The proof of quantum chaos conjecture, the distribution of distances between adjacent fractional parts of polynomial values, and generalized continuous fractions

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Abstract

The distribution of distances between adjacent quasienergy levels for a broad class of quantum systems including "kicked rotator" model as a special case is found. This distribution differs from Poisson distribution by third order term of smallness. The proof essentially uses the results on the distribution of distances between adjacent fractional parts of polynomial values and the estimate of remainder term is based on the new theory of generalized continued fractions for vectors.

In this paper the proof of quantum chaos conjecture is given for a broad class of quantum systems, which includes the well-known model of a spinning particle submitted to rotate around a fixed axis and which is under the action of short pulses depending only on the phase ("kicked rotator"). In many papers this special case is regarded as the main objects in connection with confirmation of this conjecture ([1]-[4]). The general setting of quantum chaos conjecture proposed in [5] is formulated as follows: the distribution of the distances between adjacent energy levels of many quantum systems follows a quasi-Poisson law, with a density which is close to $e^{-\sigma}$. This conjecture has not been proved for any system up to now. The proof of quantum chaos conjecture proposed here essentially uses the results on the distribution of distances between adjacent fractional parts of polynomial values [6], and the estimate of remainder term is based on a new theory of generalized continuous fractions for number vectors [7].

Consider the one-dimensional nonlinear oscillator given by the Hamiltonians $H = = H(\phi, I, t) = H_0(I) + H_1(\phi, t)$:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H}{\partial I} = \frac{\mathrm{d}H_0}{\mathrm{d}I}, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{\partial H}{\partial \phi} = -\frac{\partial H_1}{\partial \phi}, \tag{1}$$

where I, ϕ are "action-angle" variables, t is time, and the function $H_1(\phi, t)$ has period 2π in ϕ, period $T > 0$ in t and is represented in the form

$$H_1(\phi, t) = F(\phi) \sum_{k=-\infty}^{\infty} \delta(t - kT), \tag{2}$$

where $F(\phi)$ is a smooth 2π-periodic function, $\delta = \delta(t)$ is the Dirac delta distribution and the summation is over all integers k. First rigorous results about the behavior of solution of the system (1) for function $H_0(I)$ of general form were obtained in [8]. Here we assume, that $H_0(I) = \sum_{s=0}^{n} b_n I^n$ is a polynomial of degree $n \geq 2$ with coefficients $b_s = a_s/(-\hbar i)^s$ $(s = 0, \ldots, n)$, where i is the imaginary unit, $\hbar$ the Planck constant and a_s are real numbers. In the special case, where $n = 2$ $a_0 = a_1 = 0$, and $F(\phi) = \gamma \cos \phi$ (γ is a constant) the system (1) is the "kicked rotator". We introduce the Hilbert space L^2 of 2π-periodic complex functions, as the space of states of the quantum system, and the momentum operator $\hat{I} = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$. The time evolution of the wave function $\Psi = \Psi(\phi, t) \in L^2$ is described by Schrödinger's equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\phi, t) = \hat{H}(t) \Psi(\phi, t), \tag{3}$$

where the operator $\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$, $\hat{H}_0 = \sum_{s=0}^{n} b_s \hat{I}^s$, and $\hat{H}_1(t)$ is the limit as $\varepsilon \to 0$ ($\varepsilon > 0$) of operators of multiplication by the function $H_1^{(\varepsilon)}$ obtained from the function H_1 in (2), by replacing the delta function δ by a smooth positive function δ_ε, with support on the interval $[0, \varepsilon]$, such that its integral equals to 1. We denote the solution of the equation (3) after the pulse at the moment $t = nT$ by $\Psi_+(\phi, nT)$ and introduce the monodromy operator $U : \Psi_+(\phi, nT) \to \Psi_+\big(\phi, (n+1)T\big)$.

Lemma 1 ([1],[4]). The operator U has the form $U = \exp(-i\frac{\hat{F}}{\hbar}) \exp(-i\frac{T\hat{H}_0}{\hbar})$, where $\hat{F}$ is the operator of multiplication by the function $F(\phi)$.

Lemma 2. For $k \in \mathbb{Z}$ we consider the function $e_k(\phi) = \exp(ik\phi) \in L^2$. Then $Ue_k(\phi) = \lambda_k(\phi) e_k(\phi)$, where $\lambda_k(\phi) = \exp(-\frac{i(F(\phi) + T\sum_{s=0}^{n} a_s k^s)}{\hbar})$.

Definition 1. The functions $e_k(\phi)$ are called generalized eigen-functions of the operator U, and the functions $\lambda_k(\phi)$ are called their generalized eigen-values.

Remark 1. If $F(\phi) = \text{const}$, then $e_k(\phi)$ are usual eigen-functions of operator U, and $\lambda_k(\phi)$ are their eigen-values.

Let us denote by $\{x\}$ the fractional part of x.

Definition 2. According to [9] the quantity $\beta_k(\phi) = \left\{-\frac{\ln \lambda_k(\phi)}{2\pi i}\right\}$ is called k-level of quasienergy of the system (3).

Definition 3. For any $k \in \mathbb{Z}$ the k-level and $(k+1)$-level of quasienergy are called adjacent.

Remark 2. The above definition expresses the fact that the generalized functions $e_k(\phi)$ and $e_{k+1}(\phi)$ correspond to their quantum states for which between their frequencies $\gamma_k = \frac{k}{2\pi}$, $\gamma_{k+1} = \frac{k+1}{2\pi}$ there are no frequencies of other states $e_s(\phi)$ for $s \neq k, k+1$.

Theorem 1. Assume that among the numbers $(\mathrm{T}a_2)/(2\pi\hbar), \dots, (Ta_n)/(2\pi\hbar)$ is an irrational number. Let σ be an arbitrary number in the interval $0 < \sigma \leq 1$, N a natural number, and $D_N(\phi, \sigma)$ the number of numbers k in the row $k = 1, \dots, N$, such that $0 \leq \beta_{k+1}(\phi) - \beta_k(\phi) < \sigma$. Then for any ϕ the limit distribution function $P(\sigma) = \lim_{N\to\infty} \frac{D_N(\phi,\sigma)}{N}$ exists, the form $P(\sigma) = \sigma - \frac{\sigma^2}{2}$ and differs from the Poisson distribution function $P_\Pi(\sigma) = 1 - \exp(-\sigma)$ of the Poisson law with density $e^{-\sigma}$ by a quantity of the third order of smallnes in σ .

The proof of Theorem 1 follows from Lemma 2, Definition 2 and also from Theorem 3 and its Corollary which was proved in [6].

Definition 4. Let α be a real number. We introduce the map $A = A(\alpha)$ on the space $\mathbf{R^n}$, such that if $x = (x_1, \dots, x_n) \in \mathbf{R^n}$, then $Ax = x' = (x'_1, \dots, x'_n)$, and $x'_s = \sum_{j=0}^{n-s} \begin{pmatrix} s+j \\ j \end{pmatrix} x_{s+j} + \begin{pmatrix} n+1 \\ n-s+1 \end{pmatrix} \alpha$, $1 \leq s \leq n$, $\begin{pmatrix} m \\ l \end{pmatrix}$ is the binomial coefficient.

Definition 5. We introduce the n-dimensional torus $T^n = \{\omega = (\omega_1, \dots, \omega_n) : 0 \leq \omega_1 < 1, \dots, 0 \leq \omega_n < 1\}$ which is the direct product of n circles of length 1.

Further in the formulation of Theorem 2 we use the theory of (A, ω)-continued fractions for vectors $x \in \mathbf{R^n}$ [7]. These continued fractions can be both finite and infinite, and they give unique symbolic representation for each vector $x \in \mathbf{R^n}$ in the form $x = [q^{(0)}, \dots, q^{(m)}]_{A,\omega}$, if the (A, ω)-continued fraction is finite, and of the form $x = [q^{(0)}, q^{(1)}, \dots]_{A,\omega}$, if this fraction is infinite.

Theorem 2. Let $\alpha = \frac{Ta_n}{2\pi(n+1)\hbar}$ be irrational number, $A = A(\alpha)$, $0 < \sigma \leq 1, 0 < \rho < \frac{1}{2}$, N be natural number, $n_0 = \frac{116-72\rho}{8-16\rho}$, $a^* = (a^*_1, \dots, a^*_n)$ be the vector with components $a^*_s = \frac{Ta_{s-1}}{2\pi\hbar s}$ $(s = 1, \dots, n)$, $D_N(\phi, \sigma)$ be the number of numbers k in the row $k = 1, \dots, N$ such that $0 \leq \beta_{k+1}(\phi) - \beta_k(\phi) < \sigma$. Then, if $n > n_0$, then there exists a set $\Gamma \subset \mathbf{T^n}$ such that the following assertions hold:

1) if there exists $\omega \in \Gamma$ such that the (A, ω)-continued fraction of the vector a^* if finite and has the form
$$a^* = [q^{(0)}, \dots, q^{(\nu)}]_{A,\omega} \ , \tag{4}$$

then for any $\epsilon > 0$ and any $N \geq 2$

$$|D_N(\phi, \sigma)/N - (\sigma - \sigma^2/2)| \leq c_1\epsilon + (c_2(c_3 + \nu))/(N^\rho \epsilon), \qquad (5)$$

where $c_1 = c_1(n), c_2 = c_2(n), c_3 = c_3(n)$ are constants not depending of $N, \epsilon, \sigma, a_0, \ldots, a_n$ and of the function $F(\phi)$;

2) the set of vectors $a = (a_1, \ldots, a_n) \in \mathbf{R^n}$, for which (4) and (5) are valid is the complement of a set of Lebesgua measure zero.

The proof of Theorem 2 is based on Theorem 9 of paper [7] and Theorem 1, and the set Γ is constructed in the section 6 of paper [7].

Summary and conclusions

The quantum chaos conjecture has been proved for a broad class of quantum systems generalizing "kicked rotator" model: the distribution of distances between adjacent quasienergy levels is close to Poisson distribution with density $\exp(-\sigma)$. This distribution is found explicitly and its distribution function differs from the Poisson distribution function by third order term of smallness with respect to σ in the interval $[0, 1)$. Moreover we obtain an estimate of the remainder term of the distribution. The proofs are based on the results on the distribution of distances between adjacent fractional parts of polynomial values and on the new theory of generalized continued fractions for vectors.

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