Spectral properties of nonrelativistic Schrödinger operators with potentials given by measures.

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Abstract

We characterize $(-\Delta)$-bounded and compact perturbations in term of boundedness and compactness of a related integral operator. Sufficient condition for the stability of the essential and the absolutely continuous spectra of $-\Delta - \mu$ is given. An estimate of the number of the negative bound states is established and also a "ground state energy representation" for such operators.

Key words: Measure, Bound state, Ground state, Schrödinger operator, Quadratic form, Spectrum.

1 Introduction

In [30], P. Stollmann and J. Voigt proved that for a certain class of measures μ, one has: For some $k^2 > 0$, there is a positive constant $C_k(\mu)$ such that every $f \in H^1(I\!\!R^d)$ satisfies the following inequality:

$$\int_{I\!\!R^d} |f|^2 d\mu \leq \ C_k(\mu)(\int_{I\!\!R^d} |\nabla f|^2 dx + \ k^2 \int_{I\!\!R^d} |f|^2 dx)\cdot \tag{1}$$

For a measure having a density with respect to the Lebesgue measure which is in the Kato class the inequality(1) was proved by M. Schechter [28]. Let us remark that if the inequality(1) is satisfied for some $k^2 > 0$ then it is satisfied for every $k^2 > 0$.
This type of inequalities is important for the study of perturbations of the form associated to the Laplace operator by measures and investigations of the so-called generalized Schrödinger operators [6], [7], [18]. Therefore it would be important to characterize those measures for which inequality(1) holds true.
To our best knowledge a sufficient condition on the measure μ was given in [30] so that (1) is satisfied. Namely it asserts that if $G^\mu = \int_{I\!\!R^d} G_k(\cdot, y) d\mu(y)$ is essentially bounded then the inequality(1) is satisfied, where G_k is the Green kernel of the operator $-\Delta + k^2$.

Later on [3] we gave the following sufficient condition: if the measure μ does not charge polar sets and if the operator

$$K_k^\mu:\ L^2(\mathbb{R}^d,\mu)\to L^2(\mathbb{R}^d,\mu),\ K_k^\mu f=\int_{\mathbb{R}^d}G_k(\cdot,y)f(y)d\mu(y) \tag{2}$$

is bounded then (1) is satisfied. In this note, we will prove that the latter condition is in fact necessary. So that the class of measures which are $(-\Delta)$-bounded is larger (even strictly) than the extended Kato class introduced in [30].
The paper will be organized as follows: First we prove the above mentioned result, then we proceed to construct the operator $H_\mu := -\Delta - \mu$ for suitable measures μ. Then we give sufficient conditions on the measure μ for the stability of the essential and the absolutely continuous spectra for the operator H_μ and give an estimate of the number of its negative bound states improving thereby the bound given in [9, Theorem 3.4]. At the end a "ground state energy representation" will be given.

First we give the useful notations and definitions: We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space. For a Radon positive measure μ, a real number $1\le p\le+\infty$ and a subset $\Omega\subset\mathbb{R}^d$, we denote by $L^p(\Omega,\mu)$ the space of measurable complex valued functions defined on Ω and which are p-integrable with respect to μ. If $\Omega=\mathbb{R}^d$, $L^p(\Omega,\mu)$ will be denoted simply by $L^p(\mu)$ and if μ is the Lebesgue measure on $\mathbb{R}^d$, which we denote by dx, the latter space will be denoted by L^p. The spaces $H_0^1(\Omega)$ and $H^1(\mathbb{R}^d)$ are the classical Sobolev spaces and $C_0^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable functions with compact support in $\mathbb{R}^d$. For every $k\in\mathbb{C}$ such that $\mathrm{Re}(k)>0$ we denote by L_k the operator $L_k:=-\Delta+k^2$ defined on $\mathbb{R}^d$, by G_k its Green function and by $\mathcal{E}_k$ the form associated to L_k:

$$D(\mathcal{E}_k)=H^1(\mathbb{R}^d),\ \mathcal{E}_k[f]=\int_{\mathbb{R}^d}|\nabla f|^2dx+k^2\int_{\mathbb{R}^d}|f|^2dx\ \cdot$$

By $\mathcal{S}^+$, we designate the set of positive measures charging no polar sets of the classical potential theory. All the measures considered in this paper are assumed to be in $\mathcal{S}^+$ and are Radon measures.

Definition 1.1 *(cf. [4]) Let $G:\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty]$ be measurable and let μ be a positive Radon measure. We define*

$$G^\mu:=\int_{\mathbb{R}^d}G(\cdot,y)d\mu(y)\cdot$$

A measure μ will be called G-bounded, if G^μ is bounded. We shall say that a G-bounded μ is a G-Kato measure provided that for every sequence (A_n) of bounded subset increasing to $\mathbb{R}^d$, $G^{1_{A_n}}$ increases to G^μ uniformly on $\mathbb{R}^d$.

A very important class of measures in the literature is the generalized Kato class [9].

The notion of capacity used here is the classical Dirichlet capacity, which we denote by cap. For an open bounded subset Ω, the capacity of Ω is defined by

$$\mathrm{cap}(\Omega)=\inf\{\int_{\mathbb{R}^d}|\nabla f|^2dx+\int_{\mathbb{R}^d}|f|^2dx,\ f\in H^1(\mathbb{R}^d),\ f\ge1,\ \text{a.e, on }\Omega\}\cdot \tag{3}$$

and extended in the usual way for arbitrary subset.
We say that a function f defined on $I\!R^d$ is quasi-continuous (q.c as notation) if for every $\epsilon > 0$, there is an open subset Ω such that $\mathrm{cap}(\Omega) < \epsilon$ and $f_{|\Omega^c}$ is continuous.
Let us recall the known fact that every function $f \in H^1(I\!R^d)$ has a quasi-continuous representative [11]. Finally we say that a property holds quasi-every (q.e) if it holds up to a polar set.
According to [11, p.353], every function $f \in H^1(I\!R^d)$ can be modified so as to become finely continuous. In what follows we shall assume implicitly that functions from $H^1(I\!R^d)$ have been modified in this sense.

2 Characterization of relatively bounded perturbations

In [3], we gave a necessary condition so that an inequality of the type: For certain $k^2 > 0$ and for every $f \in H^1(I\!R^d)$,

$$\int_{I\!R^d} |f|^2 d\mu \ \leq \ C_k(\mu)\Big(\int_{I\!R^d} |\nabla f|^2 dx + \ k^2 \int_{I\!R^d} |f|^2 dx\Big) \tag{4}$$

holds true, where $C_k(\mu)$ is a positive constant depending eventually on k and on μ. For the convenience of the reader we will reproduce the statement and the proof of this result.

Theorem 2.1 *Let $k^2 > 0$ and μ a positive Radon measure such that the operator*

$$K_k^\mu \ : L^2(\mu) \to L^2(\mu), \ K_k^\mu f \ = \ \int_{I\!R^d} G_k(\cdot, y) f(y) d\mu(y) \tag{5}$$

is bounded. Then the inequality(4) is satisfied and we have $C_k(\mu) = \|K_k^\mu\|_{L^2(\mu)}$.

Proof
For every bounded $\Omega \subset I\!R^d$, set $-\Delta_\Omega$ the Dirichlet-Laplacian on Ω, and for every $k^2 > 0$ denote by $L_{k,\Omega}$ the operator $L_{k,\Omega} := -\Delta_\Omega + k^2$, by $G_{k,\Omega}$ its Green kernel which we extend by zero on the complementary of $\Omega \times \Omega$ and by $\mathcal{E}_{k,\Omega}$ the form associated to $L_{k,\Omega}$. Since for every $f \in H^1(I\!R^d)$, $|f| \in H^1(I\!R^d)$ and [23, p.164] $\int_{I\!R^d} |\nabla |f||^2 dx \leq \int_{I\!R^d} |\nabla f|^2 dx$, it suffices to prove(4) for real valued positive functions. Let f be such a function. Suppose first that $f \in C_0^\infty(I\!R^d)$, and let Ω be an open ball containing the support of f. Set $\nu = f\mu$ and $G^\nu := \int_\Omega G_{k,\Omega}(\cdot, y) d\nu(y) = K_{k,\Omega}^\mu f$, then G^ν has finite energy on Ω i.e: $\int_\Omega G^\nu d\nu < +\infty$. Indeed: $\int_\Omega G^\nu d\nu = \int_\Omega f K_\Omega^\mu f d\mu = \int_{I\!R^d} f K_{k,\Omega}^\mu f d\mu$. Since $0 \leq G_{k,\Omega} \leq G_k$ we get $\int_{I\!R^d} f K_{k,\Omega}^\mu f d\mu \leq \int_{I\!R^d} f K_k^\mu f d\mu$ which by the boundedness of K_k^μ leads to $\int_\Omega G^\nu d\nu \leq \|K_k^\mu\| \int_{I\!R^d} f^2 d\mu < +\infty$. Now by the characterization of potentials which are in $H_0^1(\Omega)$, [20, Théorème 10] or [19, Théorème 9] we conclude that $G^\nu \in H_0^1(\Omega)$ and $\int_\Omega f d\nu = \mathcal{E}_{k,\Omega}(G^\nu, f) = \mathcal{E}_{k,\Omega}(K_{k,\Omega}^\mu f, f)$. Thereby we get

$$\int_{I\!R^d} f^2 d\mu \ = \ \mathcal{E}_{k,\Omega}(K_{k,\Omega}^\mu f, f) \leq \ (\mathcal{E}_{k,\Omega}[K_{k,\Omega}^\mu f])^{\frac{1}{2}} (\mathcal{E}_{k,\Omega}[f])^{\frac{1}{2}}. \tag{6}$$

Using an other time [20, Théorème 10] or [19, Théorème 9], we get $\mathcal{E}_{k,\Omega}[G^\nu] = \int_\Omega G^\nu d\nu \leq \|K_k^\mu\|_{L^2(\mu)} \int_{I\!R^d} f^2 d\mu$. replacing this in(6), we get

$$\int_{I\!R^d} f^2 d\mu \leq \ \|K_k^\mu\|_{L^2(\mu)} \mathcal{E}_{k,\Omega}[f] = \|K_k^\mu\|_{L^2(\mu)} \mathcal{E}_k[f]\cdot \tag{7}$$

Hence the inequality is proved for functions in $C_0^\infty(I\!R^d)$.
Now let $f \in H^1(I\!R^d)$, then there is a sequence $(f_n) \subset H^1(I\!R^d)$ such that $\mathcal{E}_k[f_n - f] \to 0$. It is known [12] that there is a subsequence (f_{n_j}) such that $f_{n_j} \to f$ q.e, since μ does not charge polar sets then $f_{n_j} \to f$ $\mu -$ a.e. So by the first part of the proof we have

$$\int_{I\!R^d} f_{n_j}^2 d\mu \leq \ \|K_k^\mu\|_{L^2(\mu)} \mathcal{E}_k[f_{n_j}] \to \|K_k^\mu\|_{L^2(\mu)} \mathcal{E}_k[f]\cdot$$

Finally by Fatou's lemma we conclude that

$$\int_{I\!R^d} f^2 d\mu \leq \liminf_{j\to\infty} \int_{I\!R^d} f_{n_j}^2 d\mu \leq \|K_k^\mu\|_{L^2(\mu)} \mathcal{E}_k[f]$$

and the proof is finished. □

In what follows we are going to prove the converse of Theorem 2.1. To this end we need the following lemma.

Lemma 2.1 *Let μ be a positive measure such that K_k^μ is bounded on $L^2(\mu)$ for some $k^2 > 0$. Then for every $f \in L^2(\mu)$, $K_k^\mu f \in H^1(I\!R^d)$ and for every $g \in H^1(I\!R^d)$ we have:*

$$\int_{I\!R^d} f\bar{g} d\mu = \int_{I\!R^d} \bar{\nabla} g \nabla K_k^\mu f dx + k^2 \int_{I\!R^d} \bar{g} K_k^\mu f dx = \mathcal{E}_k(K_k^\mu f, g)\ \cdot \tag{8}$$

It follows in particular that the operator $K_k^\mu : L^2(\mu) \to H^1(I\!R^d)$ is bounded and $\mathcal{E}_k[K_k^\mu f] \leq \|K_k^\mu\|_{L^2(\mu)} \int_{I\!R^d} f^2 d\mu$.

Proof
It suffices to prove the identity(8) for real valued functions f and g. Let μ be such that K_k^μ is bounded on $L^2(\mu)$ for some $k^2 > 0$. We shall denote by K the integral operator K_k^μ from $L^2(\mu)$ to $(H^1(I\!R^d), \mathcal{E}_k^{\frac{1}{2}})$. For a fixed real valued function $f \in L^2(\mu)$, we define the mapping $\mathcal{L}_f$, as follows:

$$\mathcal{L}_f \ : (H^1(I\!R^d), \mathcal{E}_k^{\frac{1}{2}}) \to I\!R,\ \mathcal{L}_f(g) = \int_{I\!R^d} fg d\mu\cdot \tag{9}$$

By Theorem 2.1, this mapping is bounded. Hence by Riesz's representation theorem, there exists a unique $\phi_{k,f}$ such that

$$\mathcal{L}_f(g) = \mathcal{E}_k(\phi_{k,f}, g),\ \text{for every } g \in H^1(I\!R^d)\cdot \tag{10}$$

It follows in particular that for every g in the Sobolev space $H^2(I\!R^d)$, we have

$$\int_{I\!R^d} \phi_{k,f}(-\Delta + k^2) g dx = \int_{I\!R^d} fg d\mu\cdot \tag{11}$$

Without loss of generality, we may suppose that $f \geq 0$. Let $h \in C_0(I\!R^d),\ h \geq 0$. Choose a sequence (h_n) of positive bounded functions with compact support such that $h_n(x) \uparrow h(x)$ for evey $x \in I\!R^d$. Since the operatore $(-\Delta + k^2)^{-1}$ has nonnegative kernel, it follows that $g_n = (-\Delta + k^2)^{-1} h_n \uparrow g = (-\Delta + k^2)^{-1} g$. Let us recall that $Kf \geq 0$, thus by the monotone convergence theorem, we get

$$\int_{I\!R^d} Kf(-\Delta + k^2) g_n dx \rightarrow \int_{I\!R^d} Kf(-\Delta + k^2) g dx, \text{ as } n \rightarrow \infty, \tag{12}$$

and

$$\int_{I\!R^d} f g_n d\mu \rightarrow \int_{I\!R^d} f g d\mu, \text{ as } n \rightarrow \infty\cdot \tag{13}$$

Let us note that by a direct computation we get

$$\int_{I\!R^d} (Kf) F dx = \int_{I\!R^d} f(-\Delta + k^2)^{-1} F d\mu, \text{ for every } F \in L^2\cdot \tag{14}$$

This leads to

$$\int_{I\!R^d} Kf(-\Delta + k^2) g_n dx = \int_{I\!R^d} f g_n d\mu$$

and then

$$\int_{I\!R^d} f g d\mu = \int_{I\!R^d} Kf(-\Delta + k^2) g dx\cdot \tag{15}$$

By Theorem 2.1, $\int_{I\!R^d} f g d\mu < \infty$. Thus by (11) and (15) and linearity we get

$$\int_{I\!R^d} (\phi_{k,f} - Kf) h dx = 0, \text{ for every } h \in C_0(I\!R^d) \tag{16}$$

By an other representation theorem of Riesz (this time from integration theory) and since $\phi_{k,f} - Kf$ is locally integrable, this implies that

$$Kf = \phi_{k,f},\ dx - \text{a.e.}\cdot$$

In particular, $Kf \in H^1(I\!R^d)$ and the equality in Lemma 2.1 holds. □

Theorem 2.2 *Let μ be a measure such that for every $f \in H^1(I\!R^d)$, we have*

$$\int_{I\!R^d} |f|^2 d\mu \ \leq \ C_k(\mu)(\int_{I\!R^d} |\nabla f|^2 dx + \ k^2 \int_{I\!R^d} |f|^2 dx) \tag{17}$$

then the operator

$$K_k^\mu \ : L^2(\mu) \rightarrow L^2(\mu),\ K_k^\mu f \ = \ \int_{I\!R^d} G_k(\cdot, y) f(y) d\mu(y) \tag{18}$$

is bounded. In these condition the smallest constant $C_k(\mu)$ for which the inequality holds is $\|K_k^\mu\|_{L^2(\mu)}$.

Proof
Let us first emphasize that since $|K_k^\mu f| \leq K_k^\mu |f|$ it suffices to prove that for every positive function $f \in L^2(\mu)$ we have

$$\int_{I\!R^d} (K_k^\mu f)^2 d\mu \leq C_k(\mu)^2 \int_{I\!R^d} f^2 d\mu.$$

Since $\mu \in \mathcal{S}^+$ then according to [14, Proposition 9.1], there is a sequence of measures (μ_n) increasing to μ and such that μ_n is G_k-bounded for every n. It follows that for every n the operator $K_k^{\mu_n}$ is bounded on $L^2(\mu_n)$ and by Lemma 2.1, for every $f \in L^2(\mu)(\subset L^2(\mu_n))$ we have $K_k^{\mu_n} f \in H^1(I\!R^d)$. So that by assumption it satisfies

$$\begin{aligned}\int_{I\!R^d} (K_k^{\mu_n} f)^2 d\mu \quad &\leq \quad C_k(\mu)(\int_{I\!R^d} |\nabla K_k^{\mu_n} f|^2 dx \; + \; k^2 \int_{I\!R^d} (K_k^{\mu_n} f)^2 dx) \\ &= C_k(\mu) \int_{I\!R^d} f K_k^{\mu_n} f d\mu_n \leq \; C_k(\mu)(\int_{I\!R^d} f^2 d\mu_n)^{\frac{1}{2}} (\int_{I\!R^d} (K_k^{\mu_n} f)^2 d\mu_n)^{\frac{1}{2}} \\ &\leq \; C_k(\mu)(\int_{I\!R^d} f^2 d\mu)^{\frac{1}{2}} (\int_{I\!R^d} (K_k^{\mu_n} f)^2 d\mu)^{\frac{1}{2}} \qquad (19)\end{aligned}$$

which leads to $\int_{I\!R^d} (K_k^{\mu_n} f)^2 d\mu \leq C_k(\mu)^2 \int_{I\!R^d} f^2 d\mu$. Now $K_k^{\mu_n}$ increases to $K_k^\mu f$ and by the monotone convergence theorem we get $\int_{I\!R^d} (K_k^\mu f)^2 d\mu \leq C_k(\mu)^2 \int_{I\!R^d} f^2 d\mu$ and the proof is completed. □

Remark 2.1 *1. There are other type of characterization of bounded perturbations for $-\Delta$ [24]. Nevertheless the characterization in term of boundedness of the associated integral operator is more convenient to study the spectral properties of $-\Delta - \mu$ as we shall see.*

2. If $d \geq 3$ then the case $k = 0$ can be treated in the same way. In this situation the space H^0 is exactly the space W introduced in [17]. So that arguing in the same way as before we get that for every $f \in L^2(\mu)$, $K^\mu f \in W$. We thus rediscover the result due to L.I. Hedberg [17, Proposition 5.6].

Let us consider the form

$$\mathcal{I}_\mu, \; D(\mathcal{I}_\mu) = H^1(I\!R^d), \; \mathcal{I}_\mu[f] = \int_{I\!R^d} |f|^2 d\mu$$

then Theorem 2.1 and 2.2 assert that $\mathcal{I}_\mu$ is $(-\Delta)$ (or equivalently $\mathcal{E}$)-bounded if and only if K_k^μ is bounded on $L^2(\mu)$ for some $k^2 > 0$. If one of the latter conditions is satisfied we will simply say that μ is $(-\Delta)$-bounded. Equivalently they assert that if μ is $(-\Delta)$-bounded then the mapping I_μ which to the dx-equivalence class of $f \in H^1(I\!R^d)$ associates the μ-equivalence class of f defines a bounded mapping from $H^1(I\!R^d)$ into $L^2(\mu)$. The question that arises now is the following: is there any relation ship between the compactness of I_μ and that of K_k^μ? The answer is affirmative.

For every $r \geq 0$, set S_r the ideal of operators in the Schatten class.

Theorem 2.3 *The mapping I_μ is compact if and only if the operator K_k^μ is compact for some (whence for every) $k^2 > 0$. Moreover I_μ belongs to the Schatten class S_r if and only if K_k^μ belongs to the Schatten class $S_{\frac{r}{2}}$ for some (whence every) $k^2 > 0$.*

Proof
For every $k^2 > 0$, put $I_{\mu,k} : (H^1(\mathbb{R}^d), \mathcal{E}_k^{\frac{1}{2}}) \to L^2(\mu)$, $I_{\mu,k} f = f$. By Lemma 2.1, $K_k^\mu = I_{\mu,k} I_{\mu,k}^*$. This implies first that K_k^μ is compact if and only if $I_{\mu,k}$ is compact hence if and only if I_μ is compact and second that $\lambda_{n,k}^2$ is an eigenvalue of K_k^μ if and only if $\lambda_{n,k}$ is an eigenvalue of $I_{\mu,k}$ and the result follows. □

3 Spectral properties of the operator $-\Delta - \mu$

Let us mention that by Theorems 2.1 and 2.2, the class of measures μ for which μ is $(-\Delta)$-bounded is larger (even strictly) than the class $\widehat{S}_K$ introduced in [30]. We have thus enlarged the class of measures for which perturbations of the Laplace operator can be considered. In fact we have:

Proposition 3.1 *Suppose that $\mu = \mu^+ - \mu^-$, where μ^+ is a positive smooth measure in the sense of Fukushima (cf.[12, p. 72]) and μ^- is a positive measure such that $\|K_k^{\mu^-}\|_{L^2(\mu)} < 1$ for some $k^2 > 0$. Then the form*

$$\mathcal{F},\ D(\mathcal{F}) = \{f \in H^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d} |f|^2 d\mu^+ < +\infty\},\ \mathcal{F}(f,g) = \int_{\mathbb{R}^d} \nabla f \bar{\nabla} g dx + \int_{\mathbb{R}^d} f \bar{g} d\mu \cdot$$

is closed and lower semi-bounded on L^2 having $-k^2 \|K_k^\mu\|_{L^2(\mu)}$ as a lower bound.

Proof
Set $\mathcal{F}^-$ the form

$$D(\mathcal{F}^-) \ = \ H^1(\mathbb{R}^d),\ \mathcal{F}^-(f,g) = \int_{\mathbb{R}^d} \nabla f \bar{\nabla} g dx - \int_{\mathbb{R}^d} f \bar{g} d\mu^- \ \cdot$$

Since $\|K_k^{\mu^-}\|_{L^2(\mu)} < 1$, then (cf. [21, p. 320]) $\mathcal{F}^-$ is closed. Now let $(f_n) \subset D(\mathcal{F})$ be such that $f_n \to f$ in L^2 and $\mathcal{F}[f_n - f_m] = \mathcal{F}(f_n - f_m, f_n - f_m) \to 0$. By the inequality(4), we get $(1 - \|K_k^{\mu^-}\|_{L^2(\mu)}) \int_{\mathbb{R}^d} |\nabla(f_n - f_m|^2 dx - k^2 \|K_k^{\mu^-}\|_{L^2(\mu)} \int_{\mathbb{R}^d} |f_n - f_m|^2 dx \ \le \ \mathcal{F}^-[f_n - f_m] \le \mathcal{F}[f_n - f_m]$. This leads to $-k^2 \|K_k^{\mu^-}\| \int_{\mathbb{R}^d} |f_n - f_m|^2 dx \ \le \ \mathcal{F}^-[f_n - f_m] \le \ \mathcal{F}[f_n - f_m]$. Thus $\mathcal{F}^-[f_n - f_m] \to 0$. It then follows that $f \in H^1(\mathbb{R}^d)$, $\mathcal{F}^-[f_n - f] \to 0$ and $\int_{\mathbb{R}^d} |f_n - f_m|^2 d\mu^+ \to 0$. Consequently $f_n \to \tilde{f}$ in $L^2(\mu^+)$ and $f = \tilde{f}$, q.p. Since μ^+ does not charge polar sets, then $f = \tilde{f}$, $\mu^+ -$p.p and then $f \in D(\mathcal{F})$ and $\mathcal{F}[f_n - f] \to 0$. Finally using the inequality (4), we get $-k^2 \|K_k^{\mu^-}\|_{L^2(\mu)} \int_{\mathbb{R}^d} |f|^2 dx \le \mathcal{F}[f]$. □

Let us denote by $\mathcal{B}^+$ the set of positive measures such that μ is $(-\Delta)$-bounded and by $\mathcal{B}_0^+$ those measures $\mu \in \mathcal{B}^+$ such that $\inf_{k^2>0} \|K_k^\mu\|_{L^2(\mu)} < 1$. From now on we fix a measure μ in $\mathcal{B}_0^+$, it follows that the symmetric form

$$\mathcal{E}^\mu,\ D(\mathcal{E}^\mu) = H^1(\mathbb{R}^d),\ \mathcal{E}^\mu[f] = \int_{\mathbb{R}^d} |\nabla f|^2 dx - \int_{\mathbb{R}^d} |f|^2 d\mu$$

is closed and lower semi-bounded with lower bound equals to $-k_0^2\|K^\mu_{k_0}\|_{L^2(\mu)}$, for some $k_0^2>0$. We shall denote by H_μ the self-adjoint operator associated to $\mathcal{E}^\mu$ via the representation theorem [21]. If $\mu=0$ then H_μ will be simply denoted by $H:=-\Delta$.
With the help of the above established results, we are going to localize the essential and the absolutely continuous spectra of the operator $-\Delta-\mu$ (which we denote respectively by $\sigma_{\rm ess}(H_\mu)$ and $\sigma_{\rm ac}(H_\mu)$) for certain class of measures μ. Let us recall that the investigation of the spectral properties of such operators was the subject of a very extensive literature [7] (and references therein), [9], [31], [15], [3] and [32].
In the work of J .F. Brasche [7], the study of spectral properties of such operators was mostly initiated under the main assumption that μ is in the generalized Kato class and is finite or has a compact support. In the recent paper [8], however the latter assumption was relaxed and replaced by assumption on the behaviour of the measure at infinity, namely that the measure μ vanishes at infinity i.e.: $\lim_{R\to+\infty}\sup_{|x|\geq R}\mu(B_1(x))=0$, where $B_1(x)$ is the ball centered at x with radius 1. Here we will make no assumptions on the support of the measure neither on its behaviour at infinity, nevertheless we will prove that many spectral properties of H_μ still hold. To prove this we need to give an explicit formula for the resolvent of H_μ.

Let us recall that by assumption there is $k_0^2>0$ such that $\|K^\mu_{k_0}\|_{L^2(\mu)}<1$, whence $\mathcal{E}^\mu[f]\geq -k_0^2\int_{I\!R^d}|f|^2dx$ for each $f\in H^1(I\!R^d)$. Now let $k^2>k_0^2$ then $-k^2\in\rho(H_\mu)$ and $G_k\leq G_{k_0}$. The latter inequality leads to $\|K^\mu_k\|_{L^2(\mu)}\leq\|K^\mu_{k_0}\|_{L^2(\mu)}<1$.

Lemma 3.1 *Let $k^2>k_0^2$ and $K_k:=(H+k^2)^{-1}$. Then*

$$(H_\mu+k^2)^{-1} \ = K_k+(I_\mu K_k)^*(I-K^\mu_k)^{-1}I_\mu K_k. \tag{20}$$

Proof
Let $f\in L^2, g\in H^1(I\!R^d)$, set $T=(I_\mu K_k)^*(I-K^\mu_k)^{-1}I_\mu K_k$. Since the operator K_k+T is well defined on the whole space L^2, we just have to prove that $\mathcal{E}^\mu_k(K_kf+Tf,g)=\int_{I\!R^d}fg dx$. By a direct computation we get $\mathcal{E}^\mu_k(K_kf+Tf,g)=\mathcal{E}^\mu_k(K_kf,g)+\mathcal{E}^\mu_k(Tf,g)=\mathcal{E}_k(K_kf,g)-\int_{I\!R^d}K_kfgd\mu+\mathcal{E}_k(Tf,g)-\int_{I\!R^d}Tfgd\mu$. By Lemma 2.1, we have

$$\int_{I\!R^d}fgdx=\mathcal{E}_k(K_kf,g),\ \mathcal{E}_k(Tf,g)=\int_{I\!R^d}(I-K^\mu_k)^{-1}K_kfgd\mu$$

Now the identity: $K^\mu_k(I-K^\mu_k)^{-1}=(I-K^\mu_k)^{-1}K^\mu_k$ implies that $\int_{I\!R^d}Tfgd\mu=\int_{I\!R^d}(I-K^\mu_k)^{-1}K^\mu_kK_kfgd\mu$. This leads to $-\int_{I\!R^d}K_kfgd\mu+\mathcal{E}_k(Tf,g)-\int_{I\!R^d}Tfgd\mu=-\int_{I\!R^d}K_kfgd\mu+\int_{I\!R^d}(I-K^\mu_k)^{-1}K_kfgd\mu-\int_{R^d}(I-K^\mu_k)^{-1}K^\mu_kK_kfgd\mu=0$, and the lemma is proved. □

Theorem 3.1 *Suppose that I_μ is compact. Then*

i) $\sigma_{\rm ess}(H_\mu)=\sigma_{\rm ess}(H)=[0,+\infty)$.

ii) The set of positive eigenvalues of H_μ is discrete.

iii) $\sigma_{\rm ac}(H_\mu)=\sigma_{\rm ac}(H)=[0,+\infty)$ *and* $\sigma_{\rm sc}(H_\mu)=\emptyset$.

Proof

The proof of assertion(i) follows from the boundedness of K_k from L^2 into $H^1(\mathbb{R}^d)$, Lemma 3.1 and Weyl's theorem.

For the proof of (ii)-(iii), we just have to show that the mapping

$$\mathbb{C}' := \{z\in\mathbb{C} \text{ such that } \mathrm{Im z}>0 \text{ or } \mathrm{Re(z)}>0\} \to \mathbb{B}(L^2(\mu)),\ k\mapsto K_k^\mu \tag{21}$$

is analytic then continue the proof as in [9].

Using the well known relation

$$G_k(x,y)-G_{k_0}(x,y)=(k_0^2-k^2)\int_{\mathbb{R}^d}G_k(x,z)G_{k_0}(z,y)dz,\ \text{for } x\neq y$$

we get

$$K_k^\mu-K_{k_0}^\mu=(k_0^2-k^2)I_\mu K_k(I_\mu K_{k_0})^*.$$

Recalling that $k\mapsto K_k$ is continuous, we get the result. □

The assumption that K_k^μ is compact on $L^2(\mu)$ is weaker than the one used in [8]. For instance if μ is a G_k-Kato measure, then K_k^μ is compact on $L^2(\mu)$ [4]. Now the assumption that μ vanishes at infinity implies that μ is G_k-Kato for every $k^2>\frac{d}{2}$(cf.proof of [8, Lemma 9]) and it follows that our assumption is weaker.

On the other hand it is known [4, Remark 2.1], that every positive measure μ which is globally in the Kato class with respect to a kernel G, i.e. G^μ is continuous and vanishes at infinity, then it is G-Kato. Hence $K^\mu f:=\int_{\mathbb{R}^d}G(\cdot,y)f(y)d\mu(y)$ is compact on $L^2(\mu)$. However, the measure μ need not to vanish at infinity, like for example measures supported by a hyper-plane.

Proposition 3.2 *Suppose that $I_\mu K_k^\mu$ is a Hilbert-Schmidt operator for some $k^2>0$. Then the wave operators $\Omega^\pm(H_\mu,H)$ exist and are complete. It follows in particular that* $\sigma_{\rm ac}(H_\mu)=[0,+\infty)$ *and it does not contain any embedded eigenvalue.*

Proof

The proof follows from the resolvent formula(20) and Kuroda-Birmann's theorem. □

Let us emphasize that if $I_\mu\in S_r$ for some $r\leq 2$ then $I_\mu K_k^\mu$ is a Hilbert-Schmidt operator and that the operator

$$\tilde K_k^\mu:\ L^2(\mu)\longrightarrow L^2,\ \tilde K_k^\mu f=\int_{\mathbb{R}^d}G_k(\cdot,y)f(y)d\mu(y)$$

is the adjoint of $I_\mu K_k$. Hence the assumption of Proposition 3.2 is equivalent to the fact that $\tilde K_k^\mu$ is a Hilbert-Schmidt operator.

We are going to investigate the behavior of the negative eigenvalues of H_μ whenever they are infinite (in number). Let us emphasize that as usually we shall arrange these eigenvalues in an increasing way.

Lemma 3.2 *Suppose that I_μ is compact and that there are infinitely many eigenvalues (E_n) of H_μ below zero. Suppose moreover that there is $\epsilon > 0$ such that $\lambda_n < -\epsilon$ for every integer n, then there is a sequence $(f_n) \subset H^1(\mathbb{R}^d)$ such that:*
i) $\|f_n\|_{H^1(\mathbb{R}^d)} \leq C_1$, for every n.
ii) $\int_{\mathbb{R}^d} f_n^2 d\mu \geq C_2 > 0$ for every n.
iii) $\int_{\mathbb{R}^d} f_n f_m d\mu = 0$ for $n \neq m$.

The proof is just an adaptation to our situation of the one given in [10]p. 448-450.

Proposition 3.3 *Suppose that I_μ is compact and that there are infinitely many eigenvalues (E_n) of H_μ below zero. Then $\lim_{n\to\infty} E_n = 0$.*

Proof

For if not, there is $\epsilon > 0$ such that $\lambda_n < -\epsilon$ for every n. Let (f_n) the sequence given by Lemma 3.2, then by property(i) and Theorem 2.3, there is a subsequence (f_{n_m}) such that $f_{n_m} \to f$ in $L^2(\mu)$. Hence $\int_{\mathbb{R}^d}(f_{n_m} - f_{n_l})^2 d\mu \to 0$, taking condition(iii) into account we get $\int_{\mathbb{R}^d} f_{n_m}^2 d\mu + \int_{\mathbb{R}^d} f_{n_l}^2 d\mu \to 0$, which contradicts condition(ii). □.

In order to derive sufficient conditions for the finiteness of the discrete spectrum, we are going first to give estimates for the number of the negative bound states of $-\Delta - \mu$.

Proposition 3.4 *Suppose that $\sigma_{\mathrm{ess}}(H_\mu) \subset [0, +\infty($. Let $E < 0$, and $\xi > 0$, then E is an eigenvalue of $H_\mu(\xi) := -\Delta - \xi\mu$ if and only if ξ^{-1} is an eigenvalue of K_k^μ on $L^2(\mu)$, where $k = \sqrt{-E}$.*

Proof

Suppose that $E < 0$ is an eigenvalue of $H_\mu(\xi)$, then there is $f \in H^1(\mathbb{R}^d) \setminus \{0\}$ such that for every $g \in H^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \nabla f \bar{\nabla} g dx - \xi \int_{\mathbb{R}^d} f\bar{g} d\mu = E \int_{\mathbb{R}^d} f\bar{g} dx$$

which implies that $\int_{\mathbb{R}^d} \nabla f \bar{\nabla} g dx - E \int_{\mathbb{R}^d} f\bar{g} dx = \xi \int_{\mathbb{R}^d} f\bar{g} d\mu$. The first part of this equality is $\mathcal{E}_k(f, g)$ and the second part is $\xi\mathcal{E}_k(K_k^\mu f, g)$ which leads to $\mathcal{E}_k(f - \xi K_k^\mu f, g) = 0$, for every $g \in H^1(\mathbb{R}^d)$. Hence $f - \xi K_k^\mu f = 0$, everywhere.
Conversely suppose that ξ^{-1} is an eigenvalue of K_k^μ, then there is $f \in L^2(\mu) \setminus \{0\}$ such that $K_k^\mu f = \xi^{-1} f, \mu -$ a.e. Set $g = K_k^\mu f$ then by Proposition 2.1, $g \in H^1(\mathbb{R}^d)$ and $g \neq 0$, furthermore it satisfies $\xi K_k^\mu g = g$ which implies that $\xi g\mu = -\Delta g - Eg$ in the sense of distributions and the result follows. □

Now suppose that I_μ is compact. Let $\xi \in [0, 1]$, by Theorem 3.1 $\sigma_{\mathrm{ess}}(H_{\xi\mu}) = [0, +\infty)$. It follows that the operator $H_{\xi\mu}$ has below 0 at most countably many eigenvalues accumulating at 0. We shall arrange them in an increasing order (counted with thier multiplicity):

$$E_0(\xi) \leq E_1(\xi) \leq .... \leq E_n(\xi) \leq ... \to 0\cdot \tag{22}$$

For $\xi = 1$, we shall denote $E_n(1)$ by E_n.

Proposition 3.5 *Suppose that I_μ is compact. Then the function: $[0,1] \longrightarrow I\!R$, $\xi \mapsto E_n(\xi)$ is a continuous non-increasing function.*

Proof
Obviously $\xi \mapsto E_n(\xi)$ is non-increasing. Now fix $k^2 > 0$ large enough so that $\|K_k^\mu\|_{L^2(\mu)} < 1$. Then $\|K_k^{\xi\mu}\|_{L^2(\xi\mu)} = \xi\|K_k^\mu\|_{L^2(\mu)} < 1$, thereby $k^2 \in \rho(H_{\xi\mu})$ and $E_n(\xi) + k^2 > 0$. It follows that $(E_n(\xi) + k^2)^{-1}$ is an isolated eigenvalue of $(H_{\xi\mu} + k^2)^{-1}$ having the same multiplicity as $E_n(\xi)$. Hence it suffices to prove that the mapping $\xi \mapsto (E_n(\xi) + k^2)^{-1}$ is continuous on $[0,1]$. By the resolvent formula(20) we have

$$(H_{\xi\mu} + k^2)^{-1} = K_k + (I_{\xi\mu})^*(I - K_k^{\xi\mu})^{-1} I_{\xi\mu} K_k.$$

By a direct computation we get $(I_{\xi\mu})^*(I - K_k^{\xi\mu})^{-1} I_{\xi\mu} K_k = \xi(I_\mu K_k)^*(I - \xi K_k^\mu)^{-1} I_\mu K_K$. Hence the selfadjoint operator $(H_{\xi\mu} + k^2)^{-1}$ depends continuously on ξ, consequently (cf. [21]) $\xi \mapsto (E_n(\xi) + k^2)^{-1}$ is continuous. □

Using Propositions 3.4 and 3.5, we establish the following estimate:

Theorem 3.2 *Suppose that I_μ is compact. Let $E < 0$, then*

$$N_E \;\leq\; \|K^\mu_{\sqrt{-E}}\|_p^p \;\cdot$$

For $d \geq 3$, it is known that $G_k(x,y) \leq\; G_0 = C_d|x - y|^{2-d}$, q.e, hence letting $E \to 0$ and setting N_0 the number of the negative bound states of $-\Delta - \mu$ we get

Proposition 3.6 *Suppose that I_μ is compact, $d \geq 3$. Denote by K_0^μ the operator whose kernel is G_0, then $N_0 \leq \|K_0\|_p^p$.*

Next we will improve the estimate given by J.F. Brasche et al. [9, Theorem 3.4]. Let us stress that we wont suppose that the measure μ is finite.

Theorem 3.3 *Assume that μ is a G_k-Kato measure for some $k^2 > 0$. Let $E < 0$, and $1 < q \leq 2$ then*

$$N_E \;\leq\; \int_{I\!R^d} \Big(\int_{I\!R^d} G^q_{\sqrt{-E}}(x,y)\,d\mu(y)\Big)^{\frac{1}{q-1}} d\mu(x)\cdot \tag{23}$$

If $d \geq 3$, then

$$N_0 \;\leq\; \int_{I\!R^d} \Big(\int_{I\!R^d} G^q_0(x,y)\,d\mu(y)\Big)^{\frac{1}{q-1}} d\mu(x)\;\cdot \tag{24}$$

Proof
We may assume that the right hand side of (23) is finite. By [4] the operator K_k^μ is in fact compact on every $L^p(\mu)$-space where $1 < p < +\infty$ furthermore its spectrum does not repent on p. Hence $\|K_k^\mu\|_r$ does not depend on p. Now applying a theorem due to H. König et al. [22, Theorem 38, p.147] we conclude that $K_k^\mu \in S_q$ and $\|K_k^\mu\|_q^q \leq \int_{I\!R^d}(\int_{I\!R^d} G_k^q(x,y)d\mu(y))^{\frac{1}{q-1}} d\mu(x)$. Finally we get the result using Theorem 3.2. If $d \geq 3$, using (23) and the fact that $G_k \leq G_0$ we get (24). □

Let us give an example where $\mu(I\!R^d) = +\infty$ and $N_0 < +\infty$.

Example 3.1 *Set S the unit sphere in $\mathbb{R}^d$ $(d \geq 3)$, $\nu = \sigma$, the surface measure on S. Consider the sequence (α_n) where $\alpha_0 = 0$ and $\alpha_n = n^2 + 2$ for $n > 0$ and the sequence. Set $x_n = (\alpha_n, 0, ..., 0)$, T_{x_n} the transformation defined by $T_{x_n}(x) = \beta_n x + x_n$, $S_n = T_{x_n}(S)$, $\nu_n = T_{x_n}(\nu)$ and $\beta_n = n^{-1}$ for every $n \geq 1$. Finally consider the measure $\mu = \sum_{n>0} \beta_n \nu_n$. Now let $1 < q < 2$, a direct computation yields*

$$(\int_{\mathbb{R}^d} G^q(x,y)d\mu(y))^{\frac{1}{q-1}} = (\sum_{n>0} \beta_n \int_{S_n} G^q(x,y)d\nu_n)^{\frac{1}{q-1}} \leq \sum_{n>0} \beta_n^{\frac{1}{q-1}} (\int_{S_n} G^q(x,y)d\nu_n)^{\frac{1}{q-1}}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^d}(\int_{\mathbb{R}^d} G^q(x,y)d\mu(y))^{\frac{1}{q-1}} d\mu(x) &= \sum_{n,k>0} \beta_k \beta_n^{\frac{1}{q-1}} \int_{S_k}(\int_{S_n} G^q(x,y)d\nu_n(y))^{\frac{1}{q-1}} d\nu_k(x) \\ &= \sum_{n>0} \beta_n \beta_n^{\frac{1}{q-1}} \int_{S_n}(\int_{S_n} G^q(x,y)d\nu_n)^{\frac{1}{q-1}} d\nu_n(x) \\ &+ \sum_{k\neq n} \beta_k \beta_n^{\frac{1}{q-1}} \int_{S_k}(\int_{S_n} G^q(x,y)d\nu_n(y))^{\frac{1}{q-1}} d\nu_k(x) \end{aligned}$$

By construction $A_n := \int_{S_n}(\int_{S_n} G^q(x,y)d\nu_n(y))^{\frac{1}{q-1}} d\nu_n(x)$ decreases with n and we have $A_n \leq \int_S(\int_S G^q(x,y)d\nu(y))^{\frac{1}{q-1}} d\nu(x) < +\infty$ (cf. [5, p. 121]), so that the first term in the RHS of the last equation is finite.
On the other hand for $n \neq k$, we have for every $x \in S_k, y \in S_n, |x-y| \geq |\alpha_k - \alpha_n - 2|$. Hence $\int_{S_k}(\int_{S_n} G^q(x,y)d\nu_n(y))^{\frac{1}{q-1}} d\nu_k(x) \leq |k^2 - n^2 - 2|^{-\frac{q(d-2)}{q-1}}$, so that by elementary computation we get $\sum_{k\neq n} \beta_k \beta_n^{\frac{1}{q-1}} \int_{S_k}(\int_{S_n} G^q(x,y)d\nu_n(y))^{\frac{1}{q-1}} d\nu_k(x) < +\infty$ and thereby $N_0 < +\infty$.

For $d \geq 3$, it may happen that $N_0 = 0$, for example if one change the measure μ by $\epsilon\mu$ where ϵ is small enough so that the right hand term of(24) is strictly smaller than one. However, for $d = 1$ or 2 it is known [25] that if $\mu = V dx$ where V is a non-positive function which is not identically zero, then $-\Delta + V$ has at least one negative eigenvalue. We will give a much more general condition that guarantees existence of negative bound states.

Proposition 3.7 *Suppose that I_μ is compact. If $\lim_{k^2\to 0} \|K_k^\mu\|_{L^2(\mu)} = +\infty$ then H_μ has at least one negative eigenvalue.*

Proof
Since K_k^μ is a compact positive self-adjoint operator, $\|K_k^\mu\|_{L^2(\mu)}$ is an eigenvalue of K_k^μ. Let $0 < \epsilon < 1$, then there is $\delta > 0$ such that for $k^2 < \delta$ we have $\|K_k^\mu\|_{L^2(\mu)}^{-1} < \epsilon$. On the other hand by Proposition 3.4, there is a sequence (f_k) such that $\|f_k\|_{L^2} = 1$ and $\int_{\mathbb{R}^d} |\nabla f_k|^2 dx - \|K_k^\mu\|_{L^2(\mu)}^{-1} \int_{\mathbb{R}^d} |f_k|^2 d\mu = -k^2$, so that for $0 < k^2 < \delta$ we have $\int_{\mathbb{R}^d} |\nabla f_k|^2 dx - \int_{\mathbb{R}^d} |f_k|^2 d\mu = -k^2 + (-1 + \|K_k^\mu\|_{L^2(\mu)}^{-1}) \int_{\mathbb{R}^d} |f_k|^2 d\mu \leq -k^2 < 0$. Hence by the min-max principle $\lambda_1 = \inf \sigma(H_\mu) < 0$. □

Corollary 3.1 *Suppose that $d = 1$ or 2 that K_k^μ is compact for every $k^2 > 0$ and that $\mu(\mathbb{R}^d) > 0$ then $N_0 \geq 1$.*

Proof
Put $\psi(x) = \mathrm{e}^{-|x|}$ then $\psi \in H^1(\mathbb{R}^d)$ for $d = 1, 2$, hence $\psi \in L^2(\mu)$. We are going to prove that $(K_k^\mu \psi, \psi)_{L^2(\mu)} \to +\infty$ as $k^2 \to 0$.
For $d = 1$, we have $(K_k^\mu \psi, \psi)_{L^2(\mu)} = \frac{1}{2k} \int_{\mathbb{R}} \mathrm{e}^{-|y|} \int_{\mathbb{R}} \mathrm{e}^{-k|x-y|} d\mu(x) d\mu(y)$. For k small enough we have $-k|x-y| \geq -|x| - |y|$ thereby $(K_k^\mu \psi, \psi)_{L^2(\mu)} \geq \frac{1}{k} (\int_{\mathbb{R}} \mathrm{e}^{-2|y|} d\mu(y))^2 \to +\infty$ as $k^2 \to 0$.
For $d = 2$, we have $(K_k^\mu \psi, \psi)_{L^2(\mu)} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \psi(y) \int_{\mathbb{R}^2} K_0(|x-y|) \psi(x) d\mu(x) d\mu(y)$, where K_0 is the Macdonald's function. By assumption, there is a compact $K \subset \mathbb{R}^2$ such that $\mu(K) > 0$, thus $\int_{\mathbb{R}^2} \psi(y) \int_{\mathbb{R}^2} K_0(k|x-y|) \psi(x) d\mu(x) d\mu(y) \geq \int_K \psi(y) \int_K K_0(k|x-y|) \psi(x) d\mu(x) d\mu(y)$. Now we have $\lim_{k^2 \to 0} K_0(k|x-y|) = +\infty$ uniformly on $K \times K$ which implies that

$$\lim_{k^2 \to 0} \int_K \psi(y) \int_K K_0(|x-y|) \psi(x) d\mu(x) d\mu(y) \geq \int_K \psi(y) \int_K \lim_{k^2 \to 0} K_0(|x-y|) \psi(x) d\mu(x) d\mu(y) = +\infty$$

and thereby we get the desired result. □

For classical Schrödinger operators $-\Delta + V$, it is known that for a large class of potentials V the ground state energy is non degenerate, i.e it is a simple eigenvalue whose associated eigenvector (up to a phase) is positive (in the L^2-setting). For operators of the form $-\Delta + \mu^+ - \mu^-$ on $\mathbb{R}^d$ with $d \geq 3$, where μ^+ is smooth in the sense of Fukushima and μ^- is in the generalized Kato class this property was proved in [6]. We will prove that this property still holds in our situation.

Proposition 3.8 *Suppose that E_0 is the ground state energy of H_μ. Then E_0 has a finely continuous strictly positive ground state.*

Suppose first that $E_0 < 0$, let f_0 be a normalized eigenfunction associated to E_0, since f_0 has a finely continuous modification and since every finely continuous modification of f_0 is still an eigenfunction of E_0, we may suppose that f_0 is finely continuous. Now, it is known that $|f_0| \in H^1(\mathbb{R}^d)$ and that ([23, p.164]) $\int_{\mathbb{R}^d} |\nabla |f_0||^2 dx \leq \int_{\mathbb{R}^d} |\nabla f_0|^2 dx$, we then have $E_0 = \mathcal{E}^\mu[|f_0|]$. Thus by the variational characterization of the ground state energy, we conclude that $|f_0|$ is also a ground state associated to E_0. Which leads by Proposition 3.4 to $|f_0| = \int_{\mathbb{R}^d} G_{\sqrt{-E_0}}(\cdot, y) |f_0(y)| d\mu(y)$ and whence $|f_0| > 0$ everywhere. Now if $E_0 \geq 0$, chose E such that $E_0 - E < 0$ and argue as before with E_0 replaced by $E_0 - E$ and μ replaced by $\mu + Edx$. □

It is known that if μ is absolutely continuous with respect to the Lebesgue measure or that the measure μ is supported by a subset whose capacity is zero, then if the operator possesses a ground energy, it admits a ground state energy representation.

Let E_0 be the ground state energy of $\mathcal{E}^\mu$ whose finely continuous positive grounde state is φ. We consider the form $\tilde{\mathcal{E}}_\varphi$ defined by

$$D(\tilde{\mathcal{E}}_\varphi) = \{f \in L^2(\varphi^2),\ f\varphi \in H^1(\mathbb{R}^d)\},\ \tilde{\mathcal{E}}_\varphi[f] = \mathcal{E}^\mu_{E_0}[f\varphi] = \mathcal{E}^\mu[f\varphi] - E_0 \int_{\mathbb{R}^d} |f\varphi|^2 dx\cdot$$

Proposition 3.9 *Suppose that H_μ has a ground energy E_0 whose finely continuous strictly positive ground state is φ, then the form*

$$\dot{\mathcal{E}}_\varphi,\ D(\dot{\mathcal{E}}_\varphi) = C_0^\infty(\mathbb{R}^d),\ \dot{\mathcal{E}}_\varphi[f] = \int_{\mathbb{R}^d} |\nabla f|^2 \varphi^2 dx \tag{25}$$

is closable in $L^2(\varphi^2)$. Moreover if we denote by $\mathcal{E}\varphi$ its closure, then we have $\mathcal{E}_\varphi = \tilde{\mathcal{E}}_\varphi$.

Althou the proof can be derived by a general result from [13], we shall here give a shorter adapted proof.

Proof

We first prove the closability. Without loss of generality we may suppose that $E_0 < 0$, for if not choose E such that $E_0 - E < 0$ and change μ by the measure $\mu + Edx$. Arguing as in the proof of Proposition 3.4, it follows that 1 is an eigenvalue of the operator $K^\mu_{\sqrt{-E_0}}$ with φ as associated eigenfunction, this is: $\varphi = \int_{\mathbb{R}^d} G_{\sqrt{-E_0}}(\cdot, y)\varphi(y)d\mu(y)$ everywhere. By the lower semi-continuity of the Green function and the Fatou's lemma, we conclude that φ is lower semi-continuous. Now since $\varphi > 0$ then $\inf_C \varphi > 0$ for every compact subset C of $\mathbb{R}^d$, and the closability follows from [27].

Now we are going to prove the second part of the statement. Let us note that $C_0^\infty(\mathbb{R}^d) \subset \tilde{\mathcal{E}}_\varphi$ and that for every $f \in C_0^\infty(\mathbb{R}^d)$, we have $\tilde{\mathcal{E}}_\varphi[f] = \dot{\mathcal{E}}_\varphi[f]$. Indeed:
Let $f \in C_0^\infty(\mathbb{R}^d)$ then $f \in L^2(\varphi^2)$ and $\nabla(f\varphi) = \varphi\nabla f + f\nabla\varphi \in L^2(\varphi^2)$. On the other hand an elementary computation yields, $\tilde{\mathcal{E}}_\varphi[f] = \int_{\mathbb{R}^d} |\nabla f|^2\varphi^2 dx + \mathcal{E}^\mu(\varphi, f^2\varphi) - E_0 \int_{\mathbb{R}^d} \varphi(f^2\varphi)dx$. Since E_0 is the ground state energy of $\mathcal{E}^\mu$, then $\mathcal{E}^\mu(\varphi, f^2\varphi) - E_0 \int_{\mathbb{R}^d} \varphi(f^2\varphi)dx = 0$, and the result follows.

Hence, $D(\mathcal{E}_\varphi) \subset D(\tilde{\mathcal{E}}_\varphi)$ and for every $f \in D(\mathcal{E}_\varphi)$, $\mathcal{E}_\varphi[f] = \tilde{\mathcal{E}}_\varphi[f]$.

Now to prove the equality it suffices to prove (cf. [21, p. 317]) that $D(\mathcal{E}_\varphi)$ is dense in $D(\tilde{\mathcal{E}}_\varphi)$ with respect to the norm $\tilde{\mathcal{E}}_{\varphi,1}[.] = (\tilde{\mathcal{E}}_\varphi[.] + \|.\|^2_{L^2(\varphi^2)})^{\frac{1}{2}}$. Let $f \in D(\tilde{\mathcal{E}}_\varphi)$, since $C_0^\infty(\mathbb{R}^d)$ is a core of $\mathcal{E}^\mu$, there is $(f_n) \subset C_0^\infty(\mathbb{R}^d)$ such that $\mathcal{E}^\mu[f_n - f\varphi] - E_0 \int_{\mathbb{R}^d} |f_n - f\varphi|^2 dx + \int_{\mathbb{R}^d} |f_n - f\varphi|^2 dx \to 0$. Hence $\tilde{\mathcal{E}}_{\varphi,1}[f_n\varphi^{-1} - f] \to 0$. Thus it suffices to prove that $g_n = f_n\varphi^{-1} \in D(\mathcal{E}_\varphi)$, or equivalently (cf. the proof of [26, Theorem 3.1]), that $\int_{\mathbb{R}^d} |\nabla g_n|^2\varphi^2 dx < \infty$. We have $\nabla g_n = -\varphi^{-2} f_n \nabla\varphi + \varphi^{-1}\nabla f_n$, since $\varphi^{-1}\nabla f_n \in L^2(\varphi^2)$, it turns out to prove that $\varphi^{-2} f_n\nabla\varphi \in L^2(\varphi^2)$. Since for every compact subset C of $\mathbb{R}^d$, we have $\inf_C \varphi > 0$, it follows that

$$\int_{\mathbb{R}^d} |\varphi^{-2} f_n \nabla\varphi|^2 \varphi^2 dx \le \sup_{\mathrm{supp}(f_n)} |f_n|^2 \sup_{\mathrm{supp}(f_n)} (\varphi^{-2}) \int_{\mathbb{R}^d} |\nabla\varphi|^2 dx < \infty$$

which completes the proof □

Remark 3.1 *Let us stress that unlike [2], we do not suppose that* $\varphi^{-1}\Delta\varphi \in L^2_{\text{loc}}$. *In fact in our situation* $\varphi^{-1}\Delta\varphi$ *may be singular with respect to the Lebesgue measure. For example, if* $d = 1$ *and* $\mu = \delta$ *the Dirac mass at* 0, *then one can easily show that* $\varphi^{-1}\Delta\varphi = -\delta - E_0$ *in the sense of distribution and the latter distribution is clearly singular with respect to the Lebesgue measure.*

For $d = 1$ or 2 we already gave a sufficient (and even necessary) condition for the existence of a ground state energy. For higher dimension the situation is much more complicated and a very general assumption for the existence of the ground state energy can not be achieved. For more discussion the reader is referred to [23] or [25]. Of course this problem turns out to the problem of finding a minimizer for

$$\inf\{\mathcal{E}^\mu[f],\ f \in H^1(I\!R^d),\ \|f\|_{L^2} = 1\}\ \cdot$$

In the absolutely continuous case, Lieb and Loss proved [23, p. 239] that for a large class of potentials, if the infimum is strictly negative then the minimizer exists. A sufficient condition for the existence of of the minimizer, is that the functional

$$J : H^1(I\!R^d) \longrightarrow I\!R^d,\ J(f) = \int_{I\!R^d} |f|^2 d\mu$$

is weakly continuous. This condition is indeed equivalent to the compactness of I_μ.

Proposition 3.10 *Suppose that* I_μ *is compact, set*

$$E_0 = \inf\{\mathcal{E}^\mu[f],\ f \in H^1(I\!R^d),\ \|f\| = 1\}\cdot$$

Suppose that $E_0 < 0$, *then* E_0 *is the ground state of* H_μ.

Proof
By the above-mentioned remark and by Theorem 3.1, $\sigma_{ess}(H_\mu) = [0, +\infty)$. It follows by the min-max principle that E_0 is the smallest eigenvalue of H_μ. Let f_0 be an eigenfunction associated to E_0, we claim that $|f_0|$ is also an eigenvalue of H_μ. Indeed: It is known that $|f_0| \in H^1(I\!R^d)$ and that (cf. [23, p.164]) $\int_{I\!R^d} |\nabla|f_0||^2 dx = \int_{I\!R^d} |\nabla f_0|^2 dx$. Hence $E_0 = \mathcal{E}^\mu[f_0]$. Let us consider the functional F:

$$F : H^1(I\!R^d) \longrightarrow I\!R,\ F(f) = \int_{I\!R^d} |\nabla f|^2 dx - \int_{I\!R^d} |f|^2 d\mu - E_0 \int_{I\!R^d} |f|^2 dx \tag{26}$$

then $F \in C^1(H^1(I\!R^d), I\!R)$ and $|f_0|$ is a critical point of F. It follows that the derivative of F at $|f_0|$ must vanishes, which leads to

$$\int_{I\!R^d} \nabla|f_0|\bar{\nabla} f dx - \int_{I\!R^d} |f_0|\bar{f} d\mu - E_0 \int_{I\!R^d} |f_0|\bar{f} dx = 0 \tag{27}$$

for every $f \in H^1(I\!R^d)$. Thus $|f_0|$ is eigenfunction associated to E_0. Now arguing as in the proof of Proposition 3.8, we conclude that $|f_0| > 0$ everywhere.
It remains to prove that E_0 is simple. Defining $g = |f_0| - f_0$, we obtain that either $g = 0 -$ a.e or $|\{g > 0\}| > 0$ (where $|\{\}|$ is the Lebesgue measure of the set under

consideration). In the latter case we conclude that g is an eigenfunction associated to E_0, thus $g > 0$ everywhere. Consequently,

$$f = |f|, \text{ or } f = -|f|, \text{ a.e} \tag{28}$$

Let f_0, g_0 be two linearly independent eigenfunctions associated to E_0. We may suppose that they are orthogonal, which leads to

$$0 = |\int_{\mathbb{R}^d} f_0 g_0 dx| = \int_{\mathbb{R}^d} |f_0||g_0| dx > 0 \tag{29}$$

and this is a contradiction, which completes the proof. □

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