Subordination operators of Dirichlet forms

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Abstract

Let $(W_\alpha)_{\alpha>0}$ be the associated resolvent to a given Dirichlet form and P an operator defined on $L^2(X,m)$. In this work, we give some conditions on P so that the family of operators $\{V_\lambda := (I-P_\lambda)^{-1}W_\lambda, \lambda > 0\}$ defines a resolvent associated to a Dirichlet form, where $P_\lambda = (I-\lambda W_\lambda)P$.

1 Preliminaries

We consider the Hilbert space $H = L^2(X,m)$, where X is a given locally compact topological space with a countable basis, and m is a positive Radon measure on X with $Supp[m] = X$ and $m(X) < +\infty$.
By $(u,v) = \int_X uvm(dx)$ we denote the inner product of H, and by $||.||$ the related norm.
If $D \in H$, we denote by D_+ the subset of m-a.e. positive functions.

Definition 1.1 ([5]) *A symmetric linear operator A on H with a dense domain $D(A)$, is said to be monotone if, for all $u \in D(A), (Au,u) \geq 0$. A is maximal if morover it satisfies $R(I+A) = H$ that is, for all $f \in H$ there exists $u \in D(A)$ such that $u+Au = f$.*

Lemma 1.1 ([7]) *The product of two commuting self adjoint monotone operators on H is a monotone operator.*

A symmetric Dirichlet form on H is a closed symmetric nonnegative bilinear form ε with domain $D(\varepsilon)$ such that $u \in D(\varepsilon)$ imlplies that $u^+ \wedge 1 \in D(\varepsilon)$ and $\varepsilon(u^+\wedge 1, u^+\wedge 1) \leq \varepsilon(u,u)$ for all $u \in D(\varepsilon)$. The last property will be called the sub-Markovian property of ε. To any Dirichlet form $(\varepsilon, D(\varepsilon))$ on H one can associate a unique self adjoint negative operator L with domain $D(L)$ which satisfies

i) $D(L) \subset D(\varepsilon)$

ii) $\varepsilon(u,v) = (-Lu, v)$; for all $u \in D(L), v \in D(\varepsilon)$

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Let $\{W_\beta := (\beta I - L)^{-1}, \beta > 0\}$ and $\{T_t, t > 0\}$ be the sub-Markovian strongly continuous resolvent and semigroup associated to the operator L. To the semigroup $\{T_t, t > 0\}$ and the resolvent $\{W_\beta, \beta > 0\}$, we associate the following approximants

$$\varepsilon^{(t)}(u,v) := \frac{1}{t}(u - T_t u, v) \ ; \quad \text{for all } u,v \in H$$

$$\varepsilon^{(\beta)}(u,v) := \beta(u - \beta W_\beta u, v) \ ; \quad \text{for all } u,v \in H.$$

We get

$$D(\varepsilon) = \{u \in H; \lim_{t\to 0} \varepsilon^{(t)}(u,u) < +\infty\}$$

$$\varepsilon(u,v) = \lim_{t\to 0} \varepsilon^{(t)}(u,v) \ ; \quad \text{for all } u,v \in D(\varepsilon)$$

and

$$D(\varepsilon) = \{u \in H \ ; \ \lim_{\beta\to+\infty} \varepsilon^{(\beta)}(u,u) < +\infty\}$$

$$\varepsilon(u,v) = \lim_{\beta\to+\infty} \varepsilon^{(\beta)}(u,v) \ ; \quad \text{for all } u,v \in D(\varepsilon).$$

Definition 1.2 ([**6**]) *Let $\alpha \in]0,+\infty[$. A function $u \in H$ is called α-excessive if $e^{-\alpha t}T_t u \leq u$ for all $t > 0$.*

Proposition 1.1 ([**9**]) *Let $u \in H$ and $\alpha > 0$. If u is α-excessive then $u \geq 0$ and $\beta W_{\beta+\alpha}u \leq u$, for all $\beta > 0$. Moreover, for $u \in D(\varepsilon)$, the following assertions are equivalent*

i) u is α-excessive.

ii) $\beta W_{\beta+\alpha}u \leq u$; for all $\beta > 0$.

iii) $\varepsilon_\alpha(u,v) \geq 0$; for all $v \in D_+(\varepsilon)$.

We denote by $\mathbf{E}_\alpha$, the set of α-excessive functions in $D(\varepsilon)$, and $\mathbf{E} = \bigcap_{\alpha>0} \mathbf{E}_\alpha$.

2 Perturbation

Let $(\varepsilon, D(\varepsilon))$ be a symmetric Dirichlet form on $H, (W_\alpha)_{\alpha>0}$ the associated resolvent and L the associated infinitesimal generator. In all this section we assume that P is a linear symmetric monotone operator on H such that $Pf \in \mathbf{E}$, for all $f \in H_+$ that it satisfies
H_1) $||P|| \leq 1$ and for all $\lambda > 0, (I - P_\lambda)$ is invertible with $(I - P_\lambda)^{-1} = \sum_{n\geq 0} P_\lambda^n$.
H_2) $Pu \in D(L)$ for all $u \in H$.
H_3) (LP) is self-adjoint, that is $(LP)^\star = LP$.

Remark:We have D(LP)= H and LP is continuous by Hellinger-Toeplitz's theorem [13].

Lemma 2.1 *$-L(I-P)$ is a maximal monotone operator.*

Proof : Let $u \in D(L)$ and $(I-L)u := f$. Using lemma 1.1, we obtain

$$\begin{array}{rcl}
(-L(I-P)u,u) & = & (((I-P)-L(I-P))u,u)-((I-P)u,u) \\
& = & ((I-P)(I-L)u,u)-((I-P)u,u) \\
& = & ((I-P)f,W_1f)-((I-P)W_1f,W_1f) \\
& = & ((I-P)(I-W_1)f,W_1f) \\
& = & (W_1(I-P)(I-W_1)f,f) \\
& \geq & 0.
\end{array}$$

Hence $-L(I-P)$ is a self-adjoint operator by the hypothesis (H_3).
Using ([4] , proposition 1.1.2) we have that $-L(I-P)$ is a maximal monotone operator.

Corollary 2.1 ([**5**]) *We have*

i) *$-L(I-P)$ is closed.*

ii) *For all $\alpha > 0$, $\alpha I - L(I-P)$ is one to one from $D(L)$ into $H, (\alpha I - L(I-P))^{-1}$ is a bounded operator and $||\alpha(\alpha I - L(I-P))^{-1}|| \leq 1$.*

Theorem 2.1 *Let $V_\alpha = (\alpha I - L(I-P))^{-1}$ for every $\alpha > 0$. Then*

i) *$V_\alpha = (I-P_\alpha)^{-1}W_\alpha$ for all $\alpha > 0$.*

ii) *$(V_\alpha)_{\alpha>0}$ is strongly continuous resolvent of contraction on H and $W_\alpha f \leq V_\alpha f$ for all $f \in H_+$.*

Proof

i) For all $\alpha > 0$, we have

$$\begin{array}{rcl}
\alpha I - L + LP & = & (\alpha I - L)(I + (\alpha I - L)^{-1}LP) \\
& = & (\alpha I - L)(I + W_\alpha(\alpha I - W_\alpha^{-1})P) \\
& = & (\alpha I - L)(I + (\alpha W_\alpha - I)P) \\
& = & (\alpha I - L)(I - P_\alpha).
\end{array}$$

Hence
$$V_\alpha = (\alpha I - L + LP)^{-1} = (I-P_\alpha)^{-1}W_\alpha \text{ for all } \alpha > 0.$$

ii) For all $\alpha > 0$ and $\beta > 0$, we have

$$\begin{array}{rcl}
(\beta-\alpha)V_\alpha V_\beta & = & V_\alpha(\beta-\alpha)V_\beta \\
& = & V_\alpha((\beta I - L(I-P)) - (\alpha I - L(I-P))V_\beta \\
& = & V_\alpha - V_\beta.
\end{array}$$

We also get

$$(\beta-\alpha)V_\beta V_\alpha = V_\alpha - V_\beta.$$

Hence

$$V_\alpha - V_\beta = (\beta-\alpha)V_\alpha V_\beta = (\beta-\alpha)V_\beta V_\alpha.$$

Remark 2.1 *1. For all $u, v \in D(L)$, ${}^P\varepsilon(u,v) := (-L(I-P)u, v)$. $({}^P\varepsilon, D(L))$ is closable, its closure $({}^P\varepsilon, D({}^P\varepsilon))$ is a symmetric closed form and*

$${}^P\varepsilon(u,v) = (-L(I-P)u,v) \;\; ; \quad \textit{for all } u \in D(L) \;\; , \;\; v \in D({}^P\varepsilon).$$

By definition $({}^P\varepsilon, D({}^P\varepsilon))$ is associated with the resolvent $(V_\alpha)_\alpha > 0$.

2. *$({}^P\varepsilon, D({}^P\varepsilon))$ preserves the positivity (i.e. ${}^P\varepsilon(u,u^+) \geq 0$, for all $u \in D({}^P\varepsilon)$), because the associated resolvent $(V_\alpha)_\alpha > 0$ preserves the positivity.*

3. *The module contraction operates on $({}^P\varepsilon, D({}^P\varepsilon))$ that is*

$${}^P\varepsilon(|u|,|u|) \leq {}^P\varepsilon(u,u) \;\; ; \quad \textit{for all } u \in D({}^P\varepsilon).$$

4. *There exists $c > 0$, such that ${}^P\varepsilon(u,u) \leq c(\varepsilon(u,u) + ||u||^2)$, for all $u \in D(\varepsilon)$.*

Remark 2.2 *If $1 - P_\alpha 1 \in \mathbf{E}_\alpha$, for all $\alpha > 0$, then $1 - P1 \in \mathbf{E}$.*

Proposition 2.1 *The following statements are equivalent*

i) $1 - P1 \in \mathbf{E}$.

ii) $(V_\alpha)_\alpha > 0$ *is sub-Markovian.*

iii) ${}^P\varepsilon(u^+ \wedge 1, u^+ \wedge 1) \leq {}^P\varepsilon(u,u)$*, for all* $u \in D({}^P\varepsilon)$.

Proof i) implies ii). We have

$$\alpha W_\alpha(1-P1) \leq 1 - P1,$$

hence

$$\alpha W_\alpha 1 \leq (I - P_\alpha)1,$$

then

$$\alpha V_\alpha 1 = \alpha(I-P_\alpha)^{-1} W_\alpha 1 \leq 1.$$

ii) implies i). Let $\alpha > 0$, there exists a sequence $(f_n)_{n\in I\!N} \subset H$ such that $f_n \geq 0$ for all $n \in I\!N$ and $\sup_n V_\alpha f_n = 1$. We have

$$V_\alpha f_n - P_\alpha V_\alpha f_n = W_\alpha f_n \in \mathbf{E}_\alpha.$$

Hence, $1 - P_\alpha 1 \in \mathbf{E}_\alpha$.
ii) and iii) are equivalent (cf the proof of theorem 4.4 [9])

Definition 2.1 ([**3**]) *Let* $(\varepsilon^1, D(\varepsilon^1))$, $(\varepsilon^2, D(\varepsilon^2))$ *be two Dirichlet forms on* H. $(\varepsilon^1, D(\varepsilon^1))$ *is said to be subordinate to* $(\varepsilon^2, D(\varepsilon^2))$ *and we write* $\varepsilon^1 << \varepsilon^2$, *if and only if* $D(\varepsilon^2) \subset D(\varepsilon^1)$ *and* $\varepsilon^1(f,g) \le \varepsilon^2(f,g)$; *for all* $f, g \in D_+(\varepsilon^2)$.

We assume that the following additional condition is satisfied
$H_4)$ $1 - P1 \in \mathbf{E}$.

Proposition 2.2 $({}^P\varepsilon, D({}^P\varepsilon))$ *is a Dirichlet form on* H *subordinate to* $(\varepsilon, D(\varepsilon))$.

Proof We have

$$ {}^P\varepsilon_\alpha(V_\alpha u, v) = (u, v) \ ; \text{ for all } u \in H \ , \ \ v \in D({}^P\varepsilon). $$

Then by Remark 2.1 and Proposition 2.1, $({}^P(\varepsilon, D({}^P\varepsilon))$ is a Dirichlet form on H.
Let $u, v \in D_+(\varepsilon) \subset D_+({}^P\varepsilon)$. We have

$$ \varepsilon(u, v) = \lim_{\beta \to +\infty} \beta(u - \beta W_\beta u, v) $$

and

$$ {}^P\varepsilon(u, v) = \lim_{\beta \to +\infty} \beta(u - \beta V_\beta u, v). $$

But

$$ W_\beta u \le V_\beta u \ ; \quad \text{for all } \beta > 0. $$

Hence

$$ {}^P\varepsilon(u, v) \le \varepsilon(u, v). $$

Therefore $({}^P\varepsilon, D({}^P\varepsilon))$ is a Dirichlet form on H subordinate to $(\varepsilon, D(\varepsilon))$.

Proposition 2.3 *Let* $\alpha \in]0, 1]$, *and set* ${}^{\alpha,P}\varepsilon(u, v) = \alpha\varepsilon(u, v) + (1-\alpha)\, {}^P\varepsilon(u, v)$, *for all* $u, v \in D(\varepsilon)$. *Then* $({}^{\alpha,P}\varepsilon, D(\varepsilon))$ *is a Dirichlet form on* H *subordinate to* $(\varepsilon, D(\varepsilon))$. *Moreover,*

$$ {}^{\alpha,P}\varepsilon_\lambda(G_{\lambda,\alpha} u, v) = (u, v), \quad \textit{for all } u \in H, v \in D(\varepsilon), $$

where

$$ G_{\lambda,\alpha} f = \sum_{n \ge 0} (1-\alpha)^n P_\lambda^n W_\lambda f \ . $$

Proof
Let $\alpha > 0$, note that $({}^{\alpha,P}\varepsilon, D(\varepsilon))$ is obviously a Dirichlet form on H subordinate to $(\varepsilon, D(\varepsilon))$.
Let $u \in D(L)$ and $v \in D(\varepsilon)$, we have

$$ \begin{aligned} {}^{\alpha,P}\varepsilon(u, v) &= \alpha\varepsilon(u, v) + (1-\alpha)\ {}^P\varepsilon(u, v) \\ &= \alpha(-Lu, v) + (1-\alpha)(-L(I-P)u, v) \\ &= (-(\alpha L + (1-\alpha)L(I-P))u, v) \\ &= (-(L - (1-\alpha)LP)u, v) \end{aligned} $$

then the associated generator to $({}^{\alpha,P}\varepsilon, D(\varepsilon))$ is $L-(1-\alpha)LP$ and

$$\begin{aligned}G_{\lambda,\alpha} &= (\lambda I - L + (1-\alpha)LP)^{-1}\\ &= [(\lambda I - L)(I + (1-\alpha)W_\alpha LP)]^{-1}\\ &= [(\lambda I - L)(I+(1-\alpha)(\lambda W_\lambda - I)P)]^{-1}\\ &= (I-(1-\alpha)(I-\lambda W_\lambda)P)^{-1}(\lambda I - L)^{-1}\\ &= (I-P_{\lambda,\alpha})^{-1}W_\lambda \ \ (where\ \ P_{\lambda,\alpha} = (1-\alpha)P_\lambda)\\ &= \sum_{n\geq 0} P^n_{\lambda,\alpha}W_\lambda\\ &= \sum_{n\geq 0}(1-\alpha)^n P^n_\lambda W_\lambda.\end{aligned}$$

Thus $(G_{\lambda,\alpha})_{\lambda>0}$ is the associated resolvent to $({}^{\alpha,P}\varepsilon, D(\varepsilon))$.

Remark 2.3 *Let $\alpha_1,\alpha_2 \in [0,1]$ such that $\alpha_1 \leq \alpha_2$, then we have*

$$^{\alpha_1,P}\varepsilon << {}^{\alpha_2,P}\varepsilon$$

Recall that ([**6**]), a Dirichlet form $(\varepsilon, D(\varepsilon))$ is said to be regular if $D(\varepsilon)\bigcap C_0(X)$ is dense in $D(\varepsilon)$ with respect to $\varepsilon_1(.) = \varepsilon(.) + (.,.)$, and dense in $C_0(X)$ with respect to the uniform convergence topology. $C_0(X)$ being the set of continuous functions with compact support.

Assume in the following that $(\varepsilon, D(\varepsilon))$ is a regular Dirichlet form on H and $C_0(X) \subset D(\varepsilon)$. Then $(\varepsilon, D(\varepsilon))$ can be expressed as follows (Beurling-Deny [6]) :

$$\varepsilon(u,v) = \varepsilon^{(c)}(u,v) \ + \int\int_{X\times X-\Delta}(u(x)-u(y))(v(x)-v(y))J(dxdy)$$
$$+\int_X uvdk(x), \text{ for all } u,v \in C_0(X)$$

where $\varepsilon^{(c)}$ is a symmetric form, $D(\varepsilon^{(c)}) = C_0(X)$ and satisfies $\varepsilon^{(c)}(u,v) = 0$ for all $u,v \in D(\varepsilon^{(c)})$, v constant on a neighborhood of supp$u$. J is a positive Radon measure on $X\times X$ of the diagonal Δ and k is a positive Radon measure on X.

Remark 2.4 *Let $B(u,v) = \varepsilon(u,v) - {}^P\varepsilon(u,v)$; for all $u,v \in D(\varepsilon)$. B is a symmetric bilinear form and $B(u,v) \geq 0$ for all $u,v \in D_+(\varepsilon)$, there exists a symmetric positive measure σ on $X\times X$ such that*

$$B(u,v) = \int\int_{X\times X} u(x)v(y)\sigma(dxdy), \quad for\ all\ u,v \in C_0(X).$$

Furthermore

$$\varepsilon(u,v) = {}^P\varepsilon(u,v) + \int\int_{X\times X} u(x)v(y)\sigma(dxdy), \quad for\ all\ u,v\in C_0(X).$$

The folloying is obvious.

Corollary 2.2 *$({}^P\varepsilon, D({}^P\varepsilon)$ is a regular Dirichlet form on H. In particular for all $u, v \in C_0(X)$ we have*

$$
{}^P\varepsilon(u,v) = {}^P\varepsilon^{(c)}(u,v) + \int\int_{X\times X-\Delta}(u(x)-u(y))(v(x)-v(y))dJ^P(x,y) + \int_X u(x)v(x)dk^P(x)
$$

where $J^P = J + \frac{\sigma}{2}\mid_{X\times X-\Delta}$.

Proposition 2.4 *We have*

$$
D(\varepsilon)\bigcap {}^P\mathbf{E}_\alpha = \{u \in \mathbf{E}_\alpha; u - P_\alpha u \in \mathbf{E}_\alpha\}.
$$

Proof Let $u \in D(\varepsilon)\bigcap^P \mathbf{E}_\alpha$, then

$$
\varepsilon_\alpha(u,v) \geq 0, \text{ for all } v \in D_+(\varepsilon).
$$

Hence $u \in \mathbf{E}_\alpha$. Conversly, if $u \in E_\alpha$ and $u - P_\alpha u \in E_\alpha$, for every $v \in D_+(\varepsilon)$, we have

$$
\begin{aligned}
{}^P\varepsilon_\alpha(u,v) &= {}^P\varepsilon(u,v) + \alpha(u,v) \\
&= \varepsilon((I-P)u,v) + \alpha(u,v) \\
&= \varepsilon((I-P)u,v) + \alpha(u,v) - \alpha(Pu,v) + \alpha(Pu,v) \\
&= \varepsilon_\alpha((I-P)u,v) + \alpha(Pu,v) \\
&= \varepsilon_\alpha((I-P_\alpha)u,v) \\
&\geq 0.
\end{aligned}
$$

Remark 2.5 *1. $D(\varepsilon)\bigcap {}^P\mathbf{E} = \bigcap_{\alpha>0}(D(\varepsilon)\bigcap {}^P\mathbf{E}_\alpha) = \{u \in \mathbf{E}; u - Pu \in \mathbf{E}\}$.*

2. Let $u \in {}^P\mathbf{E}$. Assume that there exists $v \in \mathbf{E}$ such that $u \leq v$ then $u \in D(\varepsilon)$.

3 Examples

1. $\varepsilon(u,v) = \int_X u(x)v(x)m(dx)$; for all $u, v \in D(\varepsilon) = H$
   Let P be a sub-Markovian operator on H, we define

   $$
   {}^P\varepsilon(u,v) = (u - Pu, v) \quad ; \quad \text{for all } u, v \in H.
   $$

   Then $({}^P\varepsilon, D({}^P\varepsilon))$ is a Dirichlet form on H subordinate to a $(\varepsilon, D(\varepsilon))$.

2. Let $(\varepsilon, D(\varepsilon))$ be a Dirichlet form on H, $(W_\alpha)_{\alpha>0}$ the associated resolvent and $P = \alpha_0 W_{\alpha_0}$. Then $(\varepsilon_{\alpha_0}, D(\varepsilon))$ is a Dirichlet form on H and we have

   $$
   \varepsilon_{\alpha_0}(u - Pu, v) = \varepsilon(u,v) \quad ; \quad \text{for all } u, v \in D(\varepsilon).
   $$

3. Let $(\varepsilon, D(\varepsilon))$ be a regular symmetric Dirichlet form on H and σ a positive Radon measure, symmetric of positive type and sub-Markovian on $X \times X$. By Beurling-Deny's theorem, we get

$$\varepsilon(u,v) = \varepsilon^{(c)}(u,v) + \int\int_{X\times X-\Delta} (u(x)-u(y))(v(x)-v(y))dJ(x,y)$$

$$+ \int_X u(x)v(x)dk(x), \quad \text{for all } u,v \in D(\varepsilon)\bigcap C_0(X).$$

Define φ by

$$\begin{array}{rcl} \varphi : \Delta & \longrightarrow & X, \\ (x,x) & \longmapsto & x \end{array}$$

and let μ be the measure on X, defined by

$$A \subset X, \ \mu(A) = \sigma(\varphi^{-1}(A)).$$

We define

$$\varepsilon^{\sigma}(u,v) = \varepsilon(u,v) - \int\int_{X\times X} u(x)v(y)\sigma(dxdy) \ ; \quad \text{for all } u,v \in D(\varepsilon^{\sigma}).$$

The folloying are easy to prove.

Proposition 3.1 *If $k \geq \mu$, then $(\varepsilon^{\sigma}, D(\varepsilon^{\sigma}))$ is a regular Dirichlet form on H.*

Corollary 3.1 *Assume that $(\varepsilon^{\sigma}, D(\varepsilon^{\sigma}))$ is a symmetric Dirichlet form on H. If $(W_\lambda)_{\lambda>0}$ and $(W_\lambda^{\sigma})_{\lambda>0}$ are the resolvents associated respectively to $(\varepsilon, D(\varepsilon))$ and $(\varepsilon^{\sigma}, D(\varepsilon^{\sigma}))$, then we get*

$$W_\lambda^{\sigma} = W_\lambda + W_\lambda A W_\lambda^{\sigma} = W_\lambda + W_\lambda^{\sigma} A W_\lambda \ m-a.e, \quad \textit{for all } \lambda > 0$$

where A is the symmetric and sub-Markovian operator associted to σ with $D(A) = H$.

Remark 3.1 *(a) $P_\lambda = W_\lambda A$.*

(b) There exists α_0 such that $(\varepsilon_{\alpha_0}^{\sigma}, D(\varepsilon^{\sigma}))$ is a Dirichlet form on H.

4 Convergence

Let $(\varepsilon, D(\varepsilon))$ be a symmetric Dirichlet form on H and $(W_\alpha)_{\alpha>0}$ the associated resolvent. Suppose that the sequence of operators $(P_n)_{n\in I\!N}$ satisfies the hypothesis of the section 2 and converges to an operator P in H. Let $\{({}^{P_n}\varepsilon, D({}^{P_n}\varepsilon)), n \in I\!N\}$ be a family of Dirichlet forms on H. Moreover, we assume that there exists a Dirichlet form $(\tilde{\varepsilon}, D(\tilde{\varepsilon}))$

wich satisfies, $D(\tilde{\varepsilon}) \supset \cap_{n\geq 0} D({}^{P_n}\varepsilon)$, $\tilde{\varepsilon}(u,u) = \lim_{n\to+\infty} {}^{P_n}\varepsilon(u,u)$ *for all* $u \in \cap_{n\geq 0} D({}^{P_n}\varepsilon)$, there exists a constant c independent of n such that $\tilde{\varepsilon}(u,u) \leq c\,({}^{P_n}\varepsilon(u,u) + (u,u))$ for all $u \in \cap_{n\geq 0} D({}^{P_n}\varepsilon)$ and the canonical injection from $D(\tilde{\varepsilon})$ into H is compact. Let us denote by $(V_\alpha^{(n)})_{\alpha>0}$, $(V_\alpha)_{\alpha>0}$ the resolvents associated to $({}^{P_n}\varepsilon, D({}^{P_n}\varepsilon))$ and $(\tilde{\varepsilon}, D(\tilde{\varepsilon}))$ respectively.
We have $sup_n||\alpha V_\alpha^{(n)}|| \leq 1$, and

$$ {}^{P_n}\varepsilon_\alpha(V_\alpha^{(n)}f, V_\alpha^{(n)}f) = (f, V_\alpha^{(n)}f) \leq \alpha^{-1}||f||^2. $$

Then $sup_n||V_\alpha^{(n)}f||_1 < +\infty$, with $||.||_1^2 = \tilde{\varepsilon}(.) + (.,.)$. By the Banach-Alaouglu theorem, there exists a subsequence $(V_\alpha^{(n_k)})_{k\in I\!N}$ that converges weakly to Vf in $D(\tilde{\varepsilon})$. We can assume that $(V_\alpha^{(n)}f)_{n\in I\!N}$ converges to Vf m-a.e. In this section we will prove for all $\alpha > 0$, the strongly convergence of the sequence of resolvent $(V_\alpha^{(n)})_{n\in I\!N}$ to the resolvent V_α in H.

Lemma 4.1 *For all* $\alpha > 0$ *and* $f \in H$*, we have*

$$ \lim_{n\to+\infty}[\tilde{\varepsilon}_\alpha(v, V_\alpha^{(n)}f) - {}^{P_n}\varepsilon_\alpha(v, V_\alpha^{(n)}f)] = 0, \ \textit{for all } v \in \cap_{n\geq 0} D({}^{P_n}\varepsilon). $$

Proof Let $\alpha > 0$, $f \in H$ and $v \in \cap_{n\geq 0} D({}^{P_n}\varepsilon)$, we have

$$ \tilde{\varepsilon}(v, V_\alpha^{(n)}f) = \lim_{m\to+\infty} {}^{P_m}\varepsilon(v, V_\alpha^{(n)}f) = \lim_{m\to+\infty} \varepsilon(v - P_m v, V_\alpha^{(n)}f). $$

Hence

$$ \begin{aligned} \tilde{\varepsilon}_\alpha(v, V_\alpha^{(n)}f) - {}^{P_n}\varepsilon_\alpha(v, V_\alpha^{(n)}f) &= \lim_{m\to+\infty}[\,{}^{P_m}\varepsilon_\alpha(v, V_\alpha^{(n)}f) - {}^{P_n}\varepsilon_\alpha(v, V_\alpha^{(n)}f)] \\ &= \lim_{m\to+\infty}[\varepsilon(v - P_m v, V_\alpha^{(n)}f) - \varepsilon(v - P_n v, V_\alpha^{(n)}f)] \\ &= \lim_{m\to+\infty} \varepsilon(P_n v - P_m v, V_\alpha^{(n)}f) \\ &= \lim_{m\to+\infty}[\varepsilon_\alpha(V_\alpha^{(n)}f, P_n v - P_m v) - \alpha(V_\alpha^{(n)}f, P_n v - P_m v)] \\ &= \lim_{m\to+\infty}[\varepsilon_\alpha(\sum_{k\geq 0} P_{\alpha,n}^k W_\alpha f, P_n v - P_m v) - \alpha(V_\alpha^{(n)}f, P_n v - P_m v)] \\ &= \lim_{m\to+\infty}[(\sum_{k\geq 0} P_{\alpha,n}^k f, P_n v - P_m v) - \alpha(V_\alpha^{(n)}f, P_n v - P_m v)]. \end{aligned} $$

Then

$$ \lim_{n\to+\infty}[\tilde{\varepsilon}_\alpha(v, V_\alpha^{(n)}f) - {}^{P_n}\varepsilon_\alpha(v, V_\alpha^{(n)}f)] = 0. $$

Lemma 4.2 *Let* $\alpha > 0$ *and* $(f_n)_{n\in I\!N} \subset H$ *such that* $(f_n)_{n\in I\!N}$ *converges to* 0 *weakly in* H*, then* $(V_\alpha^{(n)}f_n)_{n\in I\!N}$ *converges to* 0 *weakly in* $D(\tilde{\varepsilon})$.

Proof Let $\alpha > 0$, $(f_n)_{n\in I\!N} \subset H$ such that $(f_n)_{n\in I\!N}$ converges to 0 weakly in H and $u \in \cap_{n\geq 0} D({}^{P_n}\varepsilon)$, we have

$$\tilde{\varepsilon}_\alpha(u, V_\alpha^{(n)} f_n) = (u, f_n) + \tilde{\varepsilon}_\alpha(u, V_\alpha^{(n)} f_n) - {}^{P_n}\varepsilon_\alpha(u, V_\alpha^{(n)} f_n).$$

Hence

$$\begin{aligned}\tilde{\varepsilon}_\alpha(u, V_\alpha^{(n)} f_n) - {}^{P_n}\varepsilon_\alpha(u, V_\alpha^{(n)} f_n) &= \lim_{m\to+\infty}[\varepsilon(u - P_m u, V_\alpha^{(n)} f_n) - \varepsilon(v - P_n u, V_\alpha^{(n)} f)_n] \\ &= \lim_{m\to+\infty} \varepsilon(P_n u - P_m u, V_\alpha^{(n)} f_n) \\ &= \lim_{m\to+\infty} \varepsilon(V_\alpha^{(n)} f_n, P_n u - P_m u) \\ &= \lim_{m\to+\infty}[(\sum_{k\geq 0} P_{\alpha,n}^k f_n, P_n u - P_m u) + \alpha(V_\alpha^{(n)} f_n, P_m u - P_n u)].\end{aligned}$$

Thus

$$\lim_{n\to+\infty}[\tilde{\varepsilon}_\alpha(u, V_\alpha^{(n)} f_n) - {}^{P_n}\varepsilon_\alpha(u, V_\alpha^{(n)} f)] = 0.$$

Then

$$\lim_{n\to+\infty} \tilde{\varepsilon}_\alpha(u, V_\alpha^{(n)} f_n) = 0.$$

Proposition 4.1 *For all $\alpha > 0$, we have that $(V_\alpha^{(n)})_{n\in I\!N}$ converges strongly to V_α in H.*

Proof Let $\alpha > 0$, $f \in H$ and $v \in \cap_{n\geq 0} D({}^{P_n}\varepsilon)$, from lemma 4.1, we have

$$\begin{aligned}\tilde{\varepsilon}_\alpha(v, Vf) &= \lim_{n\to+\infty} \tilde{\varepsilon}_\alpha(v, V_\alpha^{(n)} f) = \lim_{n\to+\infty} {}^{P_n}\varepsilon_\alpha(v, V_\alpha^{(n)} f) \\ &= (v, f) = \tilde{\varepsilon}_\alpha(v, V_\alpha f).\end{aligned}$$

Then, we have

$$Vf = V_\alpha f.$$

As

$$\begin{aligned}||V_\alpha^{(n)} f||^2 &= (V_\alpha f, V_\alpha^{(n)} f) + (V_\alpha^{(n)} f - V_\alpha f, V_\alpha^{(n)} f) \\ &= (V_\alpha f, V_\alpha^{(n)} f, V_\alpha^{(n)} f) + \int_X V_\alpha^{(n)}(V_\alpha^{(n)} f - V_\alpha f) f dm.\end{aligned}$$

We deduce from lemma 4.2, that

$$V_\alpha^{(n)} f \longrightarrow V_\alpha f \text{ in } H \text{ as } n \to +\infty.$$

Example
Let (a,D(a)) be a regular Dirichlet form on H, $(U_\alpha)_{\alpha>0}$ be the associated resolvent such that the canonical injection from D(a) into H is compact. We define the regular Dirichlet form $(\varepsilon, D(\varepsilon))$ on H by

$$\varepsilon(u, v) = a(u, v) + (u, v)\ ; \quad \text{for all } u, v \in D(\varepsilon) = D(a).$$

Note that the resolvent given below

$$W_\alpha = U_{\alpha+1} \ ; \ \text{for all} \ ; \ \alpha > 0$$

is associated to $(\varepsilon, D(\varepsilon))$. Let $\mathcal{L}_S(H)$ be the set of positive, symmetric and sub-Markovian operators on H. Let us consider a sequence $(A_n)_{n\in I\!N} \subset \mathcal{L}_S(H)$ and $A \in \mathcal{L}_S(H)$ such that $||A_n - A|| \to 0$ as $n \to +\infty$.
For every $n \in I\!N$, we set

$$^{P_n}\varepsilon(u,v) = \varepsilon(u,v) - \int\int_{X\times X} u(x)v(y)\sigma_n(dxdy); \text{for all} \quad u,v \in D(^{P_n}\varepsilon) = D(\varepsilon)$$

$$\tilde{\varepsilon}(u,v) = \varepsilon(u,v) - \int\int_{X\times X} u(x)v(y)\sigma(dxdy); \text{for all} \quad u,v \in D(\tilde{\varepsilon}) = D(\varepsilon)$$

where σ_n (resp. σ) describe the Radon measure defined on $X \times X$ associated to A_n (resp. A). It's easy to check that $(\tilde{\varepsilon}, D(\tilde{\varepsilon}))$ and $(^{P_n}\varepsilon, D(^{P_n}\varepsilon))$ are Dirichlet forms on H satisfying

$$\tilde{\varepsilon}(u,u) = \lim_{n\to+\infty} {}^{P_n}\varepsilon(u,u) \ \text{ for all } u \in D(\varepsilon).$$

Let $(V_\alpha^{(n)})_{\alpha>0}$ and $(V_\alpha)_{\alpha>0}$ be associated resovents respectively to $(^{P_n}\varepsilon, D(^{P_n}\varepsilon))$, then we have

$$V_\alpha^{(n)} = W_\alpha + W_\alpha A_n V_\alpha^{(n)} = W_\alpha + V_\alpha^{(n)} A_n W_\alpha \ \ m - a.e, \ \text{for all} \ \ \alpha > 0$$

and

$$V_\alpha = W_\alpha + W_\alpha A V_\alpha = W_\alpha + V_\alpha A W_\alpha \ \ m - a.e, \ \text{for all } \alpha > 0.$$

Hence,

$$P_n = WA_n, \ \ P = WA \quad \text{where} \quad W = U_1 \quad \text{and} \quad ||P_n - P|| \to 0 \quad \text{as} \quad n \to +\infty.$$

Moreover,

$$\tilde{\varepsilon}(u,u) \le 2\,(^{P_n}\varepsilon(u,u) + (u,u)); \ \text{for all} \quad u \in D(\varepsilon).$$

Finally, we conclude that for all $\alpha > 0$, $(V_\alpha^{(n)})_{n\in I\!N}$ converges strongly to V_α in H.

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