EXACTLY SOLVABLE MODELS OF RELATIVISTIC δ-SPHERE INTERACTIONS

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Abstract

We discuss the quantum Hamiltonian H_G describing a δ-sphere interaction introduced in [J. Math. Phys **30**,2275-2288 (1989)] and formally given by $H_G = H_D + G\delta(|x| - R)$, where H_D is the Dirac Hamiltonian and G is a 2×2 matrix defined by $G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $A, B \in \mathbb{R}$. We also consider the case of a δ-sphere plus a Coulomb interaction. We obtain new results for both models, in particular the resolvent equation, the nonrelativistic limit and the various quantities related to the scattering theory.

1 INTRODUCTION

In the last two decades, a lot of research has been carried out on point and sphere interactions in quantum mechanics, both from the mathematical point of view and for their applications in physics [1]-[15].

For a long time, this research focused on nonrelativistic interactions. The first paper providing a rigorous mathematical analysis of relativistic δ sphere interactions was published in 1989 by Dittrich et al [7]. Using the theory of self-adjoint extensions of symmetric closed operators in Hilbert spaces, the paper defines the quantum Hamiltonian corresponding to the formal expression:

$$H_G = H_D + G\delta(|x| - R), x \in \mathbb{R}^3, R \in \mathbb{R} \tag{1}$$

where G is a 2×2 matrix of the form $G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$; $A, B \in \mathbb{R}$ and H_D is the Dirac Hamiltonian. The paper also contains a detailed discussion of the spectral properties of H_G.

In [12], the above results were extended to the case of a δ-sphere plus a Coulomb interaction. In this publication, the authors also provide a numerical analysis of the point spectrum and compare the definition of H_G used in [7, 12] with an alternative definition proposed by Dominguez-Adame[8].

More recently, Hounkonnou and Avossevou published a series of papers in which they claim to provide rigorous mathematical definitions and to perform a systematic study of the δ, δ' and finitely

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many δ-sphere interactions in the particular cases where $A \neq 0,\ B = 0$ and $A = 0,\ B \neq 0$[17]-[19]. Unfortunately, as pointed out in [20], the boundary conditions used in these papers do not correspond to any of the above mentioned δ-sphere interactions. Actually these boundary conditions do not define any self-adjoint operator. Therefore, the conditions required for the application of the Krein Formula and the Weyl theorem for the computation of the resolvent equation and the analysis of spectral properties of H_G are not fulfilled which means that all the results presented in [17]-[19] are false.

The aim of this paper is to provide new results on the relativistic δ-sphere interaction in the general case $A \neq 0$ and $B \neq 0$. We obtain the resolvent equation, the nonrelativistic limit, and the spectral properties and study the scattering theory, both for a pure δ-sphere and a δ-sphere plus a Coulomb interaction.

The special cases $A \neq 0,\ B = 0$ and $A = 0,\ B \neq 0$ yield the relativistic δ-sphere interactions of the first and second type respectively. Indeed, as indicated in[21], the nonrelativistic limits of the Hamiltonians corresponding to these interactions converge in the norm resolvent topology to the Hamiltonians describing the nonrelativistic δ-sphere interactions of the first and the second type respectively[2, 3, 4].

In a series of forthcoming papers [22], we extend the results presented in this publication to the case of a δ' and finitely many sphere interactions and discuss the approximation of H_G by local scaled short range and momentum cutt off Hamiltonians.

2 THE δ-SPHERE INTERACTION

A. Definition of the Hamiltonian

In this section, using the theory of self-adjoint extensions of symmetric closed operators in Hilbert spaces, we provide the mathematical definitions of a quantum Hamiltonian describing a relativistic δ-sphere interaction formally given by:

$$H_G = H_D + G\delta(|x| - R); \quad x, \in \mathbb{R}^3 \ , R \in \mathbb{R} \tag{2}$$

where[7]
(i) G is a 2×2 matrix of the form

$$G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A, B \in \mathbb{R}. \tag{3}$$

The radial operators

Consider in the state Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \bigotimes \mathbb{C}^4$ the Dirac Hamiltonian H_D defined by

$$\begin{aligned} H_D &= -i\underline{\underline{\alpha}}\nabla + \underline{\underline{\beta}}\frac{c^2}{2}; \\ \mathcal{D}(H_D) &= H^{1,2}(\mathbb{R}^3) \bigotimes \mathbb{C}^4, \end{aligned} \tag{4}$$

where we have used the following definitions and notations:
(i) c is the velocity of the light,
(ii) $H^{m,n}(\Omega)$ is the Sobolev space of indices (m, n),
(iii) $\underline{\underline{\alpha}}$ and $\underline{\underline{\beta}}$ are 4×4 Dirac matrices given as

$$\underline{\underline{\alpha}} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \underline{\underline{\beta}} = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}. \tag{5}$$

Here σ are Pauli's spin matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

Consider in $L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^4$ the symmetric closed operator $\overline{H}_D$ defined by:

$$\begin{aligned} \overline{H}_D &= H_D, \\ D(\overline{H}_D) &= \{\psi \in H^{1,2}(\mathbb{R}^3)\bigotimes\mathbb{C}^4/\psi(S_R) = 0\} \end{aligned} \tag{7}$$

where $S_R = \{x \in \mathbb{R}^3 : |x| = R\}$ is the sphere of radius R in $\mathbb{R}^3$ centered at the origin.

As indicated in [7], the operators $\overline{H}_D$ admits a large number of self-adjoint extensions. In this paper, we consider those extensions of $\overline{H}_D$ which correspond to H_G.

For this purpose, we restrict ourselves to those extensions of $\overline{H}_D$ which are rotationally and space-reflection symmetric. With these assumptions, one may decompose the space $\mathcal{H}$ in the following way:

$$\mathcal{H} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} \bigoplus_{\mu=-j}^{j} \mathcal{H}_{jl\mu} \tag{8}$$

where we have used the following definitions and notations:

(i) $\mathcal{H}_{jl\mu} = \left\{\psi \in \mathcal{H} | \psi(r,n) = \begin{pmatrix} f(r)\Omega_{jl\mu}(n) \\ g(r)\Omega_{jl'\mu}(n) \end{pmatrix}; f,g \in L^2((0,\infty), r^2dr)\right\}$ (9)

(ii) $\Omega_{jl\mu}$ are spherical spinors defined by[23]

$$\Omega_{jl\mu}(\theta,\varphi) = \begin{pmatrix} \sqrt{\frac{j+\mu}{2l+1}}Y_{l,\mu-\frac{1}{2}}(\theta,\varphi) \\ \sqrt{\frac{j-\mu}{2l+1}}Y_{l,\mu+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \quad \text{for } l = j - \frac{1}{2}, \tag{10}$$

$$\Omega_{jl\mu}(\theta,\varphi) = \begin{pmatrix} -\sqrt{\frac{j-\mu+1}{2l+1}}Y_{l,\mu-\frac{1}{2}}(\theta,\varphi) \\ \sqrt{\frac{j+\mu+1}{2l+1}}Y_{l,\mu+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \quad \text{for } l = j + \frac{1}{2} \tag{11}$$

(iii) $l' = j \mp \frac{1}{2}$ for $l = j \pm \frac{1}{2}$.

Next we introduce the isomorphisms U_{jl} defined by

$$U_{jl} : L^2((0,\infty); r^2dr)\bigotimes\mathbb{C}^2 \to \mathcal{H}_{jl} \equiv L^2((0,\infty); dr)\bigotimes\mathbb{C}^2, \tag{12}$$

$$(U_{jl}\psi)(r) = \begin{pmatrix} rf(r) \\ (-1)^{j-l-\frac{1}{2}}rg(r) \end{pmatrix} \tag{13}$$

which enable us to represent $\mathcal{H}$ in the form

$$\mathcal{H} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} \bigoplus_{\mu=-j}^{j} [U_{jl}^{-1}\mathcal{H}_{jl}]\bigotimes[\Omega_{jl\mu}(\theta,\varphi)] \tag{14}$$

where $[\Omega_{jl\mu}(n)]$ stands for the vector space generated by the spherical spinors.

With respect to the decomposition (14),$\overline{H}_D$ reads

$$\overline{H}_D = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} [U_{jl}^{-1}h_{jl}U_{jl}]\bigotimes \mathbb{1}. \tag{15}$$

The operators h_{jl} in $L^2((0,\infty))\bigotimes\mathbb{C}^2$ are given by

$$\begin{aligned}
h_{jl} &= \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix} \equiv \tau, && (16)\\
\kappa_{jl} &= (-1)^{j-l+\frac{1}{2}}(j+\frac{1}{2}), && (17)\\
D(h_{jl}) &= \{\psi\in L^2((0,\infty))\bigotimes\mathbb{C}^2|\psi\in \mathrm{AC}_{loc}((0,\infty));\ \ \psi(R\pm)=0;\\
\tau\psi &\in L^2((0,\infty))\bigotimes\mathbb{C}^2\} && (18)
\end{aligned}$$

where $\mathrm{AC}_{loc}(\Omega)$ denotes the set of locally absolutely continuous functions on Ω and

$$f(x\pm)=\lim_{\epsilon\to 0+}f(x\pm\epsilon).$$

The adjoint h^*_{jl} of h_{jl} reads:

$$\begin{aligned}
h^*_{jl} &= \tau,\\
D(h^*_{jl}) &= \{\psi\in L^2((0,\infty))\bigotimes\mathbb{C}^2|\psi\in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}),\\
\tau\psi &\in L^2((0,\infty))\bigotimes\mathbb{C}^2\}. && (19)
\end{aligned}$$

The self-adjoint extensions of h_{jl}

Consider the equation

$$(h^*_{jl}-z)\psi=0,\ \ \psi\in\mathcal{D}(h^*_{jl}), z\in\mathbb{C}\backslash(-\infty,-\frac{c^2}{2}]\cup[\frac{c^2}{2},\infty) \tag{20}$$

which may be written in the form:

$$\begin{cases} \psi_1''+[k(z)^2-\frac{\kappa_{jl}(\kappa_{jl}+1)}{r^2}]\psi_1=0\\ \psi_2''+[k(z)^2-\frac{\kappa_{jl}(\kappa_{jl}-1)}{r^2}]\psi_2=0 \end{cases} \tag{21}$$

where

$$k(z)=\frac{1}{c}\sqrt{z^2-\frac{c^4}{4}}\equiv k. \tag{22}$$

A straightforward computation shows that eq (20) has two linearly independent solutions

$$\psi^{(1)}_{jl,z}(r)=\begin{cases} \begin{pmatrix} F_{jl}(z,r)\\ \tilde{F}_{jl}(z,r)\end{pmatrix}; & r<R,\\ \binom{0}{0}; & r>R, \end{cases} \tag{23}$$

$$\psi^{(2)}_{jl,z}(r)=\begin{cases} \binom{0}{0}; & r<R,\\ \begin{pmatrix} G_{jl}(z,r)\\ \tilde{G}_{jl}(z,r)\end{pmatrix}; & r<R \end{cases} \tag{24}$$

where

$$\begin{aligned}
F_{jl}(z,r) &= \left(\frac{k(z)}{2}\right)^{-\kappa_{jl}-\frac{1}{2}}\Gamma\left(\kappa_{jl}+\frac{3}{2}\right)r^{\frac{1}{2}}J_{\kappa_{jl}+\frac{1}{2}}(k(z)r), && (25)\\
\tilde{F}_{jl}(z,r) &= \frac{1}{c}\left(\frac{1}{2}\right)^{-\kappa_{jl}-\frac{1}{2}}k(z)^{-\kappa_{jl}+\frac{1}{2}}\Gamma\left(\kappa_{jl}+\frac{3}{2}\right)r^{\frac{1}{2}}J_{\kappa_{jl}-\frac{1}{2}}(k(z)r), && (26)\\
G_{jl}(z,r) &= \mathrm{i}\frac{\pi}{2}\left(\frac{k(z)}{2}\right)^{\kappa_{jl}+\frac{1}{2}}\Gamma\left(\kappa_{jl}+\frac{3}{2}\right)^{-1}r^{\frac{1}{2}}H^{(1)}_{\kappa_{jl}+\frac{1}{2}}(k(z)r), && (27)\\
\tilde{G}_{jl}(z,r) &= \mathrm{i}\frac{\pi}{2c}\left(\frac{1}{2}\right)^{\kappa_{jl}+\frac{1}{2}}k(z)^{\kappa_{jl}+\frac{3}{2}}\Gamma\left(\kappa_{jl}+\frac{3}{2}\right)^{-1}r^{\frac{1}{2}}H^{(1)}_{\kappa_{jl}-\frac{1}{2}}(k(z)r),\ \ \mathrm{Im}k(z)>0. && (28)
\end{aligned}$$

$J_\nu(.)$ is the Bessel function and $H_\nu^{(1)}(.)$ the Hankel function of the first type of order ν.
The solutions (23) and (24) have been normalized in such a way that:

$$\det\begin{bmatrix} G_{jl}(z,r) & F_{jl}(z,r) \\ \tilde{G}_{jl}(z,r) & \tilde{F}_{jl}(z,r) \end{bmatrix} = G_{jl}(z,r)\tilde{F}_{jl}(z,r) - \tilde{G}_{jl}(z,r)F_{jl}(z,r) = \frac{1}{c}. \tag{29}$$

Therefore, h_{jl} has deficiency indices $(2,2)$ and consequently, all self-adjoint(s.a) extensions of h_{jl} are given by a 4-parameter family of self-adjoint operators[24]. Since the matrix G in eq (3) depends on 2 parameters, it follows that the s.a extension $h_{jl,G}$ of h_{jl} corresponding to the interaction $V(r) = G\delta(r-R)$ is a special 2-parameters family of s.a extensions of h_{jl}.
The relation $h_{jl,G} \subset h_{jl}^*$ implies that the domain $\mathcal{D}(h_{jl,G})$ contains those functions $\psi \in \mathcal{D}(h_{jl}^*)$ which satisfy suitable boundary conditions at $r = R$.

Theorem 2.1 *[7].* Any self-adjoint extension $\hat{h}_{jl}$ of h_{jl} reads

$$\begin{aligned} \hat{h}_{jl} &= \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix}, \\ D(\hat{h}_{jl}) &= \{\psi \in L^2((0,\infty)) \bigotimes \mathbb{C}^2 / \psi \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}); \\ &\qquad \hat{h}_{jl}\psi \in L^2((0,\infty)) \bigotimes \mathbb{C}^2;\ \psi \text{ satisfy } \mathrm{cond}_1 \text{ or } \mathrm{cond}_2\}. \end{aligned} \tag{30}$$

$$\mathrm{cond}_1 : \psi(R-) = e^{i\theta}\hat{A}\psi(R_+),\quad \theta \in [0,\pi) \tag{31}$$

and $\hat{A}$ is a 2×2 matrix with $\det \hat{A} = 1$;

$$\mathrm{cond}_2 : \begin{pmatrix} c_1 & c_2 \\ 0 & 0 \end{pmatrix}\psi(R-) + \begin{pmatrix} 0 & 0 \\ d_1 & d_2 \end{pmatrix}\psi(R+) = 0 \tag{32}$$

where c_1, c_2, d_1 and d_2 are real and both matrices are nonzero. Conversely, any operator of this form is self-adjoint extension of h_{jl}.

Theorem 2.2 *[7].* The general form of boundary conditions (31)and (32) reads

$$C_3\psi(R-) + D_3\psi(R+) = 0 \tag{33}$$

where C_3 and D_3 are 2×2 matrices such that the 2×4 matrix (C_3, D_3) has rank 2.

Let us now construct the self-adjoint extension corresponding to the radial Dirac operator with the potentiel

$$V(r) = G_{jl}\delta(r-R),\quad G_{jl} = \begin{pmatrix} A_{jl} & 0 \\ 0 & B_{jl} \end{pmatrix};\quad A_{jl}, B_{jl} \in \mathbb{R}.$$

Suppose that f satisfies the equation

$$[\tau + G_{jl}\delta(r-R)]f = zf, \tag{34}$$

$$\tau = \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix},\qquad G_{jl} = \begin{pmatrix} A_{jl} & 0 \\ 0 & B_{jl} \end{pmatrix},\quad A_{jl},\ B_{jl} \in \mathbb{R} \tag{35}$$

and the limits $f(R\pm)$ exist. Integrating eq (34) over $(R-\epsilon, R+\epsilon)$, and taking the limit $\epsilon \to 0$ we get [7]:

$$\left(1 - \tau_0\frac{G_{jl}}{2c}\right)f(R+) - \left(1 + \tau_0\frac{G_{jl}}{2c}\right)f(R-) = 0 \tag{36}$$

where

$$\tau_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{37}$$

We will accept only those matrices G for which eq (36) is compatible with eqs (31) and (32). As indicated in [7] the boundary condition (36) defines a self-adjoint extension of h_{jl} iff $G = G^{+}$.

Consider in $L^2((0,\infty))\bigotimes\mathbb{C}^2$ the operator $h_{jl,G_{jl}}$ defined by:

$$\begin{aligned} h_{jl,G_{jl}} &= \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix} \equiv \tau, \\ D(h_{jl,G_{jl}}) &= \left\{ g \in \mathcal{D}(h^*_{jl}) \middle| \left(1 - \tau_0 \frac{G_{jl}}{2c}\right) g(R+) - \left(1 + \tau_0 \frac{G_{jl}}{2c}\right) g(R-) = 0 \right\}; \\ & \quad l \in [j - \frac{1}{2}, j + \frac{1}{2}],\ j \in [\frac{1}{2}, \infty). \end{aligned} \tag{38}$$

The boundary condition (36) may be written in the form

$$C_3\psi(R-) + D_3\psi(R+) = 0 \tag{39}$$

with

$$C_3 = -\begin{pmatrix} 1 & -\frac{B_{jl}}{2c} \\ \frac{A_{jl}}{2c} & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 & \frac{B_{jl}}{2c} \\ -\frac{A_{jl}}{2c} & 1 \end{pmatrix}. \tag{40}$$

Therefore, according to theorem 2.2, the operator $h_{jl,G_{jl}}$ is a self-adjoint extension of h_{jl}.

The operator $h_{jl,G_{jl}}$ provides the mathematical definition of the formal expression

$$h_{G_{jl}} = h_D + G_{jl}\delta(r - R) \tag{41}$$

where h_D is the radial Dirac Hamiltonian defined by

$$\begin{aligned} h_D &= \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix} \\ \mathcal{D}(h_D) &= \{g \in L^2((0,\infty))\bigotimes\mathbb{C}^2 | g \in \mathrm{AC}_{loc}((0,\infty)), \\ & \quad h_D g \in L^2((0,\infty))\bigotimes\mathbb{C}^2\};\ \ l \in [j - \frac{1}{2}, j + \frac{1}{2}], j \in [\frac{1}{2}, \infty). \end{aligned} \tag{42}$$

The case $G_{jl} = 0$ in eq (38) yields the radial Dirac Hamiltonian $h_{jl,0} \equiv h_D$.
The case $A_{jl} \neq 0,\ \ B_{jl} = 0$ in eq (38) gives the δ-sphere interaction of the first type.
The case $A_{jl} = 0,\ \ B_{jl} \neq 0$ leads to a δ-sphere interaction of the second type.

The operator H_G in $L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^4$ defined by

$$H_G = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} U_{jl}^{-1} h_{jl,G_{jl}} U_{jl} \bigotimes 1\!\!1 \tag{43}$$

provides the mathematical formulation of the general δ-sphere interaction formally given by eq (2).
The case $G = 0$ i.e $G_{jl} = 0$ for all j and l yields the Dirac Hamiltonian H_D defined by eq (4).

B. The resolvent equation

The resolvent of $h_{jl,G_{jl}}$ and H_G are given by the following theorem

Theorem 2.3 *:*

(i) The resolvent of $h_{jl,G_{jl}}$ is given by

$$\begin{aligned} (h_{jl,G_{jl}} - z)^{-1} &= (h_D - z)^{-1} + \vartheta_{jl}(z)\left\{ A_{jl}\left(\overline{\tilde{M}_z^{(jl)}(.)},.\right)\tilde{M}_z^{(jl)}(.) + B_{jl}\left(\overline{\hat{M}_z^{(jl)}(.)},.\right)\hat{M}_z^{(jl)}(.) + \right. \\ &+ \left. \frac{A_{jl}B_{jl}}{2c}\left(\overline{\tilde{M}_z^{(jl)}(.)},.\right)\tilde{W}_z^{(jl)}(.) - \frac{A_{jl}B_{jl}}{2c}\left(\overline{\hat{M}_z^{(jl)}(.)},.\right)\hat{W}_z^{(jl)}(.)\right\}, \end{aligned}$$

$$z \in \rho(h_{jl,G_{jl}}),\ \ \mathrm{Im}k(z) > 0,\ \ l \in [j-\frac{1}{2}, j+\frac{1}{2}],\ \ j \in [\frac{1}{2}, \infty) \tag{44}$$

($\rho(.)$ is the resolvent set)

where $(h_D - z)^{-1}$, $\mathrm{Im}k(z) > 0$ is the radial Dirac resolvent with kernel

$$G^{(jl)}(z,r,r') = \begin{pmatrix} G^{(jl)}_{11}(z,r,r') & G^{(jl)}_{12}(z,r,r') \\ G^{(jl)}_{21}(z,r,r') & G^{(jl)}_{22}(z,r,r') \end{pmatrix} \tag{45}$$

where

$$G^{(jl)}_{11}(z,r,r') = \begin{cases} G_{jl}(z,r')F_{jl}(z,r); & r < r' \\ F_{jl}(z,r')G_{jl}(z,r); & r > r', \end{cases} \tag{46}$$

$$G^{(jl)}_{12}(z,r,r') = \begin{cases} \tilde{G}_{jl}(z,r')F_{jl}(z,r); & r < r' \\ \tilde{F}_{jl}(z,r')G_{jl}(z,r); & r > r', \end{cases} \tag{47}$$

$$G^{(jl)}_{21}(z,r,r') = \begin{cases} G_{jl}(z,r')\tilde{F}_{jl}(z,r); & r < r' \\ F_{jl}(z,r')\tilde{G}_{jl}(z,r); & r > r', \end{cases} \tag{48}$$

$$G^{(jl)}_{22}(z,r,r') = \begin{cases} \tilde{G}_{jl}(z,r')\tilde{F}_{jl}(z,r); & r < r' \\ \tilde{F}_{jl}(z,r')\tilde{G}_{jl}(z,r); & r > r', \end{cases} \tag{49}$$

and

$$\vartheta_{jl}(z) = -[1 - \frac{A_{jl}B_{jl}}{4c^2} + B_{jl}\tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,R) + A_{jl}F_{jl}(z,R)G_{jl}(z,R)]^{-1}, \tag{50}$$

$$\hat{M}^{(jl)}_z(r) = \begin{cases} \begin{pmatrix} \tilde{G}_{jl}(z,R)F_{jl}(z,r) \\ \tilde{G}_{jl}(z,R)\tilde{F}_{jl}(z,r) \end{pmatrix}; & r < R \\ \begin{pmatrix} \tilde{F}_{jl}(z,R)G_{jl}(z,r) \\ \tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,r) \end{pmatrix}; & r > R, \end{cases} \tag{51}$$

$$\tilde{M}^{(jl)}_z(r) = \begin{cases} \begin{pmatrix} G_{jl}(z,R)F_{jl}(z,r) \\ G_{jl}(z,R)\tilde{F}_{jl}(z,r) \end{pmatrix}; & r < R \\ \begin{pmatrix} F_{jl}(z,R)G_{jl}(z,r) \\ F_{jl}(z,R)\tilde{G}_{jl}(z,r) \end{pmatrix}; & r > R, \end{cases} \tag{52}$$

$$\tilde{W}^{(jl)}_z(r) = \begin{cases} \begin{pmatrix} \tilde{G}_{jl}(z,R)F_{jl}(z,r) \\ \tilde{G}_{jl}(z,R)\tilde{F}_{jl}(z,r) \end{pmatrix}; & r < R \\ -\begin{pmatrix} \tilde{F}_{jl}(z,R)G_{jl}(z,r) \\ \tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,r) \end{pmatrix}; & r > R, \end{cases} \tag{53}$$

$$\hat{W}^{(jl)}_z(r) = \begin{cases} \begin{pmatrix} G_{jl}(z,R)F_{jl}(z,r) \\ G_{jl}(z,R)\tilde{F}_{jl}(z,r) \end{pmatrix}; & r < R \\ -\begin{pmatrix} F_{jl}(z,R)G_{jl}(z,r) \\ F_{jl}(z,R)\tilde{G}_{jl}(z,r) \end{pmatrix}; & r > R \end{cases} \tag{54}$$

with $F_{jl}(z,r)$, $\tilde{F}_{jl}(z,r)$, $G_{jl}(z,r)$, $\tilde{G}_{jl}(z,r)$ and $k(z)$ defined by (25)-(28) and (22) respectively.

(ii) The resolvent of H_G is given by

$$\begin{aligned}
(H_G - z)^{-1} = (H_D - z)^{-1} + \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} \bigoplus_{\mu=-j}^{j} \vartheta_{jl}(z) \Big\{ A_{jl} \Big(|.|^{-1}\overline{\tilde{M}^{(jl)}_z(.)} \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi), . \Big) \times \\
\times\ |.|^{-1}\tilde{M}^{(jl)}_z(.) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) + B_{jl} \Big(|.|^{-1}\overline{\hat{M}^{(jl)}_z(.)} \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi), . \Big) |.|^{-1}\hat{M}^{(jl)}_z(.) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) + \\
+\ \frac{A_{jl}B_{jl}}{2c} \Big(|.|^{-1}\overline{\tilde{M}^{(jl)}_z(.)} \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi), . \Big) |.|^{-1}\tilde{W}^{(jl)}_z(.) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) - \\
-\ \frac{A_{jl}B_{jl}}{2c} \Big(|.|^{-1}\overline{\hat{M}^{(jl)}_z(.)} \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi), . \Big) |.|^{-1}\hat{W}^{(jl)}_z(.) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) \Big\}, \\
z \in \rho(H_G),\ \ \mathrm{Im}k(z) > 0,\ \ A_{jl}, B_{jl} \in \mathbb{R}
\end{aligned} \tag{55}$$

where we have used the notation

$$\hat{M}_z^{(jl)}(r) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) = \begin{pmatrix} \hat{M}_{z,1}^{(jl)}(r)\Omega_{jl\mu}(\theta,\varphi) \\ \hat{M}_{z,2}^{(jl)}(r)\Omega_{jl'\mu}(\theta,\varphi) \end{pmatrix} \tag{56}$$

with

$$\hat{M}_z^{(jl)} = \begin{pmatrix} \hat{M}_{z,1}^{(jl)} \\ \hat{M}_{z,2}^{(jl)} \end{pmatrix}, \quad \tilde{\Omega}_{jl\mu} = \begin{pmatrix} \Omega_{jl\mu} \\ \Omega_{jl'\mu} \end{pmatrix}. \tag{57}$$

The notation $\tilde{M}_z^{(jl)}(r) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi)$, $\tilde{W}_z^{(jl)}(r) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi)$ and $\hat{W}_z^{(jl)}(r) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi)$ are defined similarly to (56).
Proof: We use the following Krein resolvent formula[24]:

$$(h_{jl,G_{jl}} - z)^{-1} = (h_D - z)^{-1} + \sum_{n,m=1}^{2} \lambda_{nm}(z) \left(\overline{\psi_{jl,z}^{(n)}(.)},. \right) \psi_{jl,z}^{(m)}(.), \; z \in \rho(h_{jl,G_{jl}}), \; \mathrm{Im}k(z) > 0 \tag{58}$$

where $\psi_{jl,z}^{(m)}(r), m = 1,2$ are given by (23) and (24) respectively.

The constants $\lambda_{nm}(z)$ are determined as follow. Let $g = \binom{g_1}{g_2} \in L^2((0,\infty)) \bigotimes \mathbb{C}^2$ and define the function $\chi_{jl}(z,r) \in \mathcal{D}(h_{jl,G_{jl}})$ by

$$\chi_{jl}(z,r) \;\; = \;\; ((h_{jl,G_{jl}} - z)^{-1} g)(r). \tag{59}$$

Since $\chi_{jl} = \binom{\chi_{jl,1}}{\chi_{jl,2}} \in \mathcal{D}(h_{jl,G_{jl}})$, it follows that χ_{jl} satisfy the boundary conditions in (38).

The implimentation of these boundary conditions gives

$$\lambda(z) = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{vmatrix} \tag{60}$$

where

$$\begin{aligned}
\lambda_{11} &= \frac{1}{\Delta_{jl}}[A_{jl}G_{jl}(z,R)G_{jl}(z,R) + B_{jl}\tilde{G}_{jl}(z,R)\tilde{G}_{jl}(z,R)] \\
\lambda_{22} &= \frac{1}{\Delta_{jl}}[A_{jl}F_{jl}(z,R)F_{jl}(z,R) + B_{jl}\tilde{F}_{jl}(z,R)\tilde{F}_{jl}(z,R)] \\
\lambda_{12} &= \frac{1}{\Delta_{jl}}[A_{jl}F_{jl}(z,R)G_{jl}(z,R) + B_{jl}\tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,R) + \\
&+ \frac{A_{jl}B_{jl}}{2c}\tilde{G}_{jl}(z,R)F_{jl}(z,R) - \frac{A_{jl}B_{jl}}{2c}\tilde{F}_{jl}(z,R)G_{jl}(z,R)] \\
\lambda_{21} &= \frac{1}{\Delta_{jl}}[A_{jl}F_{jl}(z,R)G_{jl}(z,R) + B_{jl}\tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,R) + \\
&+ \frac{A_{jl}B_{jl}}{2c}\tilde{G}_{jl}(z,R)F_{jl}(z,R) - \frac{A_{jl}B_{jl}}{2c}\tilde{F}_{jl}(z,R)G_{jl}(z,R)]
\end{aligned} \tag{61}$$

with

$$\Delta_{jl}(z) \;\; = \;\; -[1 - \frac{A_{jl}B_{jl}}{4c^2} + B_{jl}\tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,R) + A_{jl}F_{jl}(z,R)G_{jl}(z,R)]. \tag{62}$$

Inserting (60) into (58), we obtain (44).
Equation (55) follows from (44) and (43).

Remark 1 *: From (44) one obtains the following results for j and l fixed*

$$(i) \qquad n.\lim_{B_{jl}\to 0+}(h_{jl,G_{jl}}-z)^{-1}=(h_{jl,A_{jl}}-z)^{-1},\ \ z\in\rho(h_{jl,G_{jl}})\cap\rho(h_{jl,A_{jl}}) \tag{63}$$

where

$$(h_{jl,A_{jl}}-z)^{-1}=(h_D-z)^{-1}\ +\ \lambda_{jl}(z)\left(\overline{D_z^{(jl)}(.)},.\right)D_z^{(jl)}(.),\ \ z\in\rho(h_{jl,A_{jl}}),\ \ \mathrm{Im}k(z)>0,$$
$$l\in[j-\frac{1}{2},j+\frac{1}{2}],\ \ j\in[\frac{1}{2},\infty) \tag{64}$$

with

$$\lambda_{jl}(z) = -\frac{A_{jl}}{1+A_{jl}F_{jl}(z,R)G_{jl}(z,R)}, \tag{65}$$

$$D_z^{(jl)}(r) = \begin{cases}\begin{pmatrix} G_{jl}(z,R)F_{jl}(z,r)\\ G_{jl}(z,R)\tilde{F}_{jl}(z,r)\end{pmatrix}, & r<R\\ \begin{pmatrix} F_{jl}(z,R)G_{jl}(z,r)\\ F_{jl}(z,R)\tilde{G}_{jl}(z,r)\end{pmatrix}, & r>R.\end{cases} \tag{66}$$

$$(ii) \qquad n.\lim_{A_{jl}\to 0+}(h_{jl,G_{jl}}-z)^{-1}=(h_{jl,B_{jl}}-z)^{-1},\ \ z\in\rho(h_{jl,G_{jl}})\cap\rho(h_{jl,B_{jl}}) \tag{67}$$

where

$$(h_{jl,B_{jl}}-z)^{-1}=(h_D-z)^{-1}\ +\ \Theta_{jl}(z)\left(\overline{T_z^{(jl)}(.)},.\right)T_z^{(jl)}(.),\ \ z\in\rho(h_{jl,B_{jl}}),\ \ \mathrm{Im}k(z)>0,$$
$$l\in[j-\frac{1}{2},j+\frac{1}{2}],\ \ j\in[\frac{1}{2},\infty) \tag{68}$$

with

$$\Theta_{jl}(z) = -\frac{B_{jl}}{1+B_{jl}\tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,R)}, \tag{69}$$

$$T_z^{(jl)}(r) = \begin{cases}\begin{pmatrix} \tilde{G}_{jl}(z,R)F_{jl}(z,r)\\ \tilde{G}_{jl}(z,R)\tilde{F}_{jl}(z,r)\end{pmatrix}, & r<R\\ \begin{pmatrix} \tilde{F}_{jl}(z,R)G_{jl}(z,r)\\ \tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,r)\end{pmatrix}, & r>R.\end{cases} \tag{70}$$

We note that the Hamiltonians $h_{jl,A_{jl}}$ and $h_{jl,B_{jl}}$ define relativistic δ-sphere interactions of the first and the second type respectively.[21]

C. Spectral properties

The spectral properties of $h_{jl,G_{jl}}$ are given by the following theorem

Theorem 2.4 *: For $A_{jl}, B_{jl}\in(-\infty,\infty)$, $l\in[j-\frac{1}{2},j+\frac{1}{2}]$, $j\in[\frac{1}{2},\infty)$ we obtain*

$$\sigma_{ess}(h_{jl,G_{jl}})=\sigma_{ac}(h_{jl,G_{jl}})=(-\infty,-\frac{c^2}{2}]\bigcup[\frac{c^2}{2},\infty) \tag{71}$$
$$\sigma_{sc}(h_{jl,G_{jl}})=\emptyset \tag{72}$$
$$\sigma_p(h_{jl,G_{jl}})\bigcap(-\infty,-\frac{c^2}{2}]\bigcup[\frac{c^2}{2},\infty)=\emptyset\ \text{ if } 1+\frac{A_{jl}B_{jl}}{4c^2}\neq 0. \tag{73}$$

Following[7], one can show that $h_{jl,G_{jl}}$ has at most two eigenvalues in $[-\frac{c^2}{2},\frac{c^2}{2}]$. The eigenvalues of $h_{jl,G_{jl}}$ are determined from the equation

$$1-\frac{A_{jl}B_{jl}}{4c^2}+A_{jl}F_{jl}(z,R)G_{jl}(z,R)+B_{jl}\tilde{F}_{jl}(z,R)\tilde{G}_{jl}(z,R)\Big|_{k(z)=i\sqrt{-E'}}$$

$$= 1 - \frac{A_{jl}B_{jl}}{4c^2} + A_{jl}RK_{\kappa_{jl}+\frac{1}{2}}(\sqrt{-E'}R)I_{\kappa_{jl}+\frac{1}{2}}(\sqrt{-E'}R) -$$
$$-B_{jl}c^{-2}E'RK_{\kappa_{jl}-\frac{1}{2}}(\sqrt{-E'}R)I_{\kappa_{jl}-\frac{1}{2}}(\sqrt{-E'}R)] = 0,\ E' < 0 \tag{74}$$

where E' is given by

$$E' = \frac{1}{c^2}\left(z^2 - \frac{c^4}{4}\right). \tag{75}$$

The resonances of $h_{jl,G_{jl}}$ are defined as the poles of the resolvent equation (44) in the unphysical sheet $\mathrm{Im}k(z) < 0$ i.e. as the solution of the equation

$$1 - \frac{A_{jl}B_{jl}}{4c^2} + B_{jl}\tilde{G}_{jl}(z,R)\tilde{F}_{jl}(z,R) + A_{jl}G_{jl}(z,R)F_{jl}(z,R) = 0,\ \ \mathrm{Im}k(z) < 0. \tag{76}$$

D. The nonrelativistic limit

Following the strategy of Gesztesy et al[1, 25], we discuss the nonrelativistic limit $c \to \infty$. The Hamiltonian $h_{jl,G_{jl}}$ defined by (38) reads

$$\begin{aligned} h_{jl,G_{jl}} &= \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix} \\ &= c\hat{\alpha} + \frac{c^2}{2}\hat{\beta} \end{aligned} \tag{77}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are self-adjoint operators in the Hilbert space $\mathcal{H}_{jl} = L^2((0,\infty)) \bigotimes \mathbb{C}^2$ defined as

$$\hat{\alpha} = \begin{pmatrix} 0 & \dot{A}^*_{jl} \\ \dot{A}_{jl} & 0 \end{pmatrix}, \hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \ \hat{\beta}^2 = 1 \tag{78}$$

with

$$\dot{A}_{jl} = \frac{d}{dr} + \frac{\kappa_{jl}}{r},\ \ \mathcal{D}(\dot{A}_{jl}) = \mathcal{D}\left(\frac{d}{dr}\Big|_{C_0^\infty((0,\infty))}\right) \tag{79}$$

$$\dot{A}^*_{jl} = -\frac{d}{dr} + \frac{\kappa_{jl}}{r},\ \ \mathcal{D}(\dot{A}^*_{jl}) = \mathcal{D}\left(-\frac{d}{dr}\Big|_{C_0^\infty((0,\infty))}\right). \tag{80}$$

Let us now introduce the operator $P_\pm$ by[25]

$$P_\pm = \frac{1}{2}(1 \pm \hat{\beta}) \tag{81}$$

which satisfies the following properties

$$P_\pm^2 = P_\pm,\quad P_-P_+ = P_+P_- = 0$$

and define

$$\mathcal{H}_{jl\pm} = P_\pm\mathcal{H}_{jl},\ \ \mathcal{H}_{jl} = \mathcal{H}_{jl+}\bigoplus\mathcal{H}_{jl-} \tag{82}$$

with $\mathcal{H}_{jl\pm} = L^2((0,\infty))$,

$$V(r) = \begin{pmatrix} A_{jl} & 0 \\ 0 & B_{jl} \end{pmatrix}\delta(r-R) = \begin{pmatrix} V_+(r) & 0 \\ 0 & V_-(r) \end{pmatrix}\delta(r-R). \tag{83}$$

Finally, let us define in $L^2((0,\infty))$ the radial Pauli operators $h_{\pm,jl}$ by

$$\begin{aligned}
h_{+,jl} &= \dot{A}^*_{jl}\dot{A}_{jl} \\
&= -\frac{d^2}{dr^2} + \frac{\kappa_{jl}(\kappa_{jl}+1)}{r^2} \\
\mathcal{D}(h_{+,jl}) &= \{f \in L^2((0,\infty)) | f, f' \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}); f(0+) = 0 \text{ if } \kappa_{jl} = 0; \\
& \quad f(R+) - f(R-) = \tfrac{\beta_{\kappa_{jl}}}{2}[f'(R+) + f'(R-)]; \\
& \quad f'(R+) - f'(R-) = \tfrac{\alpha_{\kappa_{jl}}}{2}[f(R+) + f(R-)]; \\
& \quad -f'' + \kappa_{jl}(\kappa_{jl}+1)r^{-2}f \in L^2((0,\infty))\}, \\
& \quad -\infty < \beta_{\kappa_{jl}}, \alpha_{\kappa_{jl}} \leq \infty; \ \kappa_{jl} > 0,
\end{aligned} \tag{84}$$

and

$$\begin{aligned}
h_{-,jl} &= \dot{A}_{jl}\dot{A}^*_{jl} \\
&= -\frac{d^2}{dr^2} + \frac{\kappa_{jl}(\kappa_{jl}-1)}{r^2} \\
\mathcal{D}(h_{-,jl}) &= \{f \in L^2((0,\infty)) | f, f' \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}); f(0+) = 0 \text{ if } \kappa_{jl} = 0; \\
& \quad f(R+) - f(R-) = \tfrac{\beta_{\kappa_{jl}}}{2}[f'(R+) + f'(R-)]; \\
& \quad f'(R+) - f'(R-) = \tfrac{\alpha_{\kappa_{jl}}}{2}[f(R+) + f(R-)]; \\
& \quad -f'' + \kappa_{jl}(\kappa_{jl}-1)r^{-2}f \in L^2((0,\infty))\}, \\
& \quad -\infty < \beta_{\kappa_{jl}}, \alpha_{\kappa_{jl}} \leq \infty; \ \kappa_{jl} > 0.
\end{aligned} \tag{85}$$

The radial Pauli operators are given as the extensions of

$$-\frac{d^2}{dr^2} + \frac{\kappa_{jl}(\kappa_{jl}\pm 1)}{r^2} + V_\pm(r)\Big|_{C_0^\infty((0,\infty))}. \tag{86}$$

The commutation formulas define by [26]

$$\begin{aligned}
(\dot{A}^*_{jl}\dot{A}_{jl} - z)^{-1}\dot{A}^*_{jl} &\subseteq \dot{A}^*_{jl}(\dot{A}_{jl}\dot{A}^*_{jl} - z)^{-1}, \\
(\dot{A}_{jl}\dot{A}^*_{jl} - z)^{-1}\dot{A}_{jl} &\subseteq \dot{A}_{jl}(\dot{A}^*_{jl}\dot{A}_{jl} - z)^{-1}, \\
\dot{A}_{jl}(\dot{A}^*_{jl}\dot{A}_{jl} - z)^{-1}\dot{A}^*_{jl} &\subseteq 1 + z(\dot{A}_{jl}\dot{A}^*_{jl} - z)^{-1}, \\
\dot{A}^*_{jl}(\dot{A}_{jl}\dot{A}^*_{jl} - z)^{-1}\dot{A}_{jl} &\subseteq 1 + z(\dot{A}^*_{jl}\dot{A}_{jl} - z)^{-1}, \quad z \in \rho(\dot{A}^*_{jl}\dot{A}_{jl})\backslash\{0\} = \rho(\dot{A}_{jl}\dot{A}^*_{jl})\backslash\{0\}
\end{aligned} \tag{87}$$

will be used later.

Theorem 2.5 *: We assume* $z \in \mathbb{C}\backslash\mathbb{R}$, then

$$\begin{aligned}
(h_{jl,G_{jl}} - \frac{c^2}{2} - z)^{-1} &= \left\{ 1 + \begin{pmatrix} 0 & c^{-1}(h_{+,jl} - z)^{-1}\dot{A}^*_{jl}(-z) \\ 0 & c^{-2}z(h_{-,jl} - z)^{-1}(-z) \end{pmatrix} \right\}^{-1} \times \\
& \times \begin{pmatrix} (h_{+,jl} - z)^{-1} & c^{-1}(h_{+,jl} - z)^{-1}\dot{A}^*_{jl} \\ c^{-1}\dot{A}_{jl}(h_{+,jl} - z)^{-1} & c^{-2}z(h_{-,jl} - z)^{-1} \end{pmatrix}
\end{aligned} \tag{88}$$

Proof:Let $z \in \mathbb{C}\backslash\mathbb{R}$, then

$$(h_{jl,G_{jl}} - \frac{c^2}{2} - z)^{-1} = \left(\begin{pmatrix} -z & c\dot{A}^*_{jl} \\ c\dot{A}_{jl} & -c^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -z \end{pmatrix} \right)^{-1}. \tag{89}$$

From (87), we have

$$\begin{pmatrix} -z & c\dot{A}^*_{jl} \\ c\dot{A}_{jl} & -c^2 \end{pmatrix}^{-1} = \begin{pmatrix} \left(\dot{A}^*_{jl}\dot{A}_{jl} - z\right)^{-1} & c^{-1}\dot{A}^*_{jl}\left(\dot{A}_{jl}\dot{A}^*_{jl} - z\right)^{-1} \\ c^{-1}\dot{A}_{jl}\left(\dot{A}^*_{jl}\dot{A}_{jl} - z\right)^{-1} & c^{-2}z\left(\dot{A}_{jl}\dot{A}^*_{jl} - z\right)^{-1} \end{pmatrix} \in \mathcal{B}(\mathcal{H}_{jl}). \tag{90}$$

Therefore (89) reads

$$(h_{jl,G_{jl}} - \frac{c^2}{2} - z)^{-1} = \left\{ 1 + \begin{pmatrix} 0 & c^{-1}\dot{A}^*_{jl}\left(\dot{A}_{jl}\dot{A}^*_{jl} - z\right)^{-1}(-z) \\ 0 & c^{-2}z\left(\dot{A}_{jl}\dot{A}^*_{jl} - z\right)^{-1}(-z) \end{pmatrix} \right\}^{-1} \times$$
$$\times \begin{pmatrix} \left(\dot{A}^*_{jl}\dot{A}_{jl} - z\right)^{-1} & c^{-1}\dot{A}^*_{jl}\left(\dot{A}_{jl}\dot{A}^*_{jl} - z\right)^{-1} \\ c^{-1}\dot{A}_{jl}\left(\dot{A}^*_{jl}\dot{A}_{jl} - z\right)^{-1} & c^{-2}z\left(\dot{A}_{jl}\dot{A}^*_{jl} - z\right)^{-1} \end{pmatrix}. \tag{91}$$

Then (91) implies (88).

Corollary 2.6 : As $c \to \infty$, the radial Dirac operator(rest energy subtracted) $h_{jl,G_{jl}} - \frac{c^2}{2}$ converges in norm resolvent sense to the Pauli operator $h_{+,jl}$ times the projector onto $\mathcal{H}_+ = L^2((0,\infty))$

$$n.\lim_{c\to\infty}(h_{jl,G_{jl}} - \frac{c^2}{2} - z)^{-1} = (h_{+,jl} - z)^{-1} \bigotimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{92}$$

where $h_{+,jl}$ is defined by (84)

Proof: Expanding the right hand side of (88) gives

$$(h_{jl,G_{jl}} - \frac{c^2}{2} - z)^{-1} = \begin{pmatrix} (h_{+,jl} - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + c^{-1}\begin{pmatrix} 0 & (h_{+,jl} - z)^{-1}\dot{A}^*_{jl} \\ \dot{A}_{jl}(h_{+,jl} - z)^{-1} & 0 \end{pmatrix} +$$
$$+ \; 0(c^{-2}). \tag{93}$$

Then (93) implies (92).

For spin-$\frac{1}{2}$ particles i.e. $l = j + \frac{1}{2}$, one obtains

$$n.\lim_{c\to\infty}(h_{jl,G_{jl}} - \frac{c^2}{2} - z)^{-1} = (h_{l,\hat{\alpha}_l} - z)^{-1} \bigotimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{94}$$

where $h_{l,\hat{\alpha}_l}$ is defined by

$$\begin{aligned} h_{l,\hat{\alpha}_l} &= -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \\ \mathcal{D}(h_{l,\hat{\alpha}_l}) &= \{f \in L^2((0,\infty)) | f, f' \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}); f(0+) = 0 \text{ if } l = 0; \\ & f(R+) - f(R-) = \tfrac{\beta_l}{2}[f'(R+) + f'(R-)]; \\ & f'(R+) - f'(R-) = \tfrac{\alpha_l}{2}[f(R+) + f(R-)]; \\ & -f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\}, \\ & \hat{\alpha} = \{\beta_l, \alpha_l\}; -\infty < \beta_l, \alpha_l \le \infty; \; l \in \mathbb{N}_0. \end{aligned} \tag{95}$$

One can easily prove that the boundary conditions in (95) define a self-adjoint extension of radial Schrödinger operator if $\alpha_l\beta_l - 4 = 0$. The Hamiltonian $h_{l,\hat{\alpha}_l}$ defines a new exactly solvable model of nonrelativistic δ-sphere interaction.

E. Scattering theory for the pair $(h_{jl,G_{jl}}; h_D)$

Let us define for $k(z) > 0$, the fonction

$$\begin{pmatrix} F_{G_{jl},1}(z,r) \\ F_{G_{jl},2}(z,r) \end{pmatrix} = \begin{pmatrix} F_{jl}(z,r) \\ \tilde{F}_{jl}(z,r) \end{pmatrix} + \vartheta_{jl}(z)\{A_{jl}F_{jl}(z,R)\tilde{M}_z^{(jl)}(r) + B_{jl}\tilde{F}_{jl}(z,R)\hat{M}_z^{(jl)}(r) +$$
$$+ \; \frac{A_{jl}B_{jl}}{2c}F_{jl}(z,R)\tilde{W}_z^{(jl)}(r) - \frac{A_{jl}B_{jl}}{2c}\tilde{F}_{jl}(z,R)\hat{W}_z^{(jl)}(r)\} \tag{96}$$

where $F_{jl}(z,r)$, $\tilde{F}_{jl}(z,r)$, $\vartheta_{jl}(z)$, $\tilde{M}_z^{(jl)}(r)$, $\hat{M}_z^{(jl)}(r)$, $\tilde{W}_z^{(jl)}(r)$ and $\hat{W}_z^{(jl)}(r)$ are defined by (25), (26), (50), (52), (51),(53)and (54) respectively.

A straightforward calculation shows that $\binom{F_{G_{jl},1}(z,r)}{F_{G_{jl},2}(z,r)}$ are the scattering wave functions of $h_{jl,G_{jl}}$. The asymptotic behavior of $\binom{F_{G_{jl},1}(z,r)}{F_{G_{jl},2}(z,r)}$ as $r \to \infty$ yields[28]

$$\begin{aligned}
\begin{pmatrix} F_{G_{jl},1}(z,r) \\ F_{G_{jl},2}(z,r) \end{pmatrix} & \xrightarrow[r\to\infty]{k(z)>0} \begin{pmatrix} \hat{A}_{jl}(z)\sin[kr-\kappa_{jl}\frac{\pi}{2}] \\ \hat{B}_{jl}(z)\sin[kr-(\kappa_{jl}-1)\frac{\pi}{2}] \end{pmatrix} + \vartheta_{jl}(z)\{A_{jl}F_{jl}(z,R)F_{jl}(z,R)+ \\
& + \quad B_{jl}\tilde{F}_{jl}(z,R)\tilde{F}_{jl}(z,R)\} \begin{pmatrix} \hat{C}_{jl}(z)\exp -\mathrm{i}[kr-\kappa_{jl}\frac{\pi}{2}] \\ \hat{O}_{jl}(z)\exp -\mathrm{i}[kr-(\kappa_{jl}-1)\frac{\pi}{2}] \end{pmatrix} \\
& = \begin{pmatrix} [A_1^2(z)+A_2^2(z)]^{\frac{1}{2}}\sin[kr-\kappa_{jl}\frac{\pi}{2}+\delta_{G_{jl}}(z)]+0(1) \\ A_3^2(z)+A_4^2(z)]^{\frac{1}{2}}\sin[kr-(\kappa_{jl}-1)\frac{\pi}{2}+\delta_{G_{jl}}(z)]+0(1) \end{pmatrix}
\end{aligned} \tag{97}$$

where $\hat{A}_{jl}(z)$, $\hat{C}_{jl}(z)$, $\hat{B}_{jl}(z)$, $\hat{O}_{jl}(z)$ are given by

$$\hat{A}_{jl}(z) = 2^{-\kappa_{jl}}k(z)^{-\kappa_{jl}-1}\Gamma(2\kappa_{jl}+2)\Gamma(\kappa_{jl}+1)^{-1}, \tag{98}$$

$$\hat{C}_{jl}(z) = 2^{\kappa_{jl}}k(z)^{\kappa_{jl}}\Gamma(2\kappa_{jl}+2)^{-1}\Gamma(\kappa_{jl}+1), \tag{99}$$

$$\hat{B}_{jl}(z) = \frac{1}{c}2^{-\kappa_{jl}}k(z)^{-\kappa_{jl}}\Gamma(2\kappa_{jl}+2)\Gamma(\kappa_{jl}+1)^{-1}, \tag{100}$$

$$\hat{O}_{jl}(z) = \frac{1}{c}2^{\kappa_{jl}}k(z)^{\kappa_{jl}+1}\Gamma(2\kappa_{jl}+2)^{-1}\Gamma(\kappa_{jl}+1). \tag{101}$$

The phase shifts is defined by

$$\begin{aligned}
\delta_{G_{jl}}(z) &= -\arctan\frac{A_2(z)}{A_1(z)} \\
&= -\arctan\frac{A_4(z)}{A_3(z)} \\
&= -\arctan\frac{\hat{C}_{jl}(z)\vartheta'_{jl}(z)}{\hat{A}_{jl}(z)-\mathrm{i}\hat{C}_{jl}(z)\vartheta'_{jl}(z)}
\end{aligned} \tag{102}$$

where

$$\vartheta'_{jl}(z) = \vartheta_{jl}(z)[A_{jl}F_{jl}(z,R)F_{jl}(z,R)+B_{jl}\tilde{F}_{jl}(z,R)\tilde{F}_{jl}(z,R)]. \tag{103}$$

The elements of the on-shell scattering matrix are given as

$$S_{G_{jl}}(z) = \exp[2\mathrm{i}\delta_{G_{jl}}(z)]. \tag{104}$$

The partial wave scattering amplitude is given by

$$f_{G_{jl}}(z) = \frac{\exp[2\mathrm{i}\delta_{G_{jl}}(z)]-1}{2\mathrm{i}k}. \tag{105}$$

3 THE ASYMMETRIC δ-SPHERE INTERACTION

We note that all the results presented in section 2 may be extended to the case of asymmetric δ-sphere interactions formally given by[7]:

$$H_{G^{(a)}} = H_D + G\delta_a(|x|-R); \quad a \in \mathbb{C} \tag{106}$$

where G is a 2×2 matrix given by $G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and δ_a is defined by:

$$\int_{R-\epsilon}^{R+\epsilon}\delta_a(r-R)\psi(r)dr = a\psi(R+)+(1-a)\psi(R-). \tag{107}$$

Following the strategy used for the construction of $h_{jl,G_{jl}}$ in Eq(38), one can show that $H_{G^{(a)}}$ is defined by:

$$H_{G^{(a)}} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} U_{jl}^{-1} h_{jl,G_{jl}^{(a)}} U_{jl} \bigotimes 1\!\!1 \tag{108}$$

with

$$\begin{aligned} h_{jl,G_{jl}^{(a)}} &= \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix} \equiv \tau \\ \mathcal{D}(h_{jl,G_{jl}^{(a)}}) &= \{g \in L^2((0,\infty)) \bigotimes \mathbb{C}^2 | g \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}), \\ & (1-aB)\, g(R+) - (1+bB)\, g(R-) = 0, \\ & \tau g \in L^2((0,\infty)) \bigotimes \mathbb{C}^2\}, l \in [j-\frac{1}{2}, j+\frac{1}{2}],\ \ j \in [\frac{1}{2}, \infty) \end{aligned} \tag{109}$$

where we have used the following definitions and notations:

$$B = \tau_0 c^{-1} G_{jl},\ \ b = 1-a,\ \ a \in \mathbb{C} \tag{110}$$

and $G_{jl} - G_{jl}^{+} = (1 - 2\mathrm{Re}\ a) c^{-1} G_{jl}^{+} \tau_0 G_{jl}$.
The case $a = \frac{1}{2}$ gives $h_{jl,G_{jl}}$ defined by Eq(38).

4 The δ-SPHERE PLUS COULOMB INTERACTION

A. Definition of the Hamiltonian

Now the Hamiltonian of the system is formally given by[12]

$$H_{\gamma,G} = H_D + \frac{\gamma}{|x|} + G\delta(|x| - R);\ \ x \in \mathbb{R}^3, R > 0 \tag{111}$$

where

$$G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},\ \ A, B \in \mathbb{R}. \tag{112}$$

Consider the decomposition (15) and introduce in $L^2((0,\infty)) \bigotimes \mathbb{C}^2$ the operator:

$$\overline{H}_{\gamma,G} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} [U_{jl}^{-1} h_{jl,\gamma} U_{jl}] \bigotimes 1\!\!1 \tag{113}$$

where $h_{jl,\gamma}$ is given by:

$$h_{jl,\gamma} = \begin{pmatrix} \frac{c^2}{2} + \frac{\gamma}{r} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} + \frac{\gamma}{r} \end{pmatrix}, \tag{114}$$

$$\begin{aligned} D(h_{jl,\gamma}) &= \{\psi \in L^2((0,\infty)) \bigotimes \mathbb{C}^2 | \psi \in \mathrm{AC}_{loc}((0,\infty)),\ \ \psi(R\pm) = 0; \\ h_{jl,\gamma}\psi &\in L^2((0,\infty)) \bigotimes \mathbb{C}^2\}. \end{aligned} \tag{115}$$

κ_{jl} is defined by eq(17).
The adjoint $\overline{H}^{*}_{\gamma,G}$ of $\overline{H}_{\gamma,G}$ is defined by

$$\overline{H}^{*}_{\gamma,G} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} [U_{jl}^{-1} h^{*}_{jl,\gamma} U_{jl}] \bigotimes 1\!\!1, \tag{116}$$

$$
\begin{aligned}
h^*_{jl,\gamma} &= \begin{pmatrix} \frac{c^2}{2}+\frac{\gamma}{r} & -c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} \\ c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2}+\frac{\gamma}{r} \end{pmatrix}, \\
D(h^*_{jl,\gamma}) &= \{\psi \in L^2((0,\infty))\bigotimes \mathbb{C}^2 | \psi \in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}); \\
h^*_{jl,\gamma}\psi &\in L^2((0,\infty))\bigotimes \mathbb{C}^2\}. \qquad (117)
\end{aligned}
$$

The self-adjoint extensions of $h_{jl,\gamma}$

Consider the equation

$$
(h^*_{jl,\gamma}-z)\varphi = 0, \quad \varphi = \begin{pmatrix} f \\ g \end{pmatrix} \in D(h^*_{jl,\gamma}), \quad z \in \mathbb{C}\backslash(-\infty,-\frac{c^2}{2}]\cup[\frac{c^2}{2},\infty) \qquad (118)
$$

and introduce the following notations

$$
\begin{aligned}
k(z) &= \frac{1}{c}\sqrt{z^2-\frac{c^4}{4}} \equiv k, && (119)\\
\xi &= (\kappa_{jl}^2c^2-\gamma^2)^{\frac{1}{2}}, && (120)\\
\tilde{\xi} &= \frac{1}{c}\xi, && (121)\\
\tilde{\gamma} &= \frac{2z\gamma}{c^2}. && (122)
\end{aligned}
$$

A straightforward computation shows that the equation (118) have two linearly independent solutions [27]

$$
\varphi^{(1)}_{\gamma,z}(r) = \begin{cases} \begin{pmatrix} f_{\gamma,1}(z,r) \\ f_{\gamma,2}(z,r) \end{pmatrix}; & r<R \\ \binom{0}{0}; & r>R, \end{cases} \qquad (123)
$$

and

$$
\varphi^{(2)}_{\gamma,z}(r) = \begin{cases} \binom{0}{0}; & r<R \\ \begin{pmatrix} g_{\gamma,1}(z,r) \\ g_{\gamma,2}(z,r) \end{pmatrix}; & r>R \end{cases} \qquad (124)
$$

where

$$
\begin{aligned}
f_{\gamma,1}(z,r) &= \left(1-\frac{\gamma^2}{(\kappa_{jl}c+\xi)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]^{-\frac{1}{2}} \times \\
&\times \left[F_{jl,\gamma}(z,r)-\frac{\gamma}{\kappa_{jl}c+\xi}\tilde{F}_{jl,\gamma}(z,r)\right], \qquad (125)
\end{aligned}
$$

$$
\begin{aligned}
f_{\gamma,2}(z,r) &= \left(1-\frac{\gamma^2}{(\kappa_{jl}c+\xi)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]^{-\frac{1}{2}} \times \\
&\times \left[\tilde{F}_{jl,\gamma}(z,r)-\frac{\gamma}{\kappa_{jl}c+\xi}F_{jl,\gamma}(z,r)\right], \qquad (126)
\end{aligned}
$$

$$
\begin{aligned}
g_{\gamma,1}(z,r) &= \left(1-\frac{\gamma^2}{(\kappa_{jl}c+\xi)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]^{-\frac{1}{2}} \times \\
&\times \left[G_{jl,\gamma}(z,r)-\frac{\gamma}{\kappa_{jl}c+\xi}\tilde{G}_{jl,\gamma}(z,r)\right], \qquad (127)
\end{aligned}
$$

$$g_{\gamma,2}(z,r) = \left(1-\frac{\gamma^2}{(\kappa_{jl}c+\xi)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]^{-\frac{1}{2}} \times$$
$$\times\left[\tilde{G}_{jl,\gamma}(z,r)-\frac{\gamma}{\kappa_{jl}c+\xi}G_{jl,\gamma}(z,r)\right], \qquad (128)$$

with

$$F_{jl,\gamma}(z,r) = r^{\tilde{\xi}+1}e^{-\mathrm{i}k(z)r}{}_1F_1\left(\tilde{\xi}+1-\mathrm{i}\frac{\tilde{\gamma}}{2k(z)},2\tilde{\xi}+2,2\mathrm{i}k(z)r\right), \qquad (129)$$

$$G_{jl,\gamma}(z,r) = \Gamma\left(2\tilde{\xi}+2\right)^{-1}\Gamma\left(\tilde{\xi}+1-\frac{\mathrm{i}\tilde{\gamma}}{2k(z)}\right)(2\mathrm{i}k(z))^{2\tilde{\xi}+1}r^{\tilde{\xi}+1}e^{-\mathrm{i}k(z)r}\times$$
$$\times\ U\left(\tilde{\xi}+1-\mathrm{i}\frac{\tilde{\gamma}}{2k(z)},2\tilde{\xi}+2,2\mathrm{i}k(z)r\right), \qquad (130)$$

$$\tilde{F}_{jl,\gamma}(z,r) = \frac{\tilde{\xi}}{c}(2\tilde{\xi}+1)\left|\Gamma\left(\tilde{\xi}+\frac{\mathrm{i}\tilde{\gamma}}{2k(z)}\right)\right|\left|\Gamma\left(\tilde{\xi}+1+\frac{\mathrm{i}\tilde{\gamma}}{2k(z)}\right)\right|^{-1}r^{\tilde{\xi}}e^{-\mathrm{i}k(z)r}\times$$
$$\times\ {}_1F_1\left(\tilde{\xi}-\mathrm{i}\frac{\tilde{\gamma}}{2k(z)},2\tilde{\xi},2\mathrm{i}k(z)r\right), \qquad (131)$$

$$\tilde{G}_{jl,\gamma}(z,r) = \frac{\tilde{\xi}^{-1}}{c}(2\tilde{\xi}+1)^{-1}\Gamma\left(2\tilde{\xi}\right)^{-1}\Gamma\left(\tilde{\xi}+\frac{\mathrm{i}\tilde{\gamma}}{2k(z)}\right)\left|\Gamma\left(\tilde{\xi}+\frac{\mathrm{i}\tilde{\gamma}}{2k(z)}\right)\right|^{-1}\times$$
$$\times\left|\Gamma\left(\tilde{\xi}+1+\frac{\mathrm{i}\tilde{\gamma}}{2k(z)}\right)\right|k(z)^2(2\mathrm{i}k(z))^{2\tilde{\xi}-1}r^{\tilde{\xi}}e^{-\mathrm{i}k(z)r}\times$$
$$\times\ U\left(\tilde{\xi}-\mathrm{i}\frac{\tilde{\gamma}}{2k(z)},2\tilde{\xi},2\mathrm{i}k(z)r\right). \qquad (132)$$

${}_1F_1(a,b,r)$ ($U(a,b,r)$) denote the regular (respectively irregular) confluent hypergeometric functions [29]. The solutions are normalized in such way that:

$$\det\begin{bmatrix} g_{\gamma,1}(z,r) & f_{\gamma,1}(z,r)\\ g_{\gamma,2}(z,r) & f_{\gamma,2}(z,r)\end{bmatrix} = g_{\gamma,1}(z,r)f_{\gamma,2}(z,r)-g_{\gamma,2}(z,r)f_{\gamma,1}(z,r)=\frac{1}{c}. \qquad (133)$$

We note that the limit $\gamma\to 0+$, yields[28]:

$$\begin{pmatrix} f_{\gamma,1}(z,r)\\ f_{\gamma,2}(z,r)\end{pmatrix}\xrightarrow[\gamma\to 0+]{}\begin{pmatrix} F_{jl}(z,r)\\ \tilde{F}_{jl}(z,r)\end{pmatrix} \qquad (134)$$

$$\begin{pmatrix} g_{\gamma,1}(z,r)\\ g_{\gamma,2}(z,r)\end{pmatrix}\xrightarrow[\gamma\to 0+]{}\begin{pmatrix} G_{jl}(z,r)\\ \tilde{G}_{jl}(z,r)\end{pmatrix} \qquad (135)$$

where $F_{jl}(z,r)$, $\tilde{F}_{jl}(z,r)$, $G_{jl}(z,r)$ and $\tilde{G}_{jl}(z,r)$ are defined by (25)-(27) and (28) respectively.

The operator $h_{jl,\gamma}$ has deficiency indices (2,2) and consequently all its self-adjoint extensions may be parametrised by a 4-parameter family of self-adjoint operators. As in section 2.A, we consider the following 2-parameter family of self-adjoint extensions of $h_{jl,\gamma}$

$$h_{jl,\gamma,G_{jl}} = \begin{pmatrix} \frac{c^2}{2}+\frac{\gamma}{r} & -c\frac{d}{dr}+c\frac{\kappa_{jl}}{r}\\ c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2}+\frac{\gamma}{r}\end{pmatrix}\equiv\tau_\gamma,$$

$$D(h_{jl,\gamma,G_{jl}}) = \{g=\begin{pmatrix} g_1\\ g_2\end{pmatrix}\in L^2((0,\infty))\bigotimes\mathbb{C}^2 \mid g\in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\}),$$
$$\left(1-\tau_0\frac{G_{jl}}{2c}\right)g(R+)-\left(1+\tau_0\frac{G_{jl}}{2c}\right)g(R-)=0,$$

$$\tau_\gamma g \in L^2((0,\infty))\bigotimes \mathbb{C}^2\},$$
$$A_{jl}, B_{jl} \in \mathbb{R};\ \ l \in [j-\frac{1}{2}, j+\frac{1}{2}];\ \ j \in [\frac{1}{2},\infty). \tag{136}$$

Following section 2.A, one can show that the operator $h_{jl,\gamma,G_{jl}}$ gives the mathematical definition of the formal expression

$$h_{\gamma,G_{jl}} = h_D + \frac{\gamma}{r} + G_{jl}\delta(r-R). \tag{137}$$

The case $G_{jl} = 0$ in eq (136) gives the radial Dirac-Coulomb Hamiltonian $h_{jl,\gamma,0} \equiv h_{\gamma,D}$

$$\begin{aligned} h_{\gamma,D} &= h_D + \frac{\gamma}{r} \equiv \tau_\gamma, \\ \mathcal{D}(h_{\gamma,D}) &= \{g \in L^2((0,\infty))\bigotimes \mathbb{C}^2 | g \in \mathrm{AC}_{loc}((0,\infty)), \\ &\tau_\gamma g \in L^2((0,\infty))\bigotimes \mathbb{C}^2\};\ \ l \in [j-\frac{1}{2}], j \in [\frac{1}{2},\infty). \end{aligned} \tag{138}$$

The case $A_{jl} \neq 0,\ \ B_{jl} = 0$ in eq (136) yields the δ- sphere interaction of the first type plus Coulomb interaction.
The case $A_{jl} = 0,\ \ B_{jl} \neq 0$ in eq (136) yields the δ- sphere interaction of the second type plus Coulomb interaction.

The model (111) is defined in $L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^4$ by

$$H_{\gamma,G} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} U_{jl}^{-1} h_{jl,\gamma,G_{jl}} U_{jl} \bigotimes \mathbb{1}. \tag{139}$$

The case $G = 0$ for all j and l leads to the Dirac-Coulomb Hamiltonian $H_{\gamma,D}$

$$H_{\gamma,D} = H_D + \frac{\gamma}{|x|},\ \ \mathcal{D}(H_{\gamma,0}) = H^{1,2}(\mathbb{R}^3)\bigotimes\mathbb{C}^4. \tag{140}$$

B. The resolvent equation

The resolvent of $h_{jl,\gamma,G_{jl}}$ and $H_{\gamma,G}$ are given by the following theorem

Theorem 4.1 :If $G_{jl} \neq 0$ we have

(i) The resolvent of $h_{jl,\gamma,G_{jl}}$ is given by

$$\begin{aligned} (h_{jl,\gamma,G_{jl}} - z)^{-1} &= (h_{\gamma,D} - z)^{-1} + \vartheta_{jl,\gamma}(z)\{A_{jl}\left(\overline{\tilde{M}^{(jl)}_{z,\gamma}(.)},.\right)\tilde{M}^{(jl)}_{z,\gamma}(.) + \\ &+ B_{jl}\left(\overline{\hat{M}^{(jl)}_{z,\gamma}(.)},.\right)\hat{M}^{(jl)}_{z,\gamma}(.) + \frac{A_{jl}B_{jl}}{2c}\left(\overline{\tilde{M}^{(jl)}_{z,\gamma}(.)},.\right)\tilde{W}^{(jl)}_{z,\gamma}(.) - \\ &- \frac{A_{jl}B_{jl}}{2c}\left(\overline{\hat{M}^{(jl)}_{z,\gamma}(.)},.\right)\hat{W}^{(jl)}_{z,\gamma}(.)\},\ \ z \in \rho(h_{jl,\gamma,G_{jl}}),\ \ \mathrm{Im}k(z) > 0, \\ &l \in [j-\frac{1}{2}, j+\frac{1}{2}], j \in [\frac{1}{2},\infty) \end{aligned} \tag{141}$$

where $G^{(jl,\gamma)} = (h_{jl,\gamma,0} - z)^{-1}$ is the radial Dirac-Coulomb resolvent with integral kernel

$$G^{(jl,\gamma)}(z,r,r') = \begin{pmatrix} G^{(jl,\gamma)}_{11}(z,r,r') & G^{(jl,\gamma)}_{12}(z,r,r') \\ G^{(jl,\gamma)}_{21}(z,r,r') & G^{(jl,\gamma)}_{22}(z,r,r') \end{pmatrix} \tag{142}$$

with

$$G^{(jl,\gamma)}_{11}(z,r,r') = \begin{cases} g_{\gamma,1}(z,r')f_{\gamma,1}(z,r); & r < r' \\ f_{\gamma,1}(z,r')g_{\gamma,1}(z,r); & r > r', \end{cases} \tag{143}$$

$$G_{12}^{(jl,\gamma)}(z,r,r') = \begin{cases} g_{\gamma,2}(z,r')f_{\gamma,1}(z,r); & r < r' \\ f_{\gamma,2}(z,r')g_{\gamma,1}(z,r); & r > r', \end{cases} \tag{144}$$

$$G_{21}^{(jl,\gamma)}(z,r,r') = \begin{cases} g_{\gamma,1}(z,r')f_{\gamma,2}(z,r); & r < r' \\ f_{\gamma,1}(z,r')g_{\gamma,2}(z,r); & r > r', \end{cases} \tag{145}$$

$$G_{22}^{(jl,\gamma)}(z,r,r') = \begin{cases} g_{\gamma,2}(z,r')f_{\gamma,2}(z,r); & r < r' \\ f_{\gamma,2}(z,r')g_{\gamma,2}(z,r); & r > r' \end{cases} \tag{146}$$

and

$$\vartheta_{jl,\gamma}(z) = -[1 - \frac{B_{jl}A_{jl}}{4c^2} + A_{jl}f_{\gamma,1}(z,R)g_{\gamma,1}(z,R) + B_{jl}f_{\gamma,2}(z,R)g_{\gamma,2}(z,R)]^{-1}, \tag{147}$$

$$\tilde{M}_{z,\gamma}^{(jl)}(r) = \begin{cases} \begin{pmatrix} g_{\gamma,1}(z,R)f_{\gamma,1}(z,r) \\ g_{\gamma,1}(z,R)f_{\gamma,2}(z,r) \end{pmatrix}; & r < R \\ \begin{pmatrix} f_{\gamma,1}(z,R)g_{\gamma,1}(z,r) \\ f_{\gamma,1}(z,R)g_{\gamma,2}(z,r) \end{pmatrix}; & r > R \end{cases} \tag{148}$$

$$\hat{M}_{z,\gamma}^{(jl)}(r) = \begin{cases} \begin{pmatrix} g_{\gamma,2}(z,R)f_{\gamma,1}(z,r) \\ g_{\gamma,2}(z,R)f_{\gamma,2}(z,r) \end{pmatrix}; & r < R \\ \begin{pmatrix} f_{\gamma,2}(z,R)g_{\gamma,1}(z,r) \\ f_{\gamma,2}(z,R)g_{\gamma,2}(z,r) \end{pmatrix}; & r > R \end{cases} \tag{149}$$

$$\tilde{W}_{z,\gamma}^{(jl)}(r) = \begin{cases} \begin{pmatrix} g_{\gamma,2}(z,R)f_{\gamma,1}(z,r) \\ g_{\gamma,2}(z,R)f_{\gamma,2}(z,r) \end{pmatrix}; & r < R \\ -\begin{pmatrix} f_{\gamma,2}(z,R)g_{\gamma,1}(z,r) \\ f_{\gamma,2}(z,R)g_{\gamma,2}(z,r) \end{pmatrix}; & r > R \end{cases} \tag{150}$$

$$\hat{W}_{z,\gamma}^{(jl)}(r) = \begin{cases} \begin{pmatrix} g_{\gamma,1}(z,R)f_{\gamma,1}(z,r) \\ g_{\gamma,1}(z,R)f_{\gamma,2}(z,r) \end{pmatrix}; & r < R \\ -\begin{pmatrix} f_{\gamma,1}(z,R)g_{\gamma,1}(z,r) \\ f_{\gamma,1}(z,R)g_{\gamma,2}(z,r) \end{pmatrix}; & r > R \end{cases} \tag{151}$$

where $f_{\gamma,1}(z,r)$, $f_{\gamma,2}(z,r)$, $g_{\gamma,1}(z,r)$ and $g_{\gamma,2}(z,r)$ are given by (125)-(127) and (128) respectively.

(ii)The resolvent of $H_{\gamma,G}$ is given by

$$\begin{aligned} (H_{\gamma,G} - z)^{-1} &= (H_{\gamma,D} - z)^{-1} + \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} \bigoplus_{\mu=-j}^{j} \vartheta_{jl,\gamma}(z)\{A_{jl}(|.|^{-1}\overline{\tilde{M}_{z,\gamma}^{(jl)}(.)} \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi),.) \times \\ &\times |.|^{-1}\tilde{M}_{z,\gamma}^{(jl)}(.) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) + B_{jl}(|.|^{-1}\overline{\hat{M}_{z,\gamma}^{(jl)}(.)} \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi),.) \times \\ &\times |.|^{-1}\hat{M}_{z,\gamma}^{(jl)}(.) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) + \frac{A_{jl}B_{jl}}{2c}(|.|^{-1}\overline{\tilde{M}_{z,\gamma}^{(jl)}(.)} \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi),.) \times \\ &\times |.|^{-1}\tilde{W}_{z,\gamma}^{(jl)}(.) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) - \frac{A_{jl}B_{jl}}{2c}(|.|^{-1}\overline{\hat{M}_{z,\gamma}^{(jl)}(.)} \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi),.) \times \\ &\times |.|^{-1}\hat{W}_{z,\gamma}^{(jl)}(.) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi)\}, \ z \in \rho(H_{\gamma,G}), \ \ \mathrm{Im}k(z) > 0, \ \ B_{jl}, A_{jl} \in \mathbb{R} \end{aligned} \tag{152}$$

where we have used the following notations

$$\tilde{M}_{z,\gamma}^{(jl)}(r) \bigotimes \tilde{\Omega}_{jl\mu}(\theta,\varphi) = \begin{pmatrix} \tilde{M}_{z,\gamma,1}^{(jl)}(r)\Omega_{jl\mu}(\theta,\varphi) \\ \tilde{M}_{z,\gamma,2}^{(jl)}(r)\Omega_{jl'\mu}(\theta,\varphi) \end{pmatrix}, \tag{153}$$

with

$$\tilde{M}_{z,\gamma}^{(jl)}(r) = \begin{pmatrix} \tilde{M}_{z,\gamma,1}^{(jl)}(r) \\ \tilde{M}_{z,\gamma,2}^{(jl)}(r) \end{pmatrix}, \ \ \tilde{\Omega}_{jl\mu}(\theta,\varphi) = \begin{pmatrix} \Omega_{jl\mu}(\theta,\varphi) \\ \Omega_{jl'\mu}(\theta,\varphi) \end{pmatrix}$$

defined by eqs (148), (10) and (11) respectively.

Similarly, notations $\hat{M}^{(jl)}_{z,\gamma}(r)\bigotimes\tilde{\Omega}_{jl\mu}$, $\tilde{W}^{(jl)}_{z,\gamma}(r)\bigotimes\tilde{\Omega}_{jl\mu}$ and $\hat{W}^{(jl)}_{z,\gamma}(r)\bigotimes\tilde{\Omega}_{jl\mu}$ are defined by

$$\hat{M}^{(jl)}_{z,\gamma}(r)\bigotimes\tilde{\Omega}_{jl\mu}(\theta,\varphi)=\begin{pmatrix}\hat{M}^{(jl)}_{z,\gamma,1}(r)\Omega_{jl\mu}(\theta,\varphi)\\ \hat{M}^{(jl)}_{z,\gamma,2}(r)\Omega_{jl'\mu}(\theta,\varphi)\end{pmatrix},\quad \tilde{W}^{(jl)}_{z,\gamma}(r)\bigotimes\tilde{\Omega}_{jl\mu}(\theta,\varphi)=\begin{pmatrix}\tilde{W}^{(jl)}_{z,\gamma,1}(r)\Omega_{jl\mu}(\theta,\varphi)\\ \tilde{W}^{(jl)}_{z,\gamma,2}(r)\Omega_{jl'\mu}(\theta,\varphi)\end{pmatrix}$$

and $\hat{W}^{(jl)}_{z,\gamma}(r)\bigotimes\tilde{\Omega}_{jl\mu}(\theta,\varphi)=\begin{pmatrix}\hat{W}^{(jl)}_{z,\gamma,1}(r)\Omega_{jl\mu}(\theta,\varphi)\\ \hat{W}^{(jl)}_{z,\gamma,2}(r)\Omega_{jl'\mu}(\theta,\varphi)\end{pmatrix}$ respectively.

C. Spectral properties

The spectral properties of $h_{jl,\gamma,G_{jl}}$ are given by the following theorem

Theorem 4.2 *: For $A_{jl},B_{jl}\in(-\infty,\infty)$, $l\in[j-\frac{1}{2},j+\frac{1}{2}]$, $j\in[\frac{1}{2},\infty)$ and $\gamma\in\mathbb{R}$ we obtain*

$$\sigma_{ess}(h_{jl,\gamma,G_{jl}})=\sigma_{ac}(h_{jl,\gamma,G_{jl}})=(-\infty,-\frac{c^2}{2}]\bigcup[\frac{c^2}{2},\infty) \tag{154}$$

$$\sigma_{sc}(h_{jl,\gamma,G_{jl}})=\emptyset \tag{155}$$

$$\sigma_{p}(h_{jl,\gamma,G_{jl}})\bigcap(-\infty,-\frac{c^2}{2}]\bigcup[\frac{c^2}{2},\infty)=\emptyset\ \text{ if }\ 1+\frac{A_{jl}B_{jl}}{4c^2}\neq 0. \tag{156}$$

The eigenvalues of $h_{jl,\gamma,G_{jl}}$ are determined from the equation

$$1-\frac{A_{jl}B_{jl}}{4c^2}+A_{jl}f_{\gamma,1}(z,R)g_{\gamma,1}(z,R)+B_{jl}f_{\gamma,2}(z,R)g_{\gamma,2}(z,R)\Big|_{k(z)=i\sqrt{-E'}}=0,\quad E'<0. \tag{157}$$

where E' is given by (75).

Following step by step section 2.D, one can obtains the following result: for spin-$\frac{1}{2}$ particles i.e. $l=j+\frac{1}{2}$

$$n.\lim_{c\to\infty}(h_{jl,\gamma,G_{jl}}-\frac{c^2}{2}-z)^{-1}=(h_{l,\gamma,\hat{\beta}_l}-z)^{-1}\bigotimes\begin{pmatrix}1&0\\0&0\end{pmatrix} \tag{158}$$

where $h_{l,\gamma,\hat{\beta}_l}$ is defined by

$$\begin{aligned} h_{l,\gamma,\hat{\beta}_l} &= -\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\frac{\gamma}{r}\\ \mathcal{D}(h_{l,\gamma,\hat{\beta}_l}) &= \{f\in L^2((0,\infty))|f,f'\in \mathrm{AC}_{loc}((0,\infty)\backslash\{R\});f(0+)=0\ \text{ if }\ l=0;\\ &\quad f'(R+)-f'(R-)=\tfrac{\alpha_l}{2}[f(R+)+f(R-)];\\ &\quad f(R+)-f(R-)=\tfrac{\beta_l}{2}[f'(R+)+f'(R-)]\\ &\quad -f''+l(l+1)r^{-2}f+\gamma r^{-1}f\in L^2((0,\infty))\},\\ &\quad -\infty<\alpha_l,\beta_l\leq\infty;\ l\in\mathbb{N}_0,\ \ \gamma\in\mathbb{R}. \end{aligned} \tag{159}$$

One can show that the boundary conditions in (159) define a self-adjoint extensions of $h_{jl,\gamma}$ if $\alpha_l\beta_l-4=0$.

D. Scattering theory for the pair $(h_{jl,\gamma,G_{jl}};h_{\gamma,D})$

For $k(z)>0$, we define the fonction

$$\begin{aligned}\begin{pmatrix}F_{G_{jl},1}(z,r)\\ F_{G_{jl},2}(z,r)\end{pmatrix} &= \begin{pmatrix}f_{\gamma,1}(z,r)\\ f_{\gamma,2}(z,r)\end{pmatrix}+\vartheta_{jl,\gamma}(z)\{A_{jl}f_{\gamma,1}(z,R)\tilde{M}^{(jl)}_{z,\gamma}(r)+B_{jl}f_{\gamma,2}(z,R)\hat{M}^{(jl)}_{z,\gamma}(r)+\\ &+\frac{A_{jl}B_{jl}}{2c}f_{\gamma,1}(z,R)\tilde{W}^{(jl)}_{z,\gamma}(r)-\frac{A_{jl}B_{jl}}{2c}f_{\gamma,2}(z,R)\hat{W}^{(jl)}_{z,\gamma}(r)\}\end{aligned} \tag{160}$$

where $f_{\gamma,1}(z,r)$, $f_{\gamma,2}(z,r)$, $\vartheta_{jl,\gamma}(z)$, $\tilde{M}^{(jl)}_{z,\gamma}(r)$, $\hat{M}^{(jl)}_{z,\gamma}(r)$, $\tilde{W}^{(jl)}_{z,\gamma}(r)$ and $\hat{W}^{(jl)}_{z,\gamma}(r)$ are defined by (125), (126), (147), (148), (149), (150) and (151) respectively. A straightforward computation proves that $\binom{F_{G_{jl},1}(z,r)}{F_{G_{jl},2}(z,r)}$ is the wave scattering function of $h_{jl,\gamma,G_{jl}}$.

The study of the asymptotic behavior of $\binom{F_{G_{jl},1}(z,r)}{F_{G_{jl},2}(z,r)}$ as $r \to \infty$ yields[28]

$$\begin{pmatrix} F_{G_{jl},1}(z,r) \\ F_{G_{jl},2}(z,r) \end{pmatrix} \xrightarrow[r\to\infty]{k(z)>0} \begin{pmatrix} V_1(z)\sin[x_1+\delta^0_{\tilde{\xi}}(z)] + V_2(z)\cos[x_1+\delta^0_{\tilde{\xi}}(z)] \\ V_3(z)\sin[x_2+\delta^0_{\tilde{\xi}-1}(z)] + V_4(z)\cos[x_2+\delta^0_{\tilde{\xi}-1}(z)] \end{pmatrix}$$
$$= \begin{pmatrix} [V_1^2(z)+V_2^2(z)]^{\frac{1}{2}}\sin[x_1+\delta^0_{\tilde{\xi}}(z)+\delta^C_{\gamma,G_{jl},1}(z)] \\ [V_3^2(z)+V_4^2(z)]^{\frac{1}{2}}\sin[x_2+\delta^0_{\tilde{\xi}-1}(z)+\delta^C_{\gamma,G_{jl},2}(z)] \end{pmatrix} \quad (161)$$

where

$$x_1 = k - \frac{\tilde{\gamma}}{2k}\ln(2kr) - \tilde{\xi}\frac{\pi}{2} \quad (162)$$
$$x_2 = k - \frac{\tilde{\gamma}}{2k}\ln(2kr) - \left(\tilde{\xi}-1\right)\frac{\pi}{2}. \quad (163)$$

and

$$\delta^0_{\tilde{\xi}}(z) = \delta^0_{\tilde{\xi}-1}(z) + \arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right) \quad (164)$$
$$\delta^0_{\tilde{\xi}}(z) = \arg\Gamma\left(\tilde{\xi}+1+\mathrm{i}\frac{\tilde{\gamma}}{2k}\right). \quad (165)$$

The Coulomb modified phase shift $\delta^C_{\gamma,G_{jl}}(z)$ is given by

$$\delta^C_{\gamma,G_{jl}}(z) = \begin{pmatrix} \delta^C_{\gamma,G_{jl},1}(z) \\ \delta^C_{\gamma,G_{jl},2}(z) \end{pmatrix} \quad (166)$$

where

$$\delta^C_{\gamma,G_{jl},1}(z) = -\arctan\frac{V_2(z)}{V_1(z)} \quad (167)$$
$$\delta^C_{\gamma,G_{jl},2}(z) = -\arctan\frac{V_4(z)}{V_3(z)}. \quad (168)$$

The contants $V_i(i=1,...,4)$ are defined by

$$\begin{aligned} V_1(z) &= d_1(z) + \left[d_2(z) - \mathrm{i}\vartheta'_{jl,\gamma}(z)d_4(z)\right]\sin\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right] \\ &- \mathrm{i}\vartheta'_{jl,\gamma}(z)d_3(z) - \vartheta'_{jl,\gamma}(z)d_4(z)\cos\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right] \end{aligned} \quad (169)$$

$$\begin{aligned} V_2(z) &= \left[d_2(z) - \mathrm{i}\vartheta'_{jl,\gamma}(z)d_4(z)\right]\cos\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right] \\ &+ \vartheta'_{jl,\gamma}(z)d_3(z) + \vartheta'_{jl,\gamma}(z)d_4(z)\sin\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right] \end{aligned} \quad (170)$$

$$\begin{aligned} V_3(z) &= \left[-\frac{\gamma}{\kappa_{jl}c+\xi}d_1(z) + \mathrm{i}\vartheta'_{jl,\gamma}(z)\frac{\gamma}{\kappa_{jl}c+\xi}d_3(z)\right]\sin\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right] \\ &- \left(\frac{\gamma}{\kappa_{jl}c+\xi}\right)^{-1}d_2(z) - \vartheta'_{jl,\gamma}(z)\frac{\gamma}{\kappa_{jl}c+\xi}d_3(z)\cos\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right] + \end{aligned}$$

$$i\vartheta'_{jl,\gamma}(z)\left(\frac{\gamma}{\kappa_{jl}c+\xi}\right)^{-1} d_4(z) \tag{171}$$

$$\begin{aligned} V_4(z) &= \left[-\frac{\gamma}{\kappa_{jl}c+\xi}d_1(z)+i\vartheta'_{jl,\gamma}(z)\frac{\gamma}{\kappa_{jl}c+\xi}d_3(z)\right]\cos\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right] \\ &- \vartheta'_{jl,\gamma}(z)\frac{\gamma}{\kappa_{jl}c+\xi}d_3(z)\sin\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]-\vartheta'_{jl,\gamma}(z)\left(\frac{\gamma}{\kappa_{jl}c+\xi}\right)^{-1}d_4(z) \end{aligned} \tag{172}$$

with $\vartheta'_{jl}(z)$ defined by

$$\vartheta'_{jl}(z)=\vartheta_{jl}(z)\left[A_{jl}f_{\gamma,1}(z,R)f_{\gamma,1}(z,R)+B_{jl}f_{\gamma,2}(z,R)f_{\gamma,2}(z,R)\right]. \tag{173}$$

The constants $d_i(z)(i=1,...,4)$ are given by

$$\begin{aligned} d_1(z) &= \left(1-\frac{\gamma^2}{(\kappa_{jl}c+\xi)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]^{-\frac{1}{2}}2^{-\tilde{\xi}}(k)^{-\tilde{\xi}-1}\Gamma\left(2\tilde{\xi}+2\right)\times \\ &\times \left|\Gamma\left(\tilde{\xi}+1+i\frac{\tilde{\gamma}}{2k}\right)\right|^{-1}e^{\pi\frac{\tilde{\gamma}}{4k}}, \end{aligned} \tag{174}$$

$$\begin{aligned} d_2(z) &= -\frac{\gamma}{c(\kappa_{jl}c+\xi)}\left(1-\frac{\gamma^2}{(\kappa_{jl}c+\xi)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]^{-\frac{1}{2}}2^{-\tilde{\xi}}(k)^{-\tilde{\xi}}\times \\ &\times \Gamma\left(2\tilde{\xi}+2\right)\left|\Gamma\left(\tilde{\xi}+1+i\frac{\tilde{\gamma}}{2k}\right)\right|^{-1}e^{\pi\frac{\tilde{\gamma}}{4k}}, \end{aligned} \tag{175}$$

$$\begin{aligned} d_3(z) &= \left(1-\frac{\gamma^2}{(\kappa_{jl}c+\xi)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]^{-\frac{1}{2}}2^{\tilde{\xi}}(k)^{\tilde{\xi}}\times \\ &\times \Gamma\left(2\tilde{\xi}+2\right)^{-1}\left|\Gamma\left(\tilde{\xi}+1+i\frac{\tilde{\gamma}}{2k}\right)\right|e^{-\pi\frac{\tilde{\gamma}}{4k}}, \end{aligned} \tag{176}$$

$$\begin{aligned} d_4(z) &= -\frac{\gamma}{c(\kappa_{jl}c+\xi)}\left(1-\frac{\gamma^2}{(\kappa_{jl}c+\xi)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\xi}}\right)\right]^{-\frac{1}{2}}2^{\tilde{\xi}}(k)^{\tilde{\xi}+1}\times \\ &\times \Gamma\left(2\tilde{\xi}+2\right)^{-1}\left|\Gamma\left(\tilde{\xi}+1+i\frac{\tilde{\gamma}}{2k}\right)\right|e^{-\pi\frac{\tilde{\gamma}}{4k}}. \end{aligned} \tag{177}$$

The limit $\gamma\to 0+$, in (167)and (168) yields:

$$\lim_{\gamma\to 0+}\delta^C_{\gamma,G_{jl},1}(z)=\lim_{\gamma\to 0+}\delta^C_{\gamma,G_{jl},2}(z)=\delta_{G_{jl}}(z) \tag{178}$$

where $\delta_{G_{jl}}(z)$ is defined by (102).

The Coulomb modified on-shell scattering matrix is given by

$$S^C_{\gamma,G_{jl},n}(z)=e^{2i\delta^C_{\gamma,G_{jl},n}(z)},\quad n=1,2. \tag{179}$$

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