$ up to order 5:

```
> assume(epsilon>0):mtaylor(mu(epsilon^4,delta^4),[epsilon=0,delta=0
  ],5);
  assume(epsilon>0):assume(delta>0):mtaylor(theta(epsilon^4,delta^4)
  ,[epsilon=0,delta=0],5);
```

$$-\frac{1}{4}\lambda_{12}\,\tilde\delta^{2}\,\tilde\varepsilon$$

$$\tilde\delta+\frac{1}{8}\tilde\delta\,\lambda_{12}\,\tilde\varepsilon-\frac{1}{128}\tilde\delta\,\lambda_{12}^{2}\,\tilde\varepsilon^{2}-\frac{1}{8}\tilde\delta^{3}\,\lambda_{22}$$
$$+\left(\frac{1}{64}\lambda_{12}\,\lambda_{11}-\frac{1}{16}\frac{\lambda_{12}}{\sqrt{u}}-\frac{1}{4}\lambda_{22}\left(\frac{1}{4}\frac{1}{\sqrt{u}}+\frac{1}{4}\frac{1}{\sqrt{v-u}}\right)+\frac{1}{1024}\lambda_{12}^{3}\right)\tilde\delta\,\tilde\varepsilon^{3}$$
$$+\left(-\frac{1}{4}\lambda_{11}\left(\frac{1}{4}\frac{1}{\sqrt{v}}+\frac{1}{4}\frac{1}{\sqrt{v-u}}\right)-\frac{1}{16}\frac{\lambda_{12}}{\sqrt{v}}+\frac{1}{64}\lambda_{12}\,\lambda_{22}\right)\tilde\delta^{3}\,\tilde\varepsilon$$
$$+\left(-\frac{1}{16}\lambda_{12}\left(\frac{1}{32}\lambda_{12}\,\lambda_{11}-\frac{1}{8}\frac{\lambda_{12}}{\sqrt{u}}-\frac{1}{2}\lambda_{22}\left(\frac{1}{4}\frac{1}{\sqrt{u}}+\frac{1}{4}\frac{1}{\sqrt{v-u}}\right)\right)-\frac{5}{32768}\lambda_{12}^{4}\right)\tilde\delta\,\tilde\varepsilon^{4}$$
$$+\left(-\frac{1}{16}\lambda_{12}\left(-\frac{1}{2}\lambda_{11}\left(\frac{1}{4}\frac{1}{\sqrt{v}}+\frac{1}{4}\frac{1}{\sqrt{v-u}}\right)-\frac{1}{8}\frac{\lambda_{12}}{\sqrt{v}}\right)-\frac{3}{1024}\lambda_{12}^{2}\,\lambda_{22}\right)\tilde\delta^{3}\,\tilde\varepsilon^{2}$$
$$+\left(\frac{1}{8}\frac{\lambda_{22}}{\sqrt{v}}+\frac{1}{2}\lambda_{12}\left(\frac{1}{4}\frac{1}{\sqrt{v}}+\frac{1}{4}\frac{1}{\sqrt{v-u}}\right)-\frac{1}{128}\lambda_{22}^{2}\right)\tilde\delta^{5}$$

Compute Z_{1}^{ε} and $Z_{2}^{\varepsilon,\delta}$:

```
> Z[1]:=epsilon-> nu(epsilon)*N[1];
  Z[2]:=(epsilon,delta)->mu(epsilon,delta)*N[1]+theta(epsilon,delta)
  *N[2];
```

$$Z_1 := \varepsilon \to \nu(\varepsilon)\,N_1$$

$$Z_2 := (\varepsilon,\delta) \to \mu(\varepsilon,\delta)\,N_1+\theta(\varepsilon,\delta)\,N_2$$

Expansion of Z_{2} as a function of $(\varepsilon^{4},\delta^{4})$ up to order 5:

```
>
  assume(epsilon>0):assume(delta>0):mtaylor(Z[2](epsilon^4,delta^4),
  [epsilon=0,delta=0],5);
```

$$\tilde\delta\,N_2+\frac{1}{8}\tilde\delta\,\lambda_{12}\,N_2\,\tilde\varepsilon-\frac{1}{128}\tilde\delta\,\lambda_{12}^{2}\,N_2\,\tilde\varepsilon^{2}-\frac{1}{4}\lambda_{12}\,\tilde\delta^{2}\,N_1\,\tilde\varepsilon-\frac{1}{8}\tilde\delta^{3}\,\lambda_{22}\,N_2$$
$$+\left(\frac{1}{64}\lambda_{12}\,\lambda_{11}-\frac{1}{16}\frac{\lambda_{12}}{\sqrt{u}}-\frac{1}{4}\lambda_{22}\left(\frac{1}{4}\frac{1}{\sqrt{u}}+\frac{1}{4}\frac{1}{\sqrt{v-u}}\right)+\frac{1}{1024}\lambda_{12}^{3}\right)N_2\,\tilde\delta\,\tilde\varepsilon^{3}$$
$$+\left(-\frac{1}{4}\lambda_{11}\left(\frac{1}{4}\frac{1}{\sqrt{v}}+\frac{1}{4}\frac{1}{\sqrt{v-u}}\right)-\frac{1}{16}\frac{\lambda_{12}}{\sqrt{v}}+\frac{1}{64}\lambda_{12}\,\lambda_{22}\right)N_2\,\tilde\delta^{3}\,\tilde\varepsilon$$

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Compute G_{3}^{ε} and $G_{4}^{\varepsilon,\delta}$:

```
> G[3]:=epsilon->a[11](epsilon)*G[1]+a[12](epsilon)*G[2]+Z[1](epsilo
  n);
  G[4]:=(epsilon,delta)->a[21](delta)*G[1]+a[22](delta)*G[2]+Z[2](ep
  silon,delta);
```

$$G_3 := \varepsilon \rightarrow a_{11}(\varepsilon)\, G_1 + a_{12}(\varepsilon)\, G_2 + Z_1(\varepsilon)$$

$$G_4 := (\varepsilon, \delta) \rightarrow a_{21}(\delta)\, G_1 + a_{22}(\delta)\, G_2 + Z_2(\varepsilon, \delta)$$

Expansion of G_{4} as a function of $(\varepsilon^{4},\delta^{4})$ up to order 5:

```
> assume(epsilon>0):assume(delta>0):mtaylor(G[4](epsilon^4,delta^4),
  [epsilon=0,delta=0],5);
```

$$\tilde{\delta}\, N_2 + \frac{1}{8}\tilde{\delta}\,\lambda_{12} N_2\,\tilde{\varepsilon} + \left(-\frac{1}{2}\lambda_{22} G_2 - \frac{1}{2}\lambda_{12} G_1\right)\tilde{\delta}^2 - \frac{1}{128}\tilde{\delta}\,\lambda_{12}^{\ 2} N_2\,\tilde{\varepsilon}^2 - \frac{1}{4}\lambda_{12}\,\tilde{\delta}^2 N_1\,\tilde{\varepsilon}$$
$$-\frac{1}{8}\tilde{\delta}^3\lambda_{22} N_2 + \left(\frac{1}{64}\lambda_{12}\lambda_{11} - \frac{1}{16}\frac{\lambda_{12}}{\sqrt{u}} - \frac{1}{4}\lambda_{22}\left(\frac{1}{4}\frac{1}{\sqrt{u}} + \frac{1}{4}\frac{1}{\sqrt{v-u}}\right) + \frac{1}{1024}\lambda_{12}^{\ 3}\right) N_2\,\tilde{\delta}\,\tilde{\varepsilon}^3$$
$$+\left(-\frac{1}{4}\lambda_{11}\left(\frac{1}{4}\frac{1}{\sqrt{v}} + \frac{1}{4}\frac{1}{\sqrt{v-u}}\right) - \frac{1}{16}\frac{\lambda_{12}}{\sqrt{v}} + \frac{1}{64}\lambda_{12}\lambda_{22}\right) N_2\,\tilde{\delta}^3\,\tilde{\varepsilon}$$
$$+\left(\left(\frac{1}{4}\frac{\lambda_{12}}{\sqrt{v}} + \lambda_{11}\left(\frac{1}{4}\frac{1}{\sqrt{v}} + \frac{1}{4}\frac{1}{\sqrt{v-u}}\right)\right) G_1 + \left(\frac{1}{4}\frac{\lambda_{22}}{\sqrt{v}} + \lambda_{12}\left(\frac{1}{4}\frac{1}{\sqrt{v}} + \frac{1}{4}\frac{1}{\sqrt{v-u}}\right)\right) G_2\right)\tilde{\delta}^4$$

Compute the products $G_{3}^{\varepsilon}G_{4}^{\varepsilon,\delta}$ and $(G_{3}^{\varepsilon})^{3}(G_{4}^{\varepsilon,\delta})^{3}$:

```
> p:=(epsilon,delta)->G[3](epsilon)*G[4](epsilon,delta);
  q:=(epsilon,delta)->p(epsilon,delta)^3;
```

$$p := (\varepsilon, \delta) \rightarrow G_3(\varepsilon)\, G_4(\varepsilon, \delta)$$

$$q := (\varepsilon, \delta) \rightarrow \mathrm{p}(\varepsilon, \delta)^3$$

Expansion of the product $G_{3}G_{4}$ as a function of $(\varepsilon^{4},\delta^{4})$, at order 5:

```
> assume(epsilon>0):assume(delta>0):mtaylor(p(epsilon^4,delta^4),[ep
  silon=0,delta=0],5);
```

$$\tilde{\varepsilon}\, N_1\,\tilde{\delta}\, N_2 + \left(\frac{1}{8} N_1\lambda_{12} N_2 + \left(-\frac{1}{2}\lambda_{11} G_1 - \frac{1}{2}\lambda_{12} G_2\right) N_2\right)\tilde{\delta}\,\tilde{\varepsilon}^2$$
$$+ N_1\left(-\frac{1}{2}\lambda_{22} G_2 - \frac{1}{2}\lambda_{12} G_1\right)\tilde{\delta}^2\,\tilde{\varepsilon}$$
$$+\left(-\frac{1}{128} N_1\lambda_{12}^{\ 2} N_2 - \frac{1}{8}\lambda_{11} N_1 N_2 + \frac{1}{8}\left(-\frac{1}{2}\lambda_{11} G_1 - \frac{1}{2}\lambda_{12} G_2\right)\lambda_{12} N_2\right)\tilde{\delta}\,\tilde{\varepsilon}^3$$
$$+\left(-\frac{1}{4} N_1^{\ 2}\lambda_{12} + \left(-\frac{1}{2}\lambda_{11} G_1 - \frac{1}{2}\lambda_{12} G_2\right)\left(-\frac{1}{2}\lambda_{22} G_2 - \frac{1}{2}\lambda_{12} G_1\right)\right)\tilde{\delta}^2\,\tilde{\varepsilon}^2 - \frac{1}{8} N_1\,\tilde{\delta}^3\lambda_{22} N_2\,\tilde{\varepsilon}$$

Expansion of the product $(G_{3})^{3}(G_{4})^{3}$ as a function of $(\varepsilon^{4},\delta^{4})$, at order 9:

```
> assume(epsilon>0):assume(delta>0):mtaylor(q(epsilon^4,delta^4),[ep
  silon=0,delta=0],9);
```

$$\varepsilon\sim^3 N_1^3\,\delta\sim^3 N_2^3+\left(\frac{3}{8}N_1^3 N_2^3\lambda_{12}+3N_1^2\left(-\frac{1}{2}\lambda_{11}G_1-\frac{1}{2}\lambda_{12}G_2\right)N_2^3\right)\delta\sim^3\varepsilon\sim^4$$
$$+3N_1^3N_2^2\left(-\frac{1}{2}\lambda_{12}G_1-\frac{1}{2}\lambda_{22}G_2\right)\delta\sim^4\varepsilon\sim^3+\left(\frac{3}{128}N_1^3N_2^3\lambda_{12}^2\right.$$
$$+\left(N_1\left(-\frac{1}{4}N_1^2\lambda_{11}+\left(-\frac{1}{2}\lambda_{11}G_1-\frac{1}{2}\lambda_{12}G_2\right)^2\right)+2\left(-\frac{1}{2}\lambda_{11}G_1-\frac{1}{2}\lambda_{12}G_2\right)^2N_1-\frac{1}{8}\lambda_{11}N_1^3\right)N_2^3$$
$$\left.+\frac{9}{8}N_1^2\left(-\frac{1}{2}\lambda_{11}G_1-\frac{1}{2}\lambda_{12}G_2\right)N_2^3\lambda_{12}\right)\delta\sim^3\varepsilon\sim^5+\left(N_1^3\left(\frac{1}{2}\left(-\frac{1}{2}\lambda_{12}G_1-\frac{1}{2}\lambda_{22}G_2\right)\lambda_{12}N_2^2\right.\right.$$
$$\left.+N_2\left(-\frac{1}{2}N_2\lambda_{12}N_1+\frac{1}{4}\left(-\frac{1}{2}\lambda_{12}G_1-\frac{1}{2}\lambda_{22}G_2\right)\lambda_{12}N_2\right)-\frac{1}{4}\lambda_{12}N_1N_2^2\right)$$
$$\left.+9N_1^2\left(-\frac{1}{2}\lambda_{11}G_1-\frac{1}{2}\lambda_{12}G_2\right)N_2^2\left(-\frac{1}{2}\lambda_{12}G_1-\frac{1}{2}\lambda_{22}G_2\right)\right)\delta\sim^4\varepsilon\sim^4+N_1^3$$
$$\left(N_2\left(-\frac{1}{4}N_2^2\lambda_{22}+\left(-\frac{1}{2}\lambda_{12}G_1-\frac{1}{2}\lambda_{22}G_2\right)^2\right)-\frac{1}{8}\lambda_{22}N_2^3+2\left(-\frac{1}{2}\lambda_{12}G_1-\frac{1}{2}\lambda_{22}G_2\right)^2N_2\right)\delta\sim^5\varepsilon\sim^3$$

Recall the expressions of \$Q_{1}\$ and \$Q_{2}\$ to understand better the last expansion

```
> Q[1]:=(-1/2)*(lambda[11]*G[1]+lambda[12]*G[2]);
  Q[2]:=(-1/2)*(lambda[12]*G[1]+lambda[22]*G[2]);
```

$$Q_1:=-\frac{1}{2}\lambda_{11}G_1-\frac{1}{2}\lambda_{12}G_2$$

$$Q_2:=-\frac{1}{2}\lambda_{12}G_1-\frac{1}{2}\lambda_{22}G_2$$

```
>
```