The quantile of a diffusion. Pricing a quantile lookback option.

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Abstract

The quantile of a general diffusion (X_t) *is deeply studied. When* (X_t) *is a Brownian motion with drift we explicitly calculate the joint distribution of the triple constituted by the quantile,* (X_t) *and its local time. We introduce a new path dependent option, baptized* **quantile lookback** *which generalizes the classical lookback option. As a significant application of the joint distribution above, we evaluate the price of quantile lookback options.*

JEL classification: G13.
AMS classification: 90A09, 62P05, 60J55.

Keywords: quantiles, diffusion process, Brownian motion with drift, lookback options, Feynman-Kac's formulae.

1 Introduction

Before expressing financial motivations, we start with a more mathematical introduction. For a real diffusion (X_s), we define the α - quantile of $(X_s)_{(0\leq s\leq t)}$ as

$$X_t^\alpha = \begin{cases} \inf\left\{x / \int_0^t 1_{(X_s\leq x)} ds > \alpha\right\} & \text{if} \quad \alpha < t \\ X_t^t = \sup_{s\in[0,t]}(X_s), \quad X_t^0 = \inf_{s\in[0,t]}(X_s) \end{cases}$$

with the convention that $\inf\emptyset = \infty$.
At our knowledge, Miura R. [20] and Akahori J. [1] were the first authors who investigated the distribution of X_t^α; they made use of the Feynman-Kac's formulae. In their paper (X_s) is a Brownian motion with drift. Pursuing Akahori's computations, Dassios [9] obtained the following remarkable identity in law

$$X_t^\alpha = \sup(X'_s)_{s\in[0,\alpha]} + \inf\left(X_s"\right)_{s\in[0,t-\alpha]}$$

where (X'_s) and $(X_s")$ are independent copies of a Brownian motion with the same drift. Those results were generalized later in [3] or [10]. In 1995, Yor M. [24] obtained the distribution of the $\alpha-$quantiles using the Lévy's result [19] about the Arc-sine laws for the Brownian motion occupation time and the property of the first hitting time at some level $x \in \mathbb{R}$.
If (X_t) is a process with stationary and independent increments with $X_0 = 0$, Dassios A. [10] obtained the following identity

$$\left(\begin{array}{cc} X_t^\alpha, & X_t \end{array} \right) \overset{(law)}{=} \left(\begin{array}{l} \sup_{\alpha\in]0,1[}(X'_{t\alpha}) + \inf_{\alpha\in]0,1[}\left(X"_{t(1-\alpha)}\right) \\ X'_{t\alpha} + X"_{t(1-\alpha)} \end{array} \right),$$

where (X'_s) and $(X_s")$ are independent copies of (X_s). To prove this identity, the author used a similar discrete time early result, by Wendel J. G. [23] and Port S. C. [21].
If (X_t) is a real diffusion, [2] provides the joint distribution of

$$\left(L_T^{X_T^W}, X_T^W\right),$$

where L is the local time of the diffusion process, (W, T) is a random vector independent of the rest, whose density is

$$1_{(0\le w\le t)}\theta\left(\theta+p\right)\exp-\left(\theta t + pw\right)dwdt \qquad \text{where} \quad (\theta, p) \in \left(\mathbb{R}^+\right)^2.$$

That paper used again essentially Feynman-Kac's formulae.

We now discuss briefly the interest of the quantile. In finance, path-dependent options have been extensively developed; in particular, the so-called Asian options involve a knowledge of the distribution of $\frac{1}{t}\int_0^t ds \exp X_s$, where (X_s) is Brownian motion with drift. This distribution was obtained by Geman H., Yor M. [13].
In 1992 Miura R. [20] introduced the "$\alpha-$ quantile option" as

$$(a) \qquad (S_0 \exp X_\tau^{\alpha\tau} - K)^+$$

where S_0 is the initial price, τ is the maturity, $\alpha \in\,]0,1[$ and $X_t^{\alpha\tau}$ is the quantile of the Brownian motion with drift related to the Black-Scholes model. The pricing of that option was performed by Akahori J. [1] and Dassios A. [9].
In 1991, Conze A., Viswanathan [8] priced another type of path-dependent option defined as

$$(b) \qquad \left(S_T - K \inf_{t\in[0,T]} S_t\right)^+.$$

A similar product was considered taking the sup.

Those derivatives were baptized with the name of lookback options. Work on standard lookback options was first done by Goldman B. M., Sosin H. B., Gatto M. A. [14] and Goldman B. M., Sosin H. B., Shepp L. A. [15].
In this paper, we define a new type path dependent option whose payoff is given by

$$(c) \qquad \left(S_T - K S_T^{\alpha}\right)^+ ,$$

where $K > 0$, S_T^{α} is the quantile associated with (S_t) and $\alpha \in [0, T]$. In this case, the strike will be equal to $K S_T^{\alpha}$. For convenience, all along this paper such options will be called **quantile lookback** options.

The paper is organized as follows. Next subsection will be devoted to motivate the quantile lookback options. In subsection 2.1, given a general diffusion process (X_t), we explicitly describe the joint distribution of

$$\left(X_T, L_T^{X_T^W}, X_T^W\right),$$

where L is the local time of the diffusion process, (W, T) is a random vector independent of the diffusion, whose density is again

$$1_{(0 \leq w \leq t)} \theta \left(\theta + p\right) \exp - \left(\theta t + p w\right) dw dt \quad \text{where} \quad (\theta, p) \in \left(\mathbb{R}^+\right)^2 .$$

In subsection 2.2, (X_t) is a diffusion semimartingale and l_t^x is the local time issued by Tanaka's formula. We give the distribution of

$$\left(X_T, l_T^{X_T^W}, X_T^W\right).$$

In subsection 2.3, (X_t) will be a geometrical Brownian motion. For (X_t) and for deterministic (t, α), we evaluate the joint law of $\left(X_t, L_t^{X_t^{\alpha}}, X_t^{\alpha}\right)$ and $\left(X_t, l_t^{X_t^{\alpha}}, X_t^{\alpha}\right)$.
Section 3 will be finally devoted to the pricing of our quantile lookback options in the context of the classical Black-Scholes model [4]. Let (S_t) the process $S_t = S_0 \exp\left(\sigma B_t + \mu t\right)$ (with $S_0 > 0$ and (B_t) being a standard Brownian motion), which satisfies the stochastic differential equation $dS_t = S_t \left(\sigma dB_t + \mu dt\right)$. Our aim will be to compute explicitly

$$e^{-r(\tau - t)} \mathbb{E}\left[\left(S_{\tau} - K S_{\tau}^{\alpha}\right)^+ / \mathcal{F}_t\right],$$

where r is a constant and τ the maturity. We use the "change of variable" and the Feynman-Kac's formulae together with the strong Markov property and the inverse Laplace transformation. In the same section, we introduce a new type of barrier options.

In section 4 we perform some numerical simulations. In particular we suggest some techniques on choosing practically the parameter α in the quantile lookback option.

1.1 Motivations.

Let us consider the lookback option whose payoff equals

$$\left(S_\tau - K \inf_{[0,\tau]} S_s \right)^+ .$$

During the fluctuation of the risky asset (S_t), the $\inf_{[0,\tau]} (S_s)$ may reach an aberrant value because of economical-political problems (wars, wildcat strikes, inflation of the oil price) or other incidents (hurricanes ...). This phenomenon might arise in a short time; however the impact could be significant for the forward contract defined by the lookback option. Since $\inf_{[0,\tau]} (S_s)$ is an instable random variable, we are first motivated by replacing it with another less sensitive path dependent variable. In fact this paper proposes the quantile S_τ^α instead of $\inf_{[0,\tau]} (S_s)$ for several reasons.

First, the quantile lookback option is a refinement of the already existing lookback options. Our option is a generalization of those classical derivatives. The quantile lookback option, whose payoff is given in (c), offers the parameter α as degree of freedom. The price of such an option decreases in α: choosing α close to zero will ensure a high level of protection to the option buyer; choosing α close to τ, will make the product more price convenient.

The second aspect, as we said, concerns the relative stability of the product. The quantile S_τ^α is defined as the inverse of an increasing continuous functions; therefore it can be visualized as an area. The appearance of an extreme isolated value in the fluctuation of the stock price (S_t) will not affect much the evolution of the quantile. The quantile lookback is not sensitive to very isolated phenomena and it is therefore more stable.

A third motivation for introducing the quantile was to dispose of a future contract whose strike belongs to the range of the past stock prices (S_t). In fact, in the classical European call option $(S_\tau - K)^+$, the strike could be outside the range fluctuation $\left[\inf_{s\in[0,\tau]} (S_s), \sup_{s\in[0,\tau]} (S_s)\right]$. Practically, the strike price is often chosen using modelling and prediction techniques, on the basis of the collected historical data on (S_t). Fixing the strike K at time zero, may not be the optimal solution for the seller and the buyer of the option. In fact, if an unusual extreme event appears, the fluctuation of S_t will be very irregular. Therefore it is reasonable that K takes partially in account such an event. If we replace the strike price K with the quantile, S_τ^α, this one will belong to $\left[\inf_{s\in[0,\tau]} (S_s), \sup_{s\in[0,\tau]} (S_s)\right]$.

1.2 Notations.

We adopt here the general approach to real diffusions presented in the book of Revuz D. and Yor M. [22].

Let I be an interval with left endpoint $l \geq -\infty$ and right endpoint $r \leq +\infty$, (X_t) a real diffusion taking values in I and starting at x_0. $\mathbb{P}_{x_0}$ and $\mathbb{E}_{x_0}$ will denote the respective underlying probability and relative expectation.
We introduce some classical notations.
$m(dx)$: (resp λ) the speed measure of the diffusion (resp. Lebesgue's measure).
s : the scale function of the diffusion. We say that the diffusion is *natural* if $s(x) = x$.
ζ : the life time of the diffusion (X_t).
$T_x = \inf\{t / X_t = x\}$: the first hitting time at level x.
In this paper, we suppose that the diffusion process is regular and the speed measure is diffuse ($\forall x \in \mathbb{R}, m\{x\} = 0$). Moreover, we set

$$M_t(x) = \int_0^t 1_{(X_u \leq x)} du.$$

We recall that $L = \frac{\partial^2}{\partial m \partial s(y)}$ is the infinitesimal generator of the diffusion in the interval $I \subset \mathbb{R}$.
$g_1^x(y, \alpha)$ (resp. $g_2^x(y, \alpha)$) will be a positive solution, increasing (resp. decreasing) of the equation

$$\frac{\partial^2 \Psi}{\partial m \partial s(y)}(y) = \alpha \Psi(y)$$

where $\alpha \in \mathbb{R}^+$ and $g_1^x(x, \alpha) = g_2^x(x, \alpha) = 1$.

For illustration, suppose for a moment that (X_t) is a solution of the following equation

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt,$$

where (W_t) is a standard Brownian motion and σ, b are locally bounded Borel functions such that σ^2 does not vanish. Then we have

$$s(x) = \int_c^x \exp\left(-\int_c^y 2b(z) \sigma^{-2}(z) dz\right) dy$$

where c being an arbitrary point in $\mathbb{R}$.
Moreover, the density of the speed measure m with respect to the Lebesgue measure is $\frac{2}{s'(x)\sigma^2(x)}$ where s' is the derivative of s and the infinitesimal generator is equal to

$$L=\frac{1}{2}\sigma^{2}\left(x\right)\frac{d^{2}}{dx^{2}}+b\left(x\right)\frac{d}{dx}.$$

2 The joint distribution of $\left(X_T, L_T^{X_T^W}, X_T^W\right)$.

All along the paper, with the exception of Remark 2.4, we will suppose that the considered diffusion process lives in $I=\mathbb{R}$ and it has a zero killing measure.

2.1 The case where the diffusion is natural.

In this subsection, we suppose (X_t) to be natural; in this case (X_t) is a local martingale and it can be constructed as a random time change of the standard Brownian motion (see [18]). We define the Markov local time L_t^x as a measurable random field which fulfills the following occupation density relation

$$\int_0^t g\left(X_u\right)du=\int_{\mathbb{R}} g\left(a\right)L_t^a dm\left(a\right),$$

see Blumenthal R.M and Getoor R.K [5] (p.228).
Moreover, such process fulfills the equality below:

$$\mathbb{E}_{x_0}\left(\exp-\left(pT_x\right)\right)=\mathbb{E}_{x_0}\left(\int_0^\infty \exp -ptdL_t^x\right) \qquad ,$$

see Blumenthal R.M and Getoor R.K [5] (p.216).

The following theorem constitutes the main result of this section.

Theorem 2.1 *For $u>0$ and $z\in\mathbb{R}$, we have*

$$\mathbb{P}_{x_0}\left(X_T>z, X_T^W\in dm(x), L_T^{X_T^W}\in du\right)=dm(x)du$$

$$\begin{cases} \frac{\theta+p}{a_1^2}ua_5e^{-\frac{a_2u}{a_1}}\left\{1_{(z<x_0<x)}\left[-\frac{a_3}{a_4}g_1^x\left(x_0,\theta+p\right)+g_2^z\left(x_0,\theta+p\right)\right]\right. \\ +1_{(z<x\le x_0)}\left(g_2^z\left(x,\theta+p\right)-\frac{a_3}{a_4}\right)g_2^x\left(x_0,\theta\right) \\ \left.+1_{(x_0\le z<x)}\left(1-\frac{a_3}{a_4}g_1^x\left(z,\theta+p\right)\right)g_1^z\left(x_0,\theta+p\right)\right\}+ \\ \frac{\theta+p}{a_7^2}ua_6e^{-\frac{a_8u}{a_7}}\left\{1_{(x<z<x_0)}\left(1-\frac{a_{10}}{a_9}g_1^x\left(z,\theta\right)\right)g_2^z\left(x_0,\theta\right)\right. \\ +1_{(x\le x_0<z)}\left(g_1^z\left(x_0,\theta\right)-\frac{a_{10}}{a_9}g_2^x\left(x_0,\theta\right)\right) \\ \left.+1_{(x_0<x<z)}\left(g_1^z\left(x,\theta\right)-\frac{a_{10}}{a_9}\right)g_1^x\left(x_0,\theta+p\right)\right\}, \end{cases}$$

with

$a_1 = 2\left[\left(g_1^x(z,\theta+p)-1\right)g_2^z(x,\theta+p)+\frac{\partial g_2^z(z,\theta+p)}{\partial y}-\frac{\partial g_1^z(z,\theta+p)}{\partial y}\right]$

$a_2 = \left(1-g_1^x(z,\theta+p)\right)\left(g_2^z(x,\theta+p)\frac{\partial g_2^x(x,\theta+p)}{\partial y}-\frac{\partial g_2^z(x,\theta+p)}{\partial y}\right)$

$-\left(\frac{\partial g_1^z(z,\theta+p)}{\partial y}-\frac{\partial g_2^z(z,\theta+p)}{\partial y}\right)\left(\frac{\partial g_2^x(x,\theta+p)}{\partial y}-\frac{\partial g_2^z(x,\theta+p)}{\partial y}\right)$

$a_3 = \frac{\partial g_2^z(z,\theta+p)}{\partial y}-\frac{\partial g_1^z(z,\theta+p)}{\partial y}$

$a_4 = \left(1-g_1^x(z,\theta+p)\right)\frac{\partial g_1^z(z,\theta+p)}{\partial y}$

$a_5 = 2\left(1-\frac{\theta}{\theta+p}\right)\left(a_4\frac{\partial g_2^z(x,\theta+p)}{\partial y}-a_3\frac{\partial g_1^z(z,\theta+p)}{\partial y}\right)+$

$\frac{\theta}{\theta+p}\frac{\partial g_1^z(z,\theta+p)}{\partial y}\left(\left(\frac{2\partial g_2^x(x,\theta)}{\partial y}-\frac{\partial g_1^z(x,\theta+p)}{\partial y}\right)g_2^z(x,\theta+p)-\frac{\partial g_2^z(x,\theta+p)}{\partial y}\right)$

$a_6 = 2\frac{\partial g_2^z(x,\theta)}{\partial y}\left(\frac{\partial g_1^z(z,\theta+p)}{\partial y}-g_1^z(x,\theta+p)\frac{\partial g_2^x(x,\theta)}{\partial y}\right)$

$a_7 = -2g_1^z(x,\theta)\left(\frac{2\partial g_2^z(z,\theta)}{\partial y}g_2^x(z,\theta)-\frac{\partial g_2^x(z,\theta)}{\partial y}\right)+\frac{\partial g_2^z(z,\theta)}{\partial y}-\frac{\partial g_1^z(z,\theta)}{\partial y}$

$a_8 = \left(\frac{2\partial g_2^z(z,\theta)}{\partial y}g_2^x(z,\theta)-\frac{\partial g_2^z(x,\theta)}{\partial y}\right)\left(\frac{\partial g_1^z(x,\theta)}{\partial y}-g_1^z(x,\theta)\frac{\partial g_2^x(x,\theta+p)}{\partial y}\right)-$

$\left(\frac{\partial g_2^z(z,\theta)}{\partial y}-\frac{\partial g_1^z(z,\theta+p)}{\partial y}\right)\left(\frac{\partial g_2^x(x,\theta)}{\partial y}-\frac{\partial g_1^x(x,\theta+p)}{\partial y}\right)$

$a_9 = \frac{\partial g_2^z(z,\theta)}{\partial y}g_2^x(z,\theta)-\frac{\partial g_2^x(z,\theta)}{\partial y}$

$a_{10} = \left(\frac{\partial g_2^z(z,\theta)}{\partial y}-\frac{\partial g_1^z(z,\theta)}{\partial y}\right)$

The proof of this theorem is based on the following two lemmas.

Lemma 2.2 *Let Φ, Ψ two measurable, positive functions with $\Phi(\infty)=0$; then for every $(\alpha,t)\in(\mathbb{R}^+)^2$.*

$$\int_{\mathbb{R}^+}\Phi\left(X_t^\alpha\right)\Psi(\alpha)\,d\alpha=\int_{\mathbb{R}}\Phi(x)\,\Psi\left(M_t(x)\right)L_t^x dm(x)$$

Proof. [2] established the continuity of X_t^α with respect to (α,t). Since X_t^α is the continuous inverse of $M_t(x)$ with respect to x, we can use the change of variables formula. ∎

Lemma 2.3 *Let $\overline{T}$ be a exponentially distributed random variable with parameter θ and independent of the diffusion. One has*

$$\mathbb{E}_{x_0}\left[1_{\left(X_{\overline{T}}>z\right)}L_{\overline{T}}^x\exp-\left(cL_{\overline{T}}^x+pM_{\overline{T}}(x)\right)\right]=$$

$$\frac{a_5}{(a_1c+a_2)^2}\left\{1_{(z<x_0\le x)}\left[-\frac{a_3}{a_4}g_1^x(x_0,\theta+p)+g_2^z(x_0,\theta+p)\right]\right.$$
$$+1_{(z<x<x_0)}\left(g_2^z(x,\theta+p)-\frac{a_3}{a_4}\right)g_2^x(x_0,\theta)$$
$$\left.+1_{(x_0 l\le z<x)}\left(1-\frac{a_3}{a_4}g_1^x(z,\theta+p)\right)g_1^z(x_0,\theta+p)\right\}$$
$$+\frac{a_6}{(a_7c+a_8)^2}\left\{1_{(x<z<x_0)}\left(1-\frac{a_{10}}{a_9}g_1^x(z,\theta)\right)g_2^z(x_0,\theta)\right.$$
$$+1_{(x\le x_0<z)}\left(g_1^z(x_0,\theta)-\frac{a_{10}}{a_9}g_2^x(x_0,\theta)\right)$$
$$\left.+1_{(x_0<x<z)}\left(g_1^z(x,\theta)-\frac{a_{10}}{a_9}\right)g_1^x(x_0,\theta+p)\right\},$$

where $(a_i)_{i\in\{1,\dots,10\}}$ *are the same quantities as in theorem 2.1.*

Proof.
Writing

$$Z_{\overline{T}}(x)=\exp-\left(cL_{\overline{T}}^x+pM_{\overline{T}}(x)\right),$$

we aim to compute $-\frac{\partial}{\partial c}\mathbb{E}_{x_0}\left[1_{\left(X_{\overline{T}}>z\right)}Z_{\overline{T}}(x)\right]$.
Let $\left(\mathcal{F}_t^{\overline{T}}\right)_{t\in\mathbb{R}^+}$ denote the smallest filtration which contains $(\mathcal{F}_t)$ and for which $\overline{T}$ is a stopping time; we look for a continuous and bounded function Φ such that

$$\Phi(X_t)Z_t(x)1_{\left(t\le\overline{T}\right)}+Z_T(x)1_{\left(t\ge\overline{T},X_{\overline{T}}>z\right)}$$
$$=\Phi\left(X_{t\wedge\overline{T}}\right)Z_{t\wedge\overline{T}}(x)+\left(1_{\left(X_{\overline{T}}>z\right)}-\Phi\left(X_{\overline{T}}\right)\right)Z_{\overline{T}}(x)1_{\left(t\ge\overline{T}\right)}\qquad(1)$$

is a $\left(\mathcal{F}_t^{\overline{T}}\right)$ martingale.
The predictable compensator of $\left(1_{\left(X_{\overline{T}}>z\right)}-\Phi\left(X_{\overline{T}}\right)\right)Z_{\overline{T}}(x)1_{\left(t\ge\overline{T}\right)}$ with respect to the filtration $\left(\mathcal{F}_t^{\overline{T}}\right)$ is

$$\theta\int_0^{t\wedge\overline{T}}\left(1_{(X_s>z)}-\Phi(X_s)\right)Z_s(x)\,ds$$

(see [2] or [11]). Therefore (1) holds if

$$(2)\qquad\Phi\left(X_{t\wedge\bar{T}}\right)Z_{t\wedge\bar{T}}(x)+\theta\int_0^{t\wedge\bar{T}}\left(1_{(X_s>z)}-\Phi(X_s)\right)Z_s(x)\,ds$$

is also a $\left(\mathcal{F}_t^{\bar{T}}\right)$ martingale. Applying Itô's and Tanaka's formulae to (2), we see that (1) will be verified provided that Φ verifies the following equalities:

$(a)\quad \frac{\partial^2\Phi(y)}{\partial m\partial y}+\theta\left(1_{(y>z)}-\Phi(y)\right)-p\Phi(y)1_{(y<x)}=0$
$(b)\quad \Phi(x+)=\Phi(x-)$
$(c)\quad \Phi'(x+)-\Phi'(x-)=2c\Phi(x)$

$(d) \quad \Phi(z+) = \Phi(z-)$
$(e) \quad \Phi'(z+) - \Phi'(z-) = 0,$

where g_1^x, g_2^x are two solutions of the equation

$$\frac{\partial^2 \Psi(y)}{\partial m \partial y} = \alpha \Psi(y), \qquad \alpha \in \mathbb{R}^+,$$

with

$$g_1^x(x, \alpha) = g_2^x(x, \alpha) = 1.$$

A solution of equation (a) is the following :

$$\Phi(y) = \begin{cases} 1_{(z<y<x)} \left(\lambda_1 g_1^x(y, v) + \lambda_2 g_2^z(y, v) + \frac{\theta}{v}\right) \\ 1_{(x<z<y)} \left(\lambda_3 g_2^z(y, \theta) + 1\right) \\ 1_{(z<x<y)} \left(\lambda_4 g_2^x(y, \theta) + 1\right) \\ 1_{(x<y<z)} \left(\lambda_5 g_2^x(y, \theta) + \lambda_6 g_1^z(y, \theta)\right) \\ 1_{(y<z<x)} \lambda_7 g_1^z(y, v) \\ 1_{(y<x<z)} \lambda_8 g_1^x(y, v) \end{cases}$$

where $v = \theta + p$ and $(\lambda_i)_{i \in \{1,\ldots,8\}}$ are given by the equations $(b), (c), (d), (e)$. This means

$$\begin{cases} \lambda_1 g_1^x(z, v) + \lambda_2 - \lambda_7 = -\frac{\theta}{v} \\ \lambda_1 \frac{\partial g_1^x(z,v)}{\partial y} + \lambda_2 \frac{\partial g_2^z(z,v)}{\partial y} - \lambda_7 \frac{\partial g_1^x(z,v)}{\partial y} = 0 \\ \lambda_3 - \lambda_5 g_2^x(z, \theta) - \lambda_6 = -1 \\ -\frac{\partial g_2^z(z,\theta)}{\partial y} \lambda_3 + \lambda_5 \frac{\partial g_2^x(z,\theta)}{\partial y} + \frac{\partial g_1^z(z,\theta)}{\partial y} \lambda_6 = 0 \\ \lambda_1 + \lambda_2 g_2^z(x, v) - \lambda_4 = 1 - \frac{\theta}{v} \\ -\lambda_1 \frac{\partial g_2^x(x,v)}{\partial y} - \lambda_2 \frac{\partial g_1^z(x,v)}{\partial y} + \lambda_4 \left(\frac{\partial g_2^x(x,\theta)}{\partial y} - 2c\right) = 2c \\ \lambda_8 - \lambda_5 - \lambda_6 g_1^z(x, \theta) = 0 \\ -\left(\frac{\partial g_2^x(x,v)}{\partial y} + 2c\right) \lambda_8 + \lambda_5 \frac{\partial g_2^x(x,\theta)}{\partial y} + \lambda_6 \frac{\partial g_2^z(x,\theta)}{\partial y} = 0 \end{cases}$$

Solving this system of equations we obtain explicitly $\Phi(y)$.
The martingale property in (1) gives us the following identity.

$$\mathbb{E}_{x_0}\left[1_{(X_{\bar{T}}>z)} Z_{\bar{T}}(x)\right] = \Phi(x_0)$$

To achieve the proof, it remains to derive this relation with respect to c. ■

Proof. (of theorem 2.1)

Let h be a measurable, positive and bounded function with $h(\infty) = 0$, $c \in \mathbb{R}^+$.
Applying lemma 2.2, we obtain

$$\mathbb{E}_{x_0}\left[h\left(X_T^W\right)1_{(X_T>z)}\exp -cL_T^{X_T^W}\right]$$

$$= \mathbb{E}_{x_0}\left[\int_{(\mathbb{R}^+)^2} h\left(X_t^w\right)1_{(X_t>z,w\leq t)}(\theta+p)\,\theta\exp -\left(cL_t^{X_t^w}+\theta t+pw\right)dwdt\right]$$
$$= \mathbb{E}_{x_0}\left[\int_{\mathbb{R}^+}(\theta+p)\,h\left(X_{\overline{T}}^w\right)1_{\left(X_{\overline{T}}>z\right)}\exp -\left(cL_{\overline{T}}^{X_{\overline{T}}^w}+pw\right)dw\right]$$
$$= \mathbb{E}_{x_0}\left[\int_{\mathbb{R}}(\theta+p)\,h(x)\,1_{\left(X_{\overline{T}}>z\right)}L_{\overline{T}}^x\exp -\left(cL_{\overline{T}}^x+pM_{\overline{T}}(x)\right)dm(x)\right]$$

where $\bar{T}$ is an exponential random variable with parameter θ which is independent from the diffusion.
Using lemma 2.3, $\mathbb{E}_{x_0}\left[1_{\left(X_{\overline{T}}>z\right)}L_{\overline{T}}^x\exp -\left(cL_{\overline{T}}^x+pM_{\overline{T}}(x)\right)\right]$ can be explicitly computed. It remains for us to invert the Laplace transformation with respect to c and the result will follow. ■

Remark 2.4 *The following observations concern the case when the diffusion lives in a subinterval of* $\mathbb{R}$.

If the diffusion is natural and takes value in $(b,+\infty[$, *we can state the analogous result to theorem 2.1 provided we add another condition on the solution* $g_1^x(y,v)$ *of* $\frac{\partial^2\Psi(y)}{\partial m\partial y} = v\Psi(y)$. *This condition depends on the nature of the point* $b \in \mathbb{R}$ *in the sense of [6] p.18 or [18].*

If $b \in I$ *the condition is*

$$vg_1^x(b,v)\,m(\{b\}) = g_1^{x\prime}(b,v)\,.$$

If $b \notin I$ *it becomes*

$$g_1^x(b+,v) = 0.$$

If b *is entrance-not- exit we have*

$$g_1^x(b+,v) > 0, g_1^{x\prime}(b+,v) = 0.$$

If b *is entrance-not-entrance, it becomes*

$$g_1^x(b+,v) = 0, g_1^{x\prime}(b+,v) > 0.$$

If b is natural we have

$$g_1^x(b+,v) = 0, g_1^{x\prime}(b+,v) = 0.$$

Remark 2.5 *This remark will be important in the following subsection.*

a) If the diffusion (X_t) is not natural, which means $s(x) \neq x$, we know that the diffusion $(Y_t) = (s(X_t))$ is natural (see [22] p.310). We denote by $\overline{Y}_t^\alpha$ the quantile associated with the diffusion (Y_t). Using the property of the scale function we have $X_t^\alpha = s^{-1}\left(\overline{Y}_t^\alpha\right)$. It means that the study of the distribution of (X_t, X_t^α) can be found via the distribution of $\left(Y_t, \overline{Y}_t^\alpha\right)$. The latter is known using theorem 2.1.

b) If we observe carefully the proof of theorem 2.1 we learn that

$$\begin{aligned}&\mathbb{P}_{x_0}\left(X_T > z, X_T^W \in dm(x), L_T^{X_T^W} \in du\right)\\&= (\theta + p)\, f_{x_{0,z}}(x, u, \theta, \theta + p)\, dm(x)\, du,\end{aligned}$$

where

$$\int_{\mathbb{R}+} e^{-cu} f_{x_{0,z}}(x, u, \theta, \theta + p)\, du =$$

$$\mathbb{E}_{x_0}\left[1_{(X_{\overline{T}} > z)} L_{\overline{T}}^x \exp -\left(cL_{\overline{T}}^x + pM_{\overline{T}}(x)\right)\right]$$

and $\overline{T}$ is a random variable independent of the diffusion having exponential law with parameter θ and $c > 0$.

2.2 The case where the diffusion is a semimartingale.

In this subsection, we suppose that the diffusion (X_t) is also a semimartingale. Our aim is to find an analogous result to theorem 2.1 statement; the idea is to proceed similarly as for subsection 2.1 but considering the local time which is issued from Tanaka formula.
We introduce this new local time (l_t^a) through

$$(X_t - a)^+ = (x_0 - a)^+ + \int_0^t 1_{(X_s \geq a)} dX_s + l_t^a.$$

For g measurable and bounded we have

$$(*)\qquad \int_0^t g(X_s)\, d\langle X,X\rangle_s = \int_{\mathbb{R}} g(a)\, l_t^a\, da.$$

In the following, we suppose that $d\langle X,X\rangle_s$ (resp. the speed measure $dm(x)$) has a density $\sigma^2(X_s) > 0$ (resp. $m(x)$) with respect to Lebesgue measure. By using $(*)$ we have

$$l_t^a = \sigma^2(a)\, m(a)\, L_t^a,$$

where (L_t^a) is the Markov local time. This relation shows that we can also study the quantile via the local time (l_t^a); for two measurable, positive functions Φ, Ψ, with $\Phi(\infty) = 0$, for every $(\alpha, t) \in (\mathbb{R}^+)^2$ we have:

$$(**)\qquad \int_{\mathbb{R}^+} \Phi(X_t^\alpha)\, \Psi(\alpha)\, d\alpha = \int_{\mathbb{R}} \Phi(x)\, \Psi(M_t(x))\, l_t^x \frac{dx)}{\sigma^2(x)};$$

this identity is the analogous of lemma 2.2 statement in subsection 2.1. Using remark 2.5 b) in subsection 2.1 and equation $(**)$ we can write

$$(***)\qquad \mathbb{P}_{x_0}\left(X_T > z, X_T^W \in \frac{dx}{\sigma^2(x)}, l_T^{X_T^W} \in du \right)$$

$$= (\theta + p)\, g_{x_0,z}(x, u, \theta, \theta + p) \frac{dx}{\sigma^2(x)} du,$$

where, for every $c > 0$,

$$\int_{\mathbb{R}^+} e^{-cu} (\theta + p)\, g_{x_0,z}(x, u, \theta, \theta + p)\, du =$$

$$\mathbb{E}_{x_0}\left[1_{(X_{\overline{T}} > z)} l_{\overline{T}}^x \exp - \left(c l_{\overline{T}}^x + p M_{\overline{T}}(x) \right) \right]$$

and where $\overline{T}$ is random variable independent of the diffusion having exponential law with parameter θ . Using the same approach as in lemma 2.3 we obtain

$$\mathbb{E}_{x_0}\left[1_{(X_{\overline{T}} > z)} l_{\overline{T}}^x \exp - \left(c l_{\overline{T}}^x + p M_{\overline{T}}(x) \right) \right] =$$

$$\begin{aligned}
&\tfrac{a_5}{(a_1 c + a_2)^2} \Big\{ 1_{(z < x_0 < x)} \left[-\tfrac{a_3}{a_4} g_1^x(x_0, \theta + p) + g_2^z(x_0, \theta + p) \right] \\
&+ 1_{(z < x < x_0)} \left(g_2^z(x, \theta + p) - \tfrac{a_3}{a_4} \right) g_2^x(x_0, \theta) \\
&+ 1_{(x_0 < z < x)} \left(1 - \tfrac{a_3}{a_4} g_1^x(z, \theta + p) \right) g_1^z(x_0, \theta + p) \Big\} \\
&+ \tfrac{a_6}{(a_7 c + a_8)^2} \Big\{ 1_{(x < z < x_0)} \left(1 - \tfrac{a_{10}}{a_9} g_1^x(z, \theta) \right) g_2^z(x_0, \theta) \\
&+ 1_{(x < x_0 < z)} \left(g_1^z(x_0, \theta) - \tfrac{a_{10}}{a_9} g_2^x(x_0, \theta) \right) \\
&+ 1_{(x_0 < x < z)} \left(g_1^z(x, \theta) - \tfrac{a_{10}}{a_9} \right) g_1^x(x_0, \theta + p) \Big\},
\end{aligned}$$

with $(a_i)_{i \in \{1,\dots,10\}}$ are the same quantities as in theorem 2.1.

Remark 2.6

a) If (X_t) is a diffusion semimartingale, to compute the following probability

$$\mathbb{P}_{x_0}\left(X_T > z, X_T^W \in \frac{dx}{\sigma^2(x)}, l_T^{X_T^W} \in du\right),$$

we do not need any condition about the scale function.

b) In general, local times (l_t^x) and (L_t^x) do not have the same behavior. For example, with respect the variable x, (l_t^x) is right continuous (see [22]) but (L_t^x) has a continuous version. In the case where the diffusion is a random time change of Brownian motion we have $L_t^x = l_t^x$.

c) A real diffusion is in general not a semimartingale; take for example

$$(X_t) = \left(\sqrt{|W_t|}\right),$$

where (W_t) is a Brownian motion(see [25]). In this case, the local time (l_t^x) defined by the Tanaka formula does not exist.

More precisely, $f(W_t)$ is a semimartingale if and only if f is a difference of two convex functions, see [7].

2.3 Applications

We will distingush two cases:
a) The case of a classical Brownian motion.
b) The case of a Brownian motion with drift.

a) In this part, (X_t) will be a classical one-dimensional Brownian motion starting at zero; its speed measure and scale function are respectively given by $m(dx) = 2dx$, $s(x) = x$; its infinitesimal generator is $L = \frac{\partial^2}{\partial x^2}$. Again we denote $v = \theta + p$. Moreover we can choose

$$g_1^x(y,\theta) = \exp\sqrt{2\theta}\,(y-x),$$

$$g_2^x(y,\theta) = \exp -\sqrt{2\theta}\,(y-x).$$

We observe that

$$\mathbb{P}\left(X_T > z, X_T^W \in dx, L_T^{X_T^W} \in du\right) = dxduv$$

$$\begin{cases} 1_{(z\leq 0<x)}\left(a_1 \exp -\sqrt{2v}x\right) \\ 1_{(x\leq z<0)}\left(a_2 \exp \sqrt{2\theta}z\right) \\ 1_{(z<x\leq 0)}\left(a_1 \exp \sqrt{2\theta}x\right) \\ 1_{(x<0\leq z)}\left(a_2 \exp -\sqrt{2\theta}z\right) \\ 1_{(0<z\leq x)} a_1 \exp -\sqrt{2v}x \\ 1_{(0\leq x<z)} a_3 \exp -\sqrt{2v}x \end{cases},$$

where

$$a_1 = \frac{2\theta\sqrt{2v}\left(1 - e^{-\sqrt{2\theta}(x-z)} + \sqrt{2\theta}\right)}{\left(\sqrt{2v} + \sqrt{2\theta} + 2c\right)^2},$$

$$a_2 = \frac{2v\sqrt{2\theta}}{\left(\sqrt{2v} + \sqrt{2\theta} + 2c\right)^2},$$

$$a_3 = \frac{2v\sqrt{2\theta}e^{-\sqrt{2\theta}(x-z)}}{\left(\sqrt{2v} + \sqrt{2\theta} + 2c\right)^2}.$$

b) In this part, let $(B_t, t \geq 0)$ be a classical Brownian motion. Let $\sigma \in \mathbb{R}^+$, $\mu \in \mathbb{R}$ and define $X_t = \sigma B_t + \mu t$ with $X_0 = x_0$. In this case the diffusion is a semimartingale with $L = \frac{\sigma^2}{2}\frac{\partial}{\partial x^2} + \mu\frac{\partial}{\partial x}$ as infinitesimal generator associated. We can choose,

$$g_1^x(y,\theta) = \exp \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\theta}}{\sigma^2}(y - x)$$

$$g_2^x(y,\theta) = \exp \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2\theta}}{\sigma^2}(y - x)$$

Let $\overline{T}$ be a random variable independent of the Brownian motion and having an exponential law with parameter θ.
The relation between the two local times is given by

$$l_t^x = 2L_t^x \exp \frac{2\mu x}{\sigma^2}.$$

We have

$$(i) \quad \mathbb{E}_{x_0}\left[1_{(X_{\overline{T}}>z)} \exp -\left(pM_{\overline{T}}(x) + cl_{\overline{T}}^x\right)\right] =$$

$$\begin{cases} 1_{(z \leq x_0 < x)} \left(\lambda_1 \exp r_1 (x_0 - x) + \lambda_2 \exp r_2 (x_0 - z) + \frac{\theta}{\theta + p}\right) \\ 1_{(x \leq z < x_0)} (\lambda_3 \exp r_3 (x_0 - z) + 1) \\ 1_{(z < x \leq x_0)} (\lambda_4 \exp r_3 (x_0 - x) + 1) \\ 1_{(x < x_0 \leq z)} (\lambda_5 \exp r_4 (x_0 - z) + \lambda_6 \exp r_3 (x_0 - x)) \\ 1_{(x_0 < z \leq x)} \lambda_7 \exp r_1 (x_0 - z) \\ 1_{(x_0 \leq x < z)} \lambda_8 \exp r_1 (x_0 - x) \end{cases}$$

where,

$$r_1 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 (\theta + p)}}{\sigma^2},$$

$$r_2 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 (\theta + p)}}{\sigma^2},$$

$$r_3 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 \theta}}{\sigma^2},$$

$$r_4 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 \theta}}{\sigma^2},$$

$$\lambda_2 = \frac{\theta r_1}{(\theta + p)(r_2 - r_1)},$$

$$\lambda_5 = \frac{r_3}{r_3 - r_4},$$

$$\lambda_7 = \lambda_1 \exp r_1 (z - x) + \lambda_2 + \frac{\theta}{\theta + p},$$

$$\lambda_4 = \frac{1}{(r_3 - r_1 - 2c)} \left[r_1 \left(1 - \frac{\theta}{\theta + p}\right) + 2c + \frac{\theta r_1}{\theta + p} e^{r_2 (x - z)} \right],$$

$$\lambda_6 = \frac{-2c + r_4 - r_1}{2c + r_1 - r_3} \lambda_5 e^{r_4 (z - x)},$$

$$\lambda_1 = 1 - \frac{\theta}{\theta + p} + \lambda_4 - \lambda_2 e^{r_2 (x - z)},$$

$$\lambda_8 = \lambda_5 \left(\frac{2c - r_4 + r_1}{2c + r_1 - r_3} + 1 \right) e^{r_4 (x - z)},$$

$$\lambda_3 = -1 + \lambda_5 \left(1 - \frac{2c + r_1 - r_4}{2c + r_1 - r_5} e^{(r_3 - r_4)(x - z)} \right).$$

The joint distribution of $\left(X_t, l_t^{X_t^\alpha}, X_t^\alpha\right)$.

For $z \in \mathbb{R}$ and $\alpha, t \in \mathbb{R}^+$ such that $0 < \alpha < t$, we have

$$\mathbb{P}_{x_0}\left[X_t > z, l_t^{X_t^\alpha} \in du, X_t^\alpha \in \frac{dx}{\sigma^2}\right] = du\frac{dx}{\sigma^2}$$

$$\left[1_{(z<x_0\leq x, x_o<x<z)} \frac{u}{2\sigma^2} e^{\frac{\mu}{\sigma^2}(x-x_0)} \left\{F_1\left(t-\alpha, \frac{2(x_0-x)-u}{2\sigma^2}\right) G\left(t,\mu,-\frac{u}{2\sigma^2}\right)\right.\right.$$

$$+F_1\left(t,-\frac{u}{2\sigma^2}\right) + G\left(t-\alpha,-\mu,\frac{2(x_0-x)-u}{2\sigma^2}\right)$$

$$\left.-e^{\frac{\mu}{\sigma^2}(z-x)} G\left(t-\alpha,-\mu,\frac{2(x_0+z)-4x-u}{2\sigma^2}\right)\right\}$$

$$+1_{(x_0\geq z\geq x)} \frac{u}{2\sigma^2} e^{\frac{\mu}{\sigma^2}(z-x_0)} F_1\left(t-\alpha,\frac{-u}{2\sigma^2}\right) G\left(t,\mu,\frac{4x-2(x_0+z)-u}{2\sigma^2}\right)$$

$$+1_{(x_0>x>z)} \frac{u}{2\sigma^2} e^{\frac{\mu}{\sigma^2}(x_0-x)} \left\{F_1\left(t,\frac{2(x-x_0)-u}{2\sigma^2}\right) G\left(t-\alpha,\mu,\frac{-u}{2\sigma^2}\right)\right.$$

$$-F_1\left(t-\alpha,\frac{-u}{2\sigma^2}\right) + G\left(t,\mu,\frac{2(x-x_0)-u}{2\sigma^2}\right)$$

$$\left.+e^{\frac{\mu}{\sigma^2}(z-x)} F_1\left(t,\frac{2(x-x_0)-u}{2\sigma^2}\right) G\left(t-\alpha,\mu,\frac{2(z-x)-u}{2\sigma^2}\right)\right\}$$

$$+1_{(x\leq x_0<z)} \frac{u}{2\sigma^2} e^{\frac{\mu}{\sigma^2}(z-x_0)} F_1\left(t-\alpha,\frac{-u}{2\sigma^2}\right) G\left(t,\mu,\frac{4x-2(z+x_0)-u}{2\sigma^2}\right)$$

$$\left.+1_{(x_0<x<z)} \frac{u}{2\sigma^2} e^{\frac{\mu}{\sigma^2}(z-x_0)} F_1\left(t-\alpha,\frac{2(x-x_0)-u}{2\sigma^2}\right) G\left(t,\mu,\frac{2(x-z)-u}{2\sigma^2}\right)\right]$$

where

$$F_1(t,a) = \frac{-2\sqrt{2}a\sigma}{t^{\frac{3}{2}}\sqrt{\pi}} \exp-\left(\frac{(a\sigma)^2}{4t} + \frac{\mu^2 t}{\sigma^2}\right),$$

$$G(t,\mu,a) = \frac{\sigma}{\sqrt{\pi}s^{\frac{3}{2}}} \int_0^t \left[\frac{\sigma^2}{\sqrt{2}s}\left(a^2 - \frac{2s}{\sigma^2}\right) + 2\sqrt{2}a\mu\right] e^{-\left(\frac{(a\sigma)^2}{4s} + \frac{s\mu^2}{\sigma^2}\right)} ds.$$

Applying result $(***)$ in subsection 2.2 with (X_t) being a Brownian motion with drift and inverting the Laplace transformation with respect to θ and $\theta + p$, we establish the result.

3 The pricing of the quantile lookback options $(S_\tau - KS_\tau^\alpha)^+$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ an underlying probability space. In this section we consider the Black-Scholes model [4]. (X_t) will be the Brownian motion with drift considered in section 2.3. More particularly we have $\mu = r - \frac{\sigma^2}{2}$ with variance coefficient σ^2 and $X_0 = 0$; $(\mathcal{F}_t)$ will be the filtration generated by (X_t).
The market consists of one stock whose price is $S_t = S_0 \exp(X_t)$ and a deterministic bond $R_t = R_0 e^{rt}$; r is a constant and S_0 is the initial price. The maturity will be a positive number τ. The considered probability $\mathbb{P}$ is in fact the neutral-risk probability measure under which the discounted stock price $S_t e^{-rt}$ is a martingale. We recall that $S_t^\alpha = S_0 \exp(X_t^\alpha)$ is the quantile associated with (S_t).
It is well known that, under no-arbitrage considerations, see e.g. Harrison [16], the price of $(S_\tau - KS_\tau^\alpha)^+$ at a fixed time $t\,(0 \le t \le \tau)$ is provided by

$$e^{-r(\tau - t)} \mathbb{E}\left[(S_\tau - KS_\tau^\alpha)^+ / \mathcal{F}_t \right],$$

In order to calculate this expression we introduce some preliminary results.

Lemma 3.1 *For $z, x \in \mathbb{R}$, $t < \tau$ and $p \in \mathbb{R}^+$ we have,*

$$\mathbb{E}_{x_0}\left[1_{(X_\tau > z)} l_\tau^x e^{-pM_\tau(x)} /_{\mathcal{F}_t} \right] = l_t^x e^{-pM_t(x)} \mathbb{E}_{X_t}\left[1_{(X_{\tau-t} > z)} e^{-pM_{\tau-t}(x)} \right]$$

$$+ e^{-pM_t(x)} E_{X_t}\left[1_{(X_{\tau-t} > z)} l_{\tau-t}^x e^{-pM_{\tau-t}(x)} \right].$$

Proof. We start observing that

$$\mathbb{E}_{x_0}\left[1_{(X_\tau > z)} l_\tau^x e^{-pM_\tau(x)} /_{\mathcal{F}_t} \right] = - \frac{\partial \mathbb{E}_{x_0}\left[1_{(X_\tau > z)} e^{-(pM_\tau(x) + c l_\tau^x)} /_{\mathcal{F}_t} \right]}{\partial c}_{|c=0}$$

We recall that (l_t^x) and $(M_t(x))$ are additive functional of the Brownian motion with drift. Using the strong Markov property of (X_t) we have

$$\mathbb{E}_{x_0}\left[1_{(X_\tau > z)} e^{-(pM_\tau(x) + c l_\tau^x)} / \mathcal{F}_t \right] =$$

$$\mathbb{E}_{x_0}\left[e^{-(pM_t(x) + c l_t^x)} \left(1_{(X_{\tau-t} > z)} e^{-\left(pM_{\tau-t}(x) + c l_{\tau-t}^x\right)} \right) \theta_t / \mathcal{F}_t \right] =$$

$$e^{-(pM_t(x) + c l_t^x)} \mathbb{E}_{X_t}\left[1_{(X_{\tau-t} > z)} e^{-\left(pM_{\tau-t}(x) + c l_{\tau-t}^x\right)} \right];$$

thus

$$\mathbb{E}_{x_0}\left[1_{(X_\tau>z)} l_\tau^x e^{-pM_\tau(x)}/\mathcal{F}_t\right] = l_t^x e^{-pM_t(x)} \mathbb{E}_{X_t}\left[1_{(X_{\tau-t}>z)} e^{-pM_{\tau-t}(x)}\right]$$

$$+e^{-pM_t(x)} \mathbb{E}_{X_t}\left[1_{(X_{\tau-t}>z)} l_{\tau-t}^x e^{-pM_{\tau-t}(x)}\right].$$

■

In the following, we recall that $\mu = r - \frac{\sigma^2}{2}$.

Theorem 3.2 *For $(0 \le t \le \tau)$ and $\alpha < t$ we have the following.*
a)

$$e^{-r(\tau-t)} \mathbb{E}_{x_0}\left[(S_\tau - K S_\tau^\alpha)^+ / \mathcal{F}_t\right] =$$

$$e^{-r(\tau-t)} \Bigg\{ \left(S_t - K S_t^{\alpha+t-\tau}\right) 1_{\left(K e^{X_t^{\alpha+t-\tau}} < S_t < e^{X_t^{\alpha+t-\tau}}\right)}$$

$$+ \left(S_t - (K \vee 1) S_t^\alpha\right) 1_{\left(K e^{X_t^{\alpha+t-\tau}} < S_t < e^{X_t^{\alpha+t-\tau}}, \alpha<t\right)}$$

$$+ (1-K)^+ S_t^\alpha 1_{(S_t > (K\vee 1) S_t^\alpha, \alpha<t)}$$

$$+S_0 \int_{\mathbb{R}} \int_{x+\ln K}^{\infty} 1_{(\alpha > M_t(x))} \left[l_t^x \Phi_{\mu,\sigma}\left(X_t, x, z, \tau-t, \alpha - M_t(x)\right)\right.$$

$$\left.+\Psi_{\mu,\sigma}\left(X_t, x, z, \tau-t, \alpha - M_t(x)\right)\right] e^z dz \frac{dx}{\sigma^2} \Bigg\}$$

b)

$$e^{-r\tau} \mathbb{E}_0\left[(S_\tau - K S_\tau^\alpha)^+\right] =$$

$$e^{-r\tau} S_0 \int_{\mathbb{R}} \int_{x+\ln K}^{\infty} \Psi_{\mu,\sigma}(0, x, z, \tau, \alpha) e^z dz \frac{dx}{\sigma^2}$$

where,

$\Phi_{\mu,\sigma}(y, x, z, t, \alpha) =$

$1_{(y<x<z)} \frac{1}{2} e^{\frac{\mu(z-y)}{\sigma^2}} \left[F_1\left(t-\alpha, \frac{y-x}{\sigma^2}\right) H\left(t, \mu, \frac{x-z}{\sigma^2}\right) - \int_{\mathbb{R}^+} F_2\left(t-\alpha, \frac{y-x}{\sigma^2} - v\right)\right.$

$$H\left(t,\mu,\tfrac{x-z}{\sigma^2}-v\right)+F_1\left(t-\alpha,\tfrac{y-x}{\sigma^2}-v\right)G\left(t,\alpha,\tfrac{x-y}{\sigma^2}-v\right)dv\Big]$$

$$+1_{(y<z<x)}e^{\frac{\mu(z-y)}{\sigma^2}}\left\{\int_0^{t-\alpha}F_1\left(s,\tfrac{y-z}{\sigma^2}\right)ds-\tfrac{1}{2}H\left(t-\alpha,\mu,\tfrac{y-z}{\sigma^2}\right)+e^{\frac{\mu(x-z)}{\sigma^2}}\right.$$

$$\Big[\int_\alpha^t F_1\left(s-\alpha,\tfrac{y-x}{\sigma^2}\right)ds-\int_0^{t-\alpha}F_1\left(s,\tfrac{y-x}{\sigma^2}\right)ds+\int_{\mathbb{R}^+}dv\int_0^t F_1(s,-v)\,ds$$

$$\left(\mu F_1\left(t-\alpha,\tfrac{y-x}{\sigma^2}-v\right)-F_2\left(t-\alpha,\tfrac{y-x}{\sigma^2}-v\right)\right)+\int_{\mathbb{R}^+}dvF_1(t,-v)$$

$$\left(G\left(t-\alpha,-\mu,-v+\tfrac{y-x}{\sigma^2}\right)-e^{\frac{\mu(z-x)}{\sigma^2}}G\left(t-\alpha,-\mu,-v+\tfrac{z+y-2x}{\sigma^2}\right)\right)$$

$$\left.+e^{\frac{\mu(z-x)}{\sigma^2}}H\left(t-\alpha,-\mu,-v+\tfrac{z+y-2x}{\sigma^2}\right)\Big]\right\}$$

$$+1_{(z<x<y)}e^{\frac{\mu(x-y)}{\sigma^2}}\Big\{\int_{\mathbb{R}^+}dv\left(\mu F_1(t-\alpha,-v)-F_2(t-\alpha,-v)\right)$$

$$\int_0^t F_1\left(s,-v+\tfrac{x-y}{\sigma^2}\right)ds-F_1\left(t,-v+\tfrac{x-y}{\sigma^2}\right)G(t-\alpha,-\mu,-v)$$

$$-e^{\frac{\mu(z-x)}{\sigma^2}}F_1\left(t,-v+\tfrac{x-y}{\sigma^2}\right)G\left(t-\alpha,-\mu,-v+\tfrac{z-x}{\sigma^2}\right)$$

$$+1_{(z<y<x)}\tfrac{1}{2}e^{\frac{\mu(z-y)}{\sigma^2}}\left[H\left(t-\alpha,-\mu,\tfrac{z+y-2x}{\sigma^2}\right)-H\left(t-\alpha,-\mu,\tfrac{z-y}{\sigma^2}\right)\right]$$

$$+e^{\frac{\mu(x-y)}{\sigma^2}}\left\{\int_\alpha^t F_1\left(s-\alpha,\tfrac{y-x}{\sigma^2}\right)ds-\int_0^{t-\alpha}F_1\left(s,\tfrac{y-x}{\sigma^2}\right)-\left[\int_0^{t-\alpha}F_1(s,-v)\,ds\right.\right.$$

$$\left(-\mu F_1\left(t-\alpha,-v-\tfrac{x-y}{\sigma^2}\right)+F_2\left(t-\alpha,-v+\tfrac{x-y}{\sigma^2}\right)\right)-F_1(t,-v)$$

$$\left.\left.\left(G\left(t-\alpha,-\mu,-v+\tfrac{y-x}{\sigma^2}\right)+e^{\frac{\mu(z-x)}{\sigma^2}}G\left(t-\alpha,-\mu,-v+\tfrac{z+y-2x}{\sigma^2}\right)\right)\right]\right\}$$

$$+1_{(x<z<y)}e^{\frac{\mu(z-y)}{\sigma^2}}\left\{\tfrac{1}{2}H\left(t,\mu,\tfrac{z-y}{\sigma^2}\right)-\int_0^t F_1\left(s,\tfrac{z-y}{\sigma^2}\right)ds+\tfrac{1}{2}e^{\frac{\mu(z-x)}{\sigma^2}}\right.$$

$$\left.\left[\int_{\mathbb{R}^+}dvF_1(t-\alpha,-v)\,G\left(t,\mu,-v+\tfrac{x-y}{\sigma^2}\right)-F_2(t-\alpha,-v)\,H\left(t,\mu,\tfrac{x-y}{\sigma^2}-v\right)\right]\right\}$$

$$+1_{(x<y<z)}\tfrac{1}{2}e^{\frac{\mu(z-y)}{\sigma^2}}\left\{H\left(t,\mu,\tfrac{z-y}{\sigma^2}\right)+e^{\frac{\mu(z-x)}{\sigma^2}}\left[\int_{\mathbb{R}^+}dvF_1(t-\alpha,-v)\right.\right.$$

$$\left.\left.G\left(t,\mu,-v+\tfrac{2x-y-z}{\sigma^2}\right)+F_2(t-\alpha,-v)\,H\left(t,\mu,\tfrac{2x-y-z}{\sigma^2}-v\right)\right]\right\}$$

and

$\Psi_{\mu,\sigma}(y,x,z,t,\alpha)=$

$$1_{(y<z<x\quad z<y<x)}2e^{\frac{\mu(x-y)}{\sigma^2}}\left\{\int_{\mathbb{R}^+}dvv\left[F_1(t,-v)\left(G\left(t-\alpha,-\mu,-v+\tfrac{y-x}{\sigma^2}\right)\right.\right.\right.$$

$$\left.-e^{\frac{\mu(z-x)}{\sigma^2}}G\left(t-\alpha,-\mu,-v+\tfrac{z+y-2x}{\sigma^2}\right)\right)$$

$$\left.\left.+G(t,\mu,-v)\,F_1\left(t-\alpha,-v+\tfrac{y-x}{\sigma^2}\right)\right]\right\}$$

$$-1_{(x<z<y)}2e^{\frac{\mu(y-z)}{\sigma^2}}\int_{\mathbb{R}^+}vF_1(t-\alpha,-v)\,G\left(t,\mu,-v+\tfrac{-y-z+2x}{\sigma^2}\right)dv$$

$+1_{(z<x<y)}2e^{\frac{\mu(x-y)}{\sigma^2}}\left\{\int_{\mathbb{R}+}v\left[F_1\left(t-\alpha,-v\right)G\left(t,\mu,-v+\frac{-y+x}{\sigma^2}\right)dv\right.\right.$

$-e^{\frac{\mu(z-x)}{\sigma^2}}F_1\left(t,-\mu,-v+\frac{x-y}{\sigma^2}\right)G\left(t-\alpha,-\mu,-v+\frac{z-x}{\sigma^2}\right)\Big\}$

$+G\left(t,-\mu,-v+\frac{x-y}{\sigma^2}\right)F_1\left(t-\alpha,-v\right)\Big]$

$-1_{(x<y<z)}2e^{\frac{\mu(x-y)}{\sigma^2}}\int_{\mathbb{R}+}vF_1\left(t-\alpha,-v\right)G\left(t,\mu,-v+\frac{-y+2x-z}{\sigma^2}\right)dv$

$e^{\frac{\mu(z-y)}{\sigma^2}}\int_{\mathbb{R}+}vF_1\left(t-\alpha,-v-\frac{-y+x}{\sigma^2}\right)G\left(t,\mu,-v+\frac{x-z}{\sigma^2}\right)dv$

with

$$F_1(t,a)=\frac{-2\sqrt{2}\sigma a}{\sqrt{\pi}t^{\frac{3}{2}}}e^{-\left(\frac{(a\sigma)^2}{2t}+\frac{\mu^2t}{2\sigma^2}\right)}$$

$$F_2(t,a)=\frac{\sigma^3}{\sqrt{2\pi}t^{\frac{5}{2}}}\left(a^2-\frac{t}{\sigma^2}\right)e^{-\left(\frac{(a\sigma)^2}{2t}+\frac{\mu^2t}{2\sigma^2}\right)}$$

$$F_3(t,a)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\left(\frac{(a\sigma)^2}{2t}+\frac{\mu^2t}{2\sigma^2}\right)}$$

$$H\left(t,\mu,a\right)=\int_0^t\mu F_3(s,a)+F_1(s,a)ds$$

$$G\left(t,\mu,a\right)=\int_0^t\mu F_1(s,a)+F_2(s,a)ds$$

Proof.

a) For $p\in\mathbb{R}^+$, we have

$$\int_{\mathbb{R}+}\mathbb{E}_{x_0}\left[(S_\tau-KS_\tau^\alpha)^+/_{\mathcal{F}_t}\right]e^{-\alpha p}d\alpha$$

$$=\int_{\mathbb{R}+}\mathbb{E}_{x_0}\left[(S_\tau-KS_0\exp X_\tau^\alpha)^+/_{\mathcal{F}_t}\right]e^{-\alpha p}d\alpha$$

$$=\mathbb{E}_{x_0}\left[\left(\int_{\mathbb{R}}(S_\tau-KS_0e^x)^+l_\tau^xe^{-pM_\tau(x)}\frac{dx}{\sigma^2}\right)/\mathcal{F}_t\right]$$

$$=\mathbb{E}_{x_0}\left[\left(\int_{\mathbb{R}}\int_{KS_0\exp x}^\infty 1_{(S_\tau>z)}l_\tau^xe^{-pM_\tau(x)}dz\frac{dx}{\sigma^2}\right)/\mathcal{F}_t\right]$$

$$=\mathbb{E}_{x_0}\left[\left(S_0\int_{\mathbb{R}}\int_{x+\ln K}^\infty 1_{(X_\tau>z)}l_\tau^xe^{-pM_\tau(x)}e^zdz\frac{dx}{\sigma^2}\right)/\mathcal{F}_t\right]$$

$$=\int_{\mathbb{R}}\int_{x+\ln K}^\infty S_0\mathbb{E}_{x_0}\left(\left(1_{(X_\tau>z)}l_\tau^xe^{-pM_\tau(x)}\right)/\mathcal{F}_t\right)e^zdz\frac{dx}{\sigma^2}$$

$$= S_0 \int_{\mathbb{R}} \int_{x+\ln K}^{\infty} \mathbb{E}_{x_0} \left(\left(1_{(X_\tau > z)} l_\tau^x e^{-pM_\tau(x)} \right) / \mathcal{F}_t \right) e^z dz \frac{dx}{\sigma^2}.$$

Because of lemma 3.1 we have

$$S_0 \int_{\mathbb{R}} \int_{x+\ln K}^{\infty} \mathbb{E}_{x_0} \left(\left(1_{(X_\tau > z)} l_\tau^x e^{-pM_\tau(x)} \right) / \mathcal{F}_t \right) e^z dz \frac{dx}{\sigma^2}$$

$$= S_0 \int_{\mathbb{R}} \int_{\ln(Ke^x)}^{\infty} \left[l_t^x e^{-pM_t(x)} \mathbb{E}_{X_t} \left(1_{(X_{\tau-t} > z)} e^{-pM_{\tau-t}(x)} \right) \right.$$

$$\left. + e^{-pM_t(x)} \mathbb{E}_{X_t} \left(1_{(X_{\tau-t} > z)} l_{\tau-t}^x e^{-pM_{\tau-t}(x)} \right) \right] e^z dz \frac{dx}{\sigma^2}.$$

Using equation (i) in 2.3 we can have explicitly the expressions of

$$\mathbb{E}_{X_t} \left(1_{(X_{\tau-t} > z)} e^{-pM_{\tau-t}(x)} \right), \quad \mathbb{E}_{X_t} \left(1_{(X_{\tau-t} > z)} l_{\tau-t}^x e^{-pM_{\tau-t}(x)} \right).$$

To get the result, we invert the Laplace transformation with respect to p and θ.

b) Using the same method as in a) one obtains

$$\mathbb{E}_{x_0} \left[(S_\tau - K S_\tau^\alpha)^+ \right]$$

$$= \mathbb{E}_{x_0} \left[(S_\tau - K S_0 \exp X_\tau^\alpha)^+ \right]$$

$$= S_0 \int_{\mathbb{R}} \int_{x+\ln K}^{\infty} \mathbb{E}_{x_0} \left(1_{(X_\tau > z)} l_\tau^x e^{-pM_\tau(x)} \right) e^z dz \frac{dx}{\sigma^2}.$$

We remark that

$$\mathbb{E}_{x_0} \left(1_{(X_\tau > z)} l_\tau^x e^{-pM_\tau(x)} \right) = - \frac{\partial \mathbb{E}_{x_0} \left(1_{(X_\tau > z)} e^{-\left(pM_\tau(x) + c l_\tau^x\right)} \right)}{\partial c}_{|c=0};$$

using again equality (i) in subsection 2.3 we can write explicitly the expression of

$$\mathbb{E}_{x_0} \left(1_{(X_\tau > z)} l_\tau^x e^{-pM_\tau(x)} \right).$$

To conclude, we invert again the Laplace transformation with respect to p and θ. ■

Proposition 3.3 *For $\alpha \in \,]0, \tau[$, we have $X_\tau^\alpha - X_\tau = -X_\tau^{\tau-\alpha}$ (in law).*

Proof. Let $x \in \mathbb{R}$, we have

$$\mathbb{P}\left(X_\tau^\alpha - X_\tau < x\right) = \mathbb{P}\left(\alpha < M_\tau\left(x + X_\tau\right)\right)$$

$$= \mathbb{P}\left(\alpha < \int_0^\tau 1_{(X_u < x + X_\tau)} du\right)$$

$$= \mathbb{P}\left(\alpha < \int_0^\tau 1_{(-x < \sigma(B_\tau - B_u + \mu(\tau - u)))} du\right)$$

$$= \mathbb{P}\left(\alpha < \int_0^\tau 1_{(-x < \sigma(B_\tau - B_{\tau - s}) + \mu s)} ds\right)$$

$$= \mathbb{P}\left(\alpha < \tau - \int_0^\tau 1_{(\sigma(B_\tau - B_{\tau - s}) + \mu s < -x)} ds\right)$$

$$= \mathbb{P}\left(\int_0^\tau 1_{(\sigma(B_\tau - B_{\tau - s}) + \mu s < -x)} ds < \tau - \alpha\right) .$$

We remark that $(B'_s) = (B_\tau - B_{\tau - s})$ is a Brownian motion. Associated with (B'_s) we define naturally a Brownian motion with drift by

$$X'_s = \mu s + \sigma B'_s.$$

X'^{α}_s denotes the quantile associated with (X'_s). We write

$$\mathbb{P}\left(\int_0^\tau 1_{(X'_s < -x)} ds < \tau - \alpha\right) = \mathbb{P}\left(X'^{\tau-\alpha}_\tau > -x\right).$$

Finally, we have $\mathbb{P}\left(X_\tau^\alpha - X_\tau < x\right) = \mathbb{P}\left(-X'^{\tau-\alpha}_\tau < x\right)$. ■

Remark 3.4

The reader may find unnatural to use local time in order to compute the price of a quantile lookback options.
However, since the quantile of a diffusion is not a Markov process, the knowledge of the distribution of (X_τ^α, X_τ) is not sufficient to compute the price

$$\mathbb{E}_{x_0}\left[\left(S_\tau - K S_0 \exp X_\tau^\alpha\right)^+ / \mathcal{F}_t\right].$$

For this reason we need to introduce the local time in theorem 2.1.

3.1 Barrier options.

We consider again the market with two assets: the riskless bond

$$S_t^0 = e^{rt}$$

and the risky asset given by

$$dS_t = S_t \left(\mu dt + \sigma dX_t\right).$$

Here (X_t) is a standard Brownian motion under the risk-neutral probability $\mathbb{P}$, $\mu = r - \frac{\sigma^2}{2}$ and we denote by

$$\Psi_{\mu,\sigma}\left(0, x, z, t, \alpha\right) \frac{dx}{\sigma^2} = \mathbb{P}\left(X_t > z, X_t^\alpha \in \frac{dx}{\sigma^2}\right).$$

Theorem 3.2 provides the explicit expression of $\Psi_{\mu,\sigma}\left(0, x, z, t, \alpha\right)$.
A peculiar type of path-dependent option is the so-called **barrier option**. We will consider two cases.

- The first barrier option considered has the following feature: its payoff depends on the quantile and on the terminal price of the underlying asset. It is as $(S_\tau^\alpha - K)^+ 1_{(S_\tau > H)}$, where K is the strike.

  The related option pricing will be denoted by $C_{t,\tau}\left(K/H, \alpha\right)$; we have

$$C_{0,\tau}\left(K/H, \alpha\right) = e^{-r\tau} S_0 \int_K^{+\infty} \left(e^x - K\right) \Psi_{\mu,\sigma}\left(0, x, \ln H, \tau, \alpha\right) \frac{dx}{\sigma^2}$$

- The second considered option will be a quantile dependent barrier whose payoff is $(S_\tau - K)^+ 1_{(X_\tau^\alpha \geq H)}$.

  The option process is denoted by $C_{t,\tau}\left(K/H, X_\tau^\alpha\right)$ and we have

$$C_{0,\tau}\left(K/H, X_\tau^\alpha\right) = e^{r\tau} S_0 \int_H^{+\infty} \int_{\ln\left(\frac{K}{S_0}\right)}^{+\infty} \Psi_{\mu,\sigma}\left(0, x, z, \tau, \alpha\right) e^z dz \frac{dx}{\sigma^2}.$$

  This type of European call option is activated whenever the quantile of the stock price hits a certain barrier before the expiration date.

3.2 Numerical examples

If $K = 1$, the value of the quantile lookback option at time $t = 0$ is equal to

$$C = e^{-r\tau} \mathbb{E}\left[\left(S_\tau - S_\tau^\alpha\right)^+\right]$$

$$
= \frac{4\sigma^3 S_0}{\sqrt{\pi}} \left\{ 16\sigma^3 e^{\frac{\alpha\left(\mu+\sigma^2\right)}{2\sigma^2}} \left[\int_\alpha^\tau ds \frac{e^{-\frac{\mu s}{2\sigma^2}}}{\sqrt{\alpha\left(s-\alpha\right)}} \left(\int_{-\frac{\left(\mu+\sigma^2\right)\sqrt{s-\alpha}}{2\sigma}}^{+\infty} dv \right.\right.\right.
$$

$$
\left(v + \frac{\left(\mu+\sigma^2\right)\sqrt{s-\alpha}}{\sqrt{2}\sigma}\right) e^{-\frac{v^2}{2}} \psi \left(\frac{2v\sigma\sqrt{s-v} + \sqrt{2}\alpha\left(\mu+\sigma^2\right)}{\sigma\sqrt{\alpha}} \right) + \frac{2\sigma^2\left(2\mu+\sigma^2\right)}{\sqrt{\alpha\left(s-\alpha\right)}}
$$

$$
\left.\left.\int_{\mathbb{R}+} dwwe^{-2w\left(\mu+\sigma^2\right)} \psi \left(\frac{w\sigma}{2\sqrt{\left(s-\alpha\right)}} \right) \psi \left(\frac{w\sigma^2 + \left(\mu+\sigma^2\right)\alpha}{2\sigma\sqrt{\alpha}} \right)\right)\right]
$$

$$
+ \frac{1}{\left(\mu+\sigma^2\right)} \int_\alpha^\tau dse^{-\frac{\mu s}{2\sigma^2}} \int_{\mathbb{R}+} \left[g\left(\frac{\sqrt{2\alpha}v_1}{\sigma}, \frac{s-\alpha}{2\sigma^2} \right) \left(e^{\sqrt{2\alpha}\sigma\left(\mu+\sigma^2\right)} - \frac{1}{\sigma^2} \right) \right.
$$

$$
\left(\mu + \frac{\mu}{\sigma^2} + 1\right) e^{-\left(\mu+\sigma^2\right)\frac{\sqrt{2\alpha}}{\sigma^3}} \int_{\frac{\sqrt{2\alpha}v_1}{\sigma}}^{+\infty} dxe^{\frac{\left(\mu+\sigma^2\right)}{\sigma^2}x} g\left(x, \frac{s-\alpha}{2\sigma^2}\right)
$$

$$
\left. + \frac{\left(\mu + \frac{\mu}{\sigma^2} + 1\right)}{\sigma^2} \int_{-\infty}^0 g\left(\frac{y + \sqrt{2\alpha}\sigma v_1}{\sigma^2}, \frac{s-\alpha}{2\sigma^2} \right) dy \right] dv_1 v_1 e^{-v_1^2} \Bigg\},
$$

where

$$
\psi\left(x\right) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{y^2}{2}} dy,
$$

$$
g\left(x,t\right) = \frac{2y}{\sqrt{\pi t}} e^{-\frac{x^2}{2}},
$$

$$
\mu = r - \frac{\sigma^2}{2}.
$$

- If α tends to zero, the quantile lookback option is a lookback option which was priced by Conze A. and Viswanathan in [8] and this price becomes

$$
C = e^{-r\tau}\mathbb{E}\left[\left(S_\tau - S_\tau^0\right)^+\right]
$$

$$
= S_0 \left[1 - \psi\left(\left(\mu+\sigma^2\right)\sqrt{\tau}\right) + \frac{\sigma^2}{\left(2\mu+\sigma^2\right)} \left(1 - \psi\left(-\left(\mu+\sigma^2\right)\sqrt{\tau}\right)\right) \right.
$$

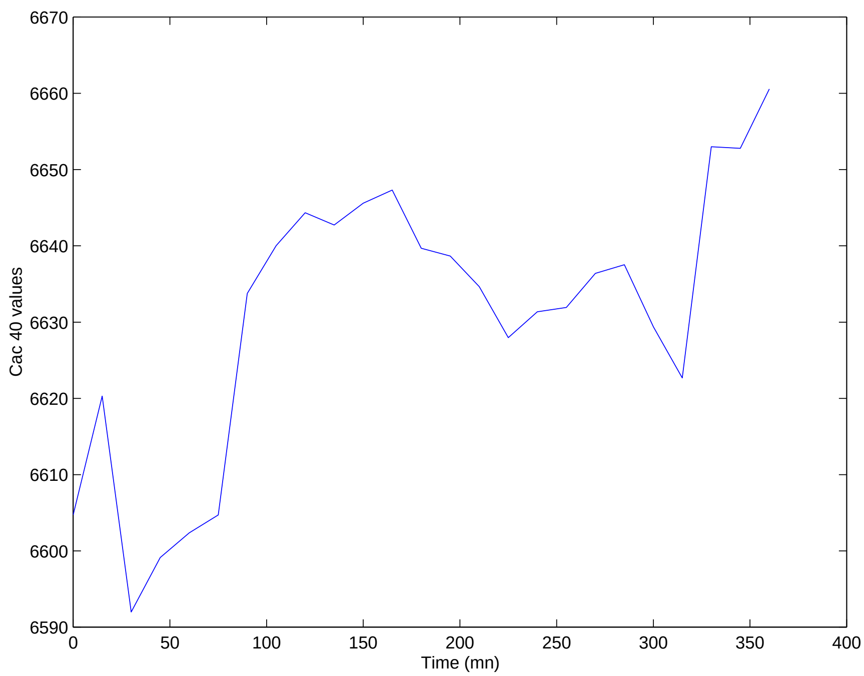
$$
\left. + e^{-r\tau} \left(1 - \psi\left(\frac{\mu\sqrt{\tau}}{\sigma}\right)\right) \left(\frac{2\mu}{2\mu+\sigma^2}\right) \right]
$$

- If α tends to τ, the quantile lookback option is a lookback option related to the maximum, and its price is equal to zero.

Next table provides prices for European path dependent option related to the quantile $(S_\tau - S_\tau^\alpha)^+$.
The short term rate r is 10%, the spot price of the security is $S_0 = 100$ and the volatility is 20%.

α	$C = e^{-r\tau}\mathbb{E}\left[(S_\tau - S_\tau^\alpha)^+\right]$
0	13.352
0.25	7.234
0.5	1.951
0.75	1.256
1	0

Next picture gives the fluctuation of the C.A.C 40 utilities index, Sept. 14 at 9h02 a.m to 14h46 p.m.(data collected in http://www.bourse.aol.fr)



Next table gives us the approximation of the quantile related to the C.A.C 40.
$\tau = 344'$ (minutes)

α	S^{α}_{344}
00′.00”	6591.99
73′.03”	6610.00
89′.29”	6620.00
95′.16”	6630.00

Now, we formulate some propositions how to choose α.

a) In practice, the value S_0 of the risky asset (S_t) at time zero, is known and we can consider it as a reference. Suppose we want the strike price KS^{α}_{τ} less than h-times S_0 $(h > 0)$ with probability $95/100$; α can be determined by the following equation :

$$(*) \quad \mathbb{P}\left(KS^{\alpha}_{\tau} < hS_0\right) = \mathbb{P}\left(X^{\alpha}_{\tau} < \ln\left(\frac{h}{K}\right)\right) = 95/100.$$

Using the joint distribution of $\left(X^{\alpha}_t, X_t, l_t^{X^{\alpha}_t}\right)$ we can deduce the law of X^{α}_{τ} and $(*)$ is equivalent to

$$\Bigg\{\Bigg[1 + \int_0^{\alpha}\int_{\mathbb{R}+} u \int_{\mathbb{R}+}\Bigg(1_{[\tau,v]}(s)\, G_{\mu}\left(u + \frac{\ln\left(\frac{hS_0}{K}\right)}{\sigma^2}, \tfrac{s-v}{2\sigma^2}\right) F\left(u, \tfrac{v}{2\sigma^2}\right) -$$
$$1_{\left[0, \frac{t-\alpha}{2\sigma^2}\right]}(s)\, G_{\mu}\left(u + \frac{\ln\left(\frac{hS_0}{K}\right)}{\sigma^2}, s\right) F\left(u, \tfrac{v}{2\sigma^2}\right)\Bigg) e^{\frac{-\mu^2 s}{2\sigma^2}}\, dsdudv\Bigg] 1_{(hS_0>K)}$$
$$+ \Bigg[1 + \int_0^{\alpha}\int_{\mathbb{R}+} u \Bigg(\int_0^{\frac{\tau-\alpha}{2\sigma^2}} G_{-\mu}(u,s)\, F\left(u - \frac{\ln\left(\frac{hS_0}{K}\right)}{\sigma^2}, \frac{v}{2\sigma^2}\right) e^{\frac{-\mu^2 \tau}{2\sigma^2}}$$
$$- \int_v^{\tau} e^{\frac{-\mu^2 s}{2\sigma^2}} F\left(u - \tfrac{x}{\sigma^2}, \tfrac{s-v}{2\sigma^2}\right) G_{-\mu}\left(u, \tfrac{v}{\sqrt{2}\sigma}\right) dsdudv\Bigg)\Bigg] 1_{(hS_0<K)}\Bigg\}\left(\tfrac{K}{hS_0}\right)^{\frac{\mu}{\sigma^2}}$$
$$= 0.05$$

where

$$F(a,b) = \left(4\pi b^3\right)^{\frac{-1}{2}} a e^{\frac{-a^2}{4b}},$$

$$G_{\mu}(a,b) = \mu F(a,b) + \frac{\partial F(a,b)}{\partial a},$$

$$\mu = r - \frac{\sigma^2}{2}.$$

b) Suppose we want to have a strike price KS^{α}_{τ} less than S_{τ} with probability $95/100$. Then α will solve the following equation

$$\mathbb{P}\left(KS^{\alpha}_{\tau} < S_{\tau}\right) = \mathbb{P}\left(X^{\alpha}_{\tau} - X_{\tau} < -\ln K\right)$$

$$= \mathbb{P}\left(X_\tau^{\tau-\alpha} > \ln K\right) = 95/100.$$

By using the same argument as in a) we can deduce α.

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