A NOTE ON A SERIES OF PAPERS ON RELATIVISTIC δ-SPHERE INTERACTIONS IN QUANTUM MECHANICS PUBLISHED BY M.N. HOUNKONNOU AND G.Y.H. AVOSSEVOU IN THE JOURNAL OF MATHEMATICAL PHYSICS

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Abstract

In this note, we show that the definitions proposed in J.Math.Phys 41, 24-39(2000); J.Math.Phys 41, 1718-1734(2000) and J.Math.Phys 41, 1735-1744(2000) for the description of relativistic δ-sphere interactions and its various generalizations are wrong.

1 INTRODUCTION

In the last two decades, several researchers were attracted by the study of δ-sphere interactions in quantum mechanics both from the mathematical point of view and for their application in modelling various physical phenomena (see e.g [1]-[8] and references therein).

For a long time, most of the studies were focusing on nonrelativistic interactions. The first rigorous mathematical definition and analysis of relativistic δ-sphere interactions were provided in [4].

More recently, Hounkonnou and Avossevou [9]-[11] published a series of papers in the Journal of Mathematical Physics on relativistic δ, δ' and finitely many δ-sphere interactions in quantum mechanics. Unfortunately, all the results presented in these papers are wrong. Indeed, although the authors refer to [4, 6], the definitions proposed in [9]-[11] for the description of relativistic δ-sphere interactions are wrong. Furthermore, the papers contain several other mistakes due to a misunderstanding of some basic concepts of the theory of self-adjoint extensions of symmetric closed operators in Hilbert spaces.

The purpose of this note is to show that the definitions proposed in [9]-[11] do not correspond to any relativistic δ-sphere interaction.

Let us consider the quantum Hamiltonian describing a relativistic δ-sphere interaction formally given in three dimensions by:

$$H_G = H_D + G\delta(|x| - R); \quad x \in \mathbb{R}^3, \quad R \in (0, \infty)$$

where G is a 2×2 matrix of the form $G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$; A, $B \in \mathbb{R}$ and H_D is the Dirac Hamiltonian.

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In Section 2, we briefly recall the definition of H_G using the theory of self-adjoint extensions of symmetric closed operators in Hilbert spaces and in Section 3 we review and analyze the definitions proposed in [9]-[11] for H_G and its generalizations, in the special cases $A = \alpha \neq 0$, $B = 0$ and $A = 0$, $B = \alpha \neq 0$.

2 DEFINITION OF THE δ-SPHERE INTERACTION IN RELATIVISTIC QUANTUM MECHANICS.

Consider the Dirac Hamiltonian with a δ-sphere interaction formally given in three dimensions by:

$$H_G = H_D + G\delta(|x| - R); \quad x \in \mathbb{R}^3, \quad R \in (0, \infty) \tag{1}$$

where G is a 2×2 matrix of the form $G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$; $A,\ B \in \mathbb{R}$ and H_D is the Dirac Hamiltonian defined in $\mathcal{H} = L^2(\mathbb{R}^3) \bigotimes \mathbb{C}^4$ by

$$H_D = -i\underline{\underline{\alpha}}\nabla + \underline{\underline{\beta}}\frac{c^2}{2}; \tag{2}$$

$$\mathcal{D}(H_D) = H^{1,2}(\mathbb{R}^3) \bigotimes \mathbb{C}^4. \tag{3}$$

In Eqs (2),(3) c is the velocity of the light, $H^{m,n}(\Omega)$ is the Sobolev space of indices (m, n) and $\underline{\underline{\alpha}}$, $\underline{\underline{\beta}}$ are the 4×4 Dirac matrices.

In the case of rotationally and space reflection symmetric operators, a mathematical definition of H_G in Eq (1) is provided in [4] using the theory of self-adjoint extensions of symmetric closed operators in Hilbert spaces.

As in the case of nonrelativistic point and δ-sphere interactions [1, 12], the definition of H_G is based on a process which involves the following stages.

First we restric the operator H_D to $\overline{H}_D$ defined by:

$$\begin{aligned} \overline{H}_D &= H_D, \\ D(\overline{H}_D) &= \{\psi \in H^{1,2}(\mathbb{R}^3) \bigotimes \mathbb{C}^4 / \psi(S_R) = 0\} \end{aligned} \tag{4}$$

where $S_R = \{x \in \mathbb{R}^3 : |x| = R\}$ is the closed ball of radius R centered at the origin in $\mathbb{R}^3$.

Then we decompose $\mathcal{H}$ with respect to the radial and angular variables and introduce a unitary transformation to remove the weight factor r^2 from the integration measure in $L^2((0, \infty))$.
The above process leads to the following representation of $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} \bigoplus_{\mu=-j}^{j} [U_{jl}^{-1}\tilde{\mathcal{H}}] \bigotimes [\Omega_{jl\mu}(\theta, \varphi)] \tag{5}$$

where

$$U_{jl} \ : \ L^2((0, \infty); r^2 dr) \bigotimes \mathbb{C}^2 \to \tilde{\mathcal{H}} \equiv L^2((0, \infty); dr) \bigotimes \mathbb{C}^2, \tag{6}$$

$$(U_{jl}\psi)(r) = \begin{pmatrix} rf(r) \\ (-1)^{j-l-\frac{1}{2}} rg(r) \end{pmatrix} \tag{7}$$

and $[\Omega_{jl\mu}(n)]$ is the space generated by the spherical spinors[13].

With respect to the decomposition (5),the operator $\overline{H}_D$ reads

$$\overline{H}_D = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} [U_{jl}^{-1} h_{jl} U_{jl}] \bigotimes \mathbb{1} \tag{8}$$

where the operators h_{jl} in $L^2((0,\infty))\bigotimes\mathbb{C}^2$ are given by

$$\begin{aligned} h_{jl} &= \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix} \equiv \tau, & (9)\\ \kappa_{jl} &= (-1)^{j-l+\frac{1}{2}}(j+\frac{1}{2}), & (10)\\ D(h_{jl}) &= \{\psi \in L^2((0,\infty))\bigotimes\mathbb{C}^2 | \psi \in \mathrm{AC}_{loc}((0,\infty));\ \ \psi(R\pm)=0; \\ \tau\psi &\in L^2((0,\infty))\bigotimes\mathbb{C}^2\} & (11)\end{aligned}$$

where $\mathrm{AC}_{loc}(\Omega)$ denotes the set of locally absolutely continuous functions on Ω and

$$f(x\pm) = \lim_{\epsilon\to 0+} f(x\pm\epsilon).$$

A straightforward calculation shows that the adjoint h^*_{jl} of h_{jl} reads:

$$\begin{aligned} h^*_{jl} &= \tau, \\ D(h^*_{jl}) &= \{\psi \in L^2((0,\infty))\bigotimes\mathbb{C}^2 | \psi \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}), \\ \tau\psi &\in L^2((0,\infty))\bigotimes\mathbb{C}^2\}. & (12)\end{aligned}$$

As in the case of nonrelativistic quantum mechanics, the relativistic quantum Hamiltonian $h_{jl,G}$ describing the interaction $V(r) = G\delta(r-R)$ is a self-adjoint extension of h_{jl} defined by $h_{jl,G} = \tau$ on the domain $\mathcal{D}(h_{jl,G})$ containg those functions $\psi \in \mathcal{D}(h^*_{jl})$ which satisfy suitable boundary conditions at $r = R$.

Theorem 2.1 *[4]*
Any self-adjoint extension $\hat{h}_{jl}$ of h_{jl} reads:

$$\begin{aligned} \hat{h}_{jl} &= \tau \\ \mathcal{D}(\hat{h}_{jl}) &= \{\psi \in \mathcal{D}(h^*_{jl}) | C\psi(R-) + D\psi(R+) = 0\} & (13)\end{aligned}$$

where C and D are 2×2 matrices such that the 2×4 matrix (C, D) has rank 2. Conversely any operator of this form is a self-adjoint extension of h_{jl}.

Theorem 2.2 *[4, 8]*
The operator H_G in Eq(1) is defined in $L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^4$ by:

$$H_G = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} U^{-1}_{jl} h_{jl,G} U_{jl} \bigotimes 1\!\!1 \qquad (14)$$

where

$$\begin{aligned} h_{jl,G} &= \tau, \\ D(h_{jl,G}) &= \left\{ g \in \mathcal{D}(h^*_{jl}) \middle| \begin{pmatrix} 1 & \frac{B}{2c} \\ -\frac{A}{2c} & 1 \end{pmatrix} g(R+) - \begin{pmatrix} 1 & -\frac{B}{2c} \\ \frac{A}{2c} & 1 \end{pmatrix} g(R-) = 0 \right\}. & (15)\end{aligned}$$

The special cases $A \neq 0$, $B = 0$ and $A = 0$, $B \neq 0$ yield the relativistic δ-sphere interactions of the first and the second type respectively. Indeed, as indicated in [8], the nonrelativistic limits of the Hamiltonians corresponding to these interactions converge in the norm resolvent topology to the Hamiltonians describing the nonrelativistic δ-sphere interactions of the first and the second type respectively [1, 2, 14].

From Eq(15), it follows that the domains $\mathcal{D}(h_{jl,A})$ and $\mathcal{D}(h_{jl,B})$ corresponding to the relativistic δ-sphere interactions of the first and the second type may be defined as follows:

$$\begin{aligned}\mathcal{D}(h_{jl,A}) &= \left\{\psi = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(h^*_{jl}) | f(R-) = f(R+) \equiv f(R),\right. \\ &\left. g(R+) - g(R-) = \frac{A}{c} f(R)\right\} \end{aligned} \tag{16}$$

$$\begin{aligned}\mathcal{D}(h_{jl,B}) &= \left\{\psi = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(h^*_{jl}) | g(R-) = g(R+) \equiv g(R),\right. \\ &\left. f(R+) - f(R-) = -\frac{B}{c} g(R)\right\}. \end{aligned} \tag{17}$$

3 THE DEFINITIONS PROPOSED BY HOUNKONNOU AND AVOSSEVOU [9]-[11]

A. Relativistic scattering theory for a δ-sphere plus a Coulomb interaction with boundary conditions of second type. J.Math. Phys,41, 24-39(2000)

In [9], Hounkonnou and Avossevou consider H_G in Eq(1) in the special cases $A = \alpha \neq 0, B = 0$ and $A = 0$, $B = \alpha \neq 0$ and propose the following two definitions for $H_\alpha = H_D + \alpha\delta(|x| - R)$ in $L^2(\mathbb{R}^3) \bigotimes \mathbb{C}^4$[[9],Eqs 2.21-2.25]

$$\begin{aligned} \text{i)}\ H_\beta &= \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} [U_{jl}^{-1} h_{jl,\beta_{jl}} U_{jl}] \bigotimes 1\!\!1, \\ & \beta = \alpha = \{\beta_{jl}, j - \frac{1}{2} \leq l \leq j + \frac{1}{2}; \frac{1}{2} \leq j \leq \infty\} \end{aligned} \tag{18}$$

where

$$\begin{aligned} h_{jl,\beta_{jl}} &= \tau \\ \mathcal{D}(h_{jl,\beta_{jl}}) &= \left\{\psi = \begin{pmatrix} f \\ g \end{pmatrix} \in L^2((0,\infty)) \bigotimes \mathbb{C}^2 | f'(R+) = f'(R-),\right. \\ & g' \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}), g'(R+) - g'(R-) = \frac{\beta_{jl}}{c} f'(R) + \tilde{A}, \\ & \left.\tau\psi \in L^2((0,\infty)) \bigotimes \mathbb{C}^2,\ \ \tilde{A} \neq 0,\ -\infty < \beta_{jl} \leq +\infty\right\} \end{aligned} \tag{19}$$

$$\begin{aligned} \text{ii)}\ H_\gamma &= \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} [U_{jl}^{-1} h_{jl,\gamma_{jl}} U_{jl}] \bigotimes 1\!\!1, \\ & \gamma = \alpha = \{\gamma_{jl}, j - \frac{1}{2} \leq l \leq j + \frac{1}{2}; \frac{1}{2} \leq j \leq \infty\} \end{aligned} \tag{20}$$

where

$$\begin{aligned} h_{jl,\gamma_{jl}} &= \tau \\ \mathcal{D}(h_{jl,\gamma_{jl}}) &= \left\{\psi = \begin{pmatrix} f \\ g \end{pmatrix} \in L^2((0,\infty)) \bigotimes \mathbb{C}^2 | g'(R+) = g'(R-),\right. \\ & f' \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}), f'(R+) - f'(R-) = -\frac{\gamma_{jl}}{c} g'(R) + \tilde{B}, \\ & \left.\tau\psi \in L^2((0,\infty)) \bigotimes \mathbb{C}^2,\ \ \tilde{B} \neq 0,\ -\infty < \gamma_{jl} \leq +\infty\right\} \end{aligned} \tag{21}$$

Comments

i) The operators $h_{jl,\beta_{jl}}$ and $h_{jl,\gamma_{jl}}$ are different. Therefore, they can not define the same formal expression H_α

ii) As indicated in Section 2, the relativistic quantum Hamiltonian $h_{jl,\alpha_{jl}}$ describing the interaction $V(r) = \alpha_{jl}\delta(r-R)$ is defined on the domain:

$$\mathcal{D}(h_{jl,\alpha_{jl}}) = \left\{ \psi = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(h^*_{jl}) | \psi \text{ satifies suitable boundary conditions at } r = R \right\}. \tag{22}$$

Therefore, the boundary conditions given in Eqs(19) and (21) which characterize the behaviour of ψ' at $r = R$ instead of that ψ can not define the interaction $V(r) = \alpha_{jl}\delta(r-R)$.

Actually, although the authors recognize in the introduction of [9] that H_α was rigorously defined and analyzed in [4], the definitions proposed in Eqs(19) and (21) are wrong since they do not correspond to any of the operators $h_{jl,A}$ and $h_{jl,B}$ given by Eqs(16) and (17).

iii) The definition of h_{jl} in [[9], Eqs 2.12 and 2.13] is wrong. The operator h_{jl} is rather defined by Eq(11).
Reference [9] contains several other mistakes related to a misunderstanding of some basic concepts of the theory of self-adjoint extensions of symmetric closed operators in Hilbert spaces. As an example, we note that contrary to the statement following [[9],Eq(2.11)], the operator $\dot{h}_{jl}$ is not self-adjoint. It is also worth mentioning that in [9, Eq(2.21)], the case $\beta_{jl} = 0$ does not coincide with the free radial Dirac Hamiltonian and $\beta_{jl} = \infty$ does not lead to the free radial Dirac Hamiltonian with Neumann boundary condition at S_R.

iv) According to theorem 2.1, $h_{jl,\beta_{jl}}$ and $h_{jl,\gamma_{jl}}$ do not define any self-adjoint extension of h_{jl}. Consequently, the conditions required for the application of Krein‘s formula in [[9],(2.26)] are not satisfied.

B. Relativistic scattering theory for finitely many δ-sphere interactions supported by concentric spheres. J. Math. Phys 41, 1735-1744(2000)

In this paper, the authors consider the quantum Hamiltonian describing a finite number of relativistic δ-sphere interactions with support on concentric spheres, formally given in three dimensions by:

$$H_{\{R\}} = H_D + \sum_{m=1}^{N} \alpha_m \delta(|x| - R_m), \quad x \in \mathbb{R}^3, \quad R_m \in (0, \infty). \tag{23}$$

They define $H_{\{R\}}$ in [[11], Eqs(2.21)-(2.24)] just by imposing the boundary conditions given by Eqs (19) and (21) at each point $R_m \in (0,\infty)$. As indicated in section 3A, these boundary conditions do not define any relativistic δ-sphere interaction.

C. Exactly solvable models of δ'-sphere interactions in relativistic quantum mechanics-J.Math. Phys 41, 1718-1734(2000)

In this paper, the authors consider the quantum Hamiltonian corresponding to a finite number of relativistic δ'-sphere interactions with support on concentric spheres, formally given in three dimensions by:

$$H_{\{R\}} = H_D + \sum_{m=1}^{N} \tilde{\alpha}_m \delta'(|x| - R_m), \quad x \in \mathbb{R}^3, R_m \in (0, \infty). \tag{24}$$

The authors propose the following two definitions for $H_{\{R\}}$ in $L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^4$ [[10],Eqs(2.18)-(2.21)]:

$$\text{i)}\ \dot{\overline{H}}_{\tilde{\beta}\{R\}} = \bigoplus_{j=\frac{1}{2}}^{\infty}\bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}}[U_{jl}^{-1}h_{jl,\tilde{\beta}_{jl}\{R\}}U_{jl}]\bigotimes 1\!\!1 \qquad (25)$$

where

$$\begin{aligned}
h_{jl,\tilde{\beta}_{jl}\{R\}} &= \tau \\
\mathcal{D}(h_{jl,\tilde{\beta}_{jl}}) &= \Big\{\psi=\begin{pmatrix} f\\ g\end{pmatrix}\in L^2((0,\infty))\bigotimes\mathbb{C}^2 | f\in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}); \\
& f'\in \mathrm{AC}_{loc}((0,\infty)); g, g'\in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}) \\
& g'(R_{m+})-g'(R_{m-}) = -\frac{\tilde{\beta}_m}{2c}[f(R_{m+})+f(R_{m-})]+A_m; \\
& \tau\psi\in L^2((0,\infty))\bigotimes\mathbb{C}^2,\ \ A_m\neq 0\Big\} \qquad (26)
\end{aligned}$$

$$\text{ii)}\ \dot{\overline{H}}_{\tilde{\gamma}\{R\}} = \bigoplus_{j=\frac{1}{2}}^{\infty}\bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}}[U_{jl}^{-1}h_{jl,\tilde{\gamma}_{jl\{R\}}}U_{jl}]\bigotimes 1\!\!1 \qquad (27)$$

where

$$\begin{aligned}
h_{jl,\tilde{\gamma}_{jl\{R\}}} &= \tau \\
\mathcal{D}(h_{jl,\tilde{\gamma}_{jl}\{R\}}) &= \Big\{\psi=\begin{pmatrix} f\\ g\end{pmatrix}\in L^2((0,\infty))\bigotimes\mathbb{C}^2 | g\in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}) \\
& g'\in \mathrm{AC}_{loc}((0,\infty)), f, f'\in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}); \\
& f'(R_{m+})-f'(R_{m-}) = \frac{\tilde{\gamma}_{jl}}{2c}[g(R_{m+})+g(R_{m-})]-B_m; \\
& \tau\psi\in L^2((0,\infty))\bigotimes\mathbb{C}^2,\ \ B_m\neq 0\Big\} \qquad (28)
\end{aligned}$$

Comments

i) The relationships between $\tilde{\alpha}_m$ in Eqs(24) and $\tilde{\beta}_{jl}\{R\}$, $\tilde{\gamma}_{jl}\{R\}$, A_m and B_m in Eqs(25)-(28) is not clear. Furthermore, the operators defined by Eqs (26) and (28) are different.
Consequently, they can not define the same formal expression $H_{\{R\}}$ in (24).

ii) In the case $m=1$, it follows from theorem 2.1 that none of the Eqs (25) and (27) define a self-adjoint extension of $\dot{\overline{H}}_{\{R\}}$ given by [[10], Eq(2.3)] since the boundary conditions (26) and (28) can not be represented in the form $C\psi(R-)+D\psi(R+)=0$.
Futhermore, the nonrelativistic limit of $h_{jl,\tilde{\beta}_{jl}\{R\}}$ and $h_{jl,\tilde{\gamma}_{jl}\{R\}}$ do not converge in any way to the Hamiltonian $h_{l,\{\alpha\}_l},\{R\}$ defined in [[15],Eq(104)] and corresponding to a finite number of nonrelativistic δ'-sphere interaction with support on concentric spheres.

The above discussion clearly shows that the definitions proposed in [11] do not correspond to the interaction given in Eq(24)

iii) As in the case of section 2.A, we note that [10] contains several mistakes due to a misunderstanding of basic concepts of the theory of self-adjoint extensions. For example, a straightforward comptutation shows that the condition $\psi'\in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\})$ does not belong to the domain $\mathcal{D}(\dot{\overline{h}}^{*}_{jl,\{R\}})$ in [[10],Eq(2.6)].

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References

[1] J.P.Antoine,F.Gesztesy and J.Shabani: J.Phys A 20, 3687-3712 (1987)

[2] J.Shabani, J.Math. Phys 29, 660-664(1988)

[3] L.Dabrowski and J.Shabani, J.Math.Phys 29, 2241-2244(1988)

[4] J.Dittrich, P.Exner and P.Seba, J.Math. Phys.30, 2275-2282(1989)

[5] F.Dominguez-Adame, J.Phys A 23, 1993(1990)

[6] J.Dittrich, P.Exner and P.Seba, J.Math. Phys.33, 2207-2214(1992).

[7] J-P Antoine, P.Exner, P.Seba and J.Shabani, Ann Phys (N.Y). 233, 1-16(1994)

[8] J.Shabani and A. Vyabandi:Exactly solvable models of relativistic δ-sphere interactions in quantum mechanics. Preprint $n^{\underline{0}}$ 00-10-27, Fakultät für Physik, Universität Bielefeld, Germany, October 2000.

[9] G.Y.H. Avossevou and M.N. Hounkonnou, J.Math. Phys 41, 24-39(2000)

[10] M.N. Hounkonnou and G.Y.H. Avossevou , J.Math. Phys 41, 1718-1734(2000)

[11] M.N. Hounkonnou and G.Y.H. Avossevou, J.Math. Phys 41, 1735-1744(2000)

[12] S.Albeverio, F.Gesztesy, R.Høegh-Kron and H.Holden: Solvable models in Quantum Mechanics. Texts and Monographs in Physics.Springer-verlag. Berlin, 1988

[13] L.D.Landau and E.Lifchitz: *Physique Théorique. Electrodynamique quantique*, Editions Mir Moscou, 1979.

[14] J.Shabani, Nuovo Cimento B 101, 429-439(1988)

[15] M.N. Hounkonnou, M. Houmkpe, and J.Shabani, J.Math.Phys.40, 4255-4273 (1999)