Convolution Calculus and Applications to Stochastic Differential Equations

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Abstract

The present work is mostly based on a functional analytic point of view. In this paper we develop a convolution calculus over a family of spaces of generalized functions. We use this calculus to discuss new solutions of some stochastic differential equations.

1 Introduction

Let X be a real nuclear Frechet space. Assume that its topology is defined by an increasing family of Hilbertian norms $\{|.|_p, p \in I\!N\}$. Then X is represented as

$$X = \bigcap_{p \in I\!N} X_p,$$

where for $p \in I\!N$ the space X_p is the completion of X with respect to the norm $|.|_p$. Denote by X_{-p} the dual space of X_p, then the dual space X' of X is represented as

$$X' = \bigcup_{p \in I\!N} X_{-p},$$

and it is equipped with the inductive limit topology. Let N (resp. N_p) be the complexification of X (resp. X_p), *i.e.* $N = X + iX$ and $N_p = X_p + iX_p$,

$p \in \mathbf{Z}$. For any $n \in I\!N$ we denote by $N^{\hat{\otimes} n}$ the n-th symetric tensor product of N equipped with the π-topology and by $N_p^{\hat{\otimes} n}$ the symetric Hilbertian tensor product of N_p. We will preserve the notation $|.|_p$ and $|.|_{-p}$ for the norms on $N_p^{\hat{\otimes} n}$ and $N_{-p}^{\hat{\otimes} n}$ respectively. Let θ be a Young function on $I\!R_+$, *i.e.* θ is continuous, convex, increasing function and satisfies $\lim_{+\infty} \dfrac{\theta(x)}{x} = +\infty$, see [9]. We define the conjugate function θ^* of θ by

$$\forall\, x \geq 0\ , \quad \theta^*(x) := \sup_{t \geq 0}(tx - \theta(t)). \tag{1}$$

For a such Young function θ we denote by $\mathcal{G}_\theta(N)$ the space of holomorphic functions on N with exponential growth of order θ and of arbitrary type, and by $\mathcal{F}_\theta(N')$ the space of holomorphic functions on N' with exponential growth of order θ and of minimal type. For every $p \in \mathbf{Z}$ and $m > 0$, we denote by $Exp(N_p, \theta, m)$ the space of entire functions f on the complex Hilbert space N_p such that $||f||_{\theta,p,m} := \sup_{x \in N_p} |f(x)| e^{-\theta(m|x|_p)} < +\infty$. Then the spaces $\mathcal{F}_\theta(N')$ and $\mathcal{G}_\theta(N)$ are represented as

$$\mathcal{F}_\theta(N') = \bigcap_{\substack{p \in I\!N \\ m>0}} Exp(N_{-p}, \theta, m)$$

$$\mathcal{G}_\theta(N) = \bigcup_{\substack{p \in I\!N \\ m>0}} Exp(N_p, \theta, m),$$

and equipped with the projective limit topology and the inductive limit topology respectively. The spaces $\mathcal{F}_\theta(N')$ and its dual $\mathcal{F}'_\theta(N')$ equipped with the strong topology are called the test functions space and the distributions space respectively.
Let $p \in I\!N$ and $m > 0$, we define the Hilbert spaces

$$F_{\theta,m}(N_p) = \{\vec{f} = (f_n)_{n=0}^{\infty}, f_n \in N_p^{\hat{\otimes} n}; \sum_{n \geq 0} \theta_n^{-2} m^{-n} |f_n|_p^2 < +\infty\}$$

$$G_{\theta,m}(N_{-p}) = \{\vec{\phi} = (\phi_n)_{n=0}^{\infty},\ \phi_n \in N_{-p}^{\hat{\otimes} n}\ ; \sum_{n \geq 0} (n!\theta_n)^2 m^n |\phi_n|_{-p}^2 < +\infty\},$$

where $\theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}$, $n \in I\!N$, and put

$$F_\theta(N) = \bigcap_{\substack{p \in I\!N \\ m>0}} F_{\theta,m}(N_p),$$

$$G_\theta(N') = \bigcup_{\substack{p\in I\!N \\ m>0}} G_{\theta,m}(N_{-p}).$$

The space $F_\theta(N)$ equipped with the projective limit topology is a nuclear Frechet space [3], and $G_\theta(N')$ carries the dual topology of $F_\theta(N)$ with respect to the $\mathbf{C}$-bilinear form $\ll . , . \gg$:

$$\ll \vec{\phi}, \vec{f} \gg = \sum_{n\geq 0} n!\langle \phi_n, f_n\rangle \ , \ \vec{\phi} = (\phi_n) \in G_\theta(N') \ , \ \vec{f} = (f_n) \in F_\theta(N).$$

It was proved in [3] that the Taylor series map, denoted by $S.T$, yields a topological isomorphism between $\mathcal{F}_\theta(N')$ (resp. $\mathcal{G}_{\theta^*}(N)$) and $F_\theta(N)$ (resp. $G_\theta(N')$). Then the action of a distribution $\phi \in \mathcal{F}'_\theta(N')$ on a test function $f \in \mathcal{F}_\theta(N')$ is given by

$$\ll \phi, f \gg := \ll \vec{\phi}, \vec{f} \gg,$$

where $\vec{\phi} = [(S.T)^*]^{-1}(\phi)$ and $\vec{f} = (S.T)(f)$. It is easy to see that for every $\xi \in N$, the exponential function $e_\xi : z \longmapsto e^{\langle z,\xi\rangle}$, $z \in N'$ belongs to the test space $\mathcal{F}_\theta(N')$ for any Young function θ. Then we define the Laplace transform of a distribution $\phi \in \mathcal{F}'_\theta(N')$ by

$$\widehat{\phi}(\xi) := \ll \phi, e_\xi \gg \ , \ \xi \in N.$$

In [3], the authors prove the important duality theorem: the Laplace transform realizes a topological isomorphism of $\mathcal{F}'_\theta(N')$ on $\mathcal{G}_{\theta^*}(N)$.

In this paper we develop a new convolution calculus over the generalized functionals spaces $\mathcal{F}'_\theta(N')$. Unlike the Wick calculus studied by many authors [7][10][12], the convolution calculus is developed independently of the gaussian analysis. In fact, we define the convolution product $\phi_1 * \phi_2$ of two distributions ϕ_1,ϕ_2 in $\mathcal{F}'_\theta(N')$ by a naturaly way using convolution operators. Then we give a sens to the expression $f^*(\phi) = \sum_n f_n \phi^{*n}$, for any entire function $f(z) = \sum_{n\geq 0} f_n z^n$, $z \in \mathbf{C}$ with exponential growth, and for any distribution $\phi \in \mathcal{F}'_\theta(N')$. In paticular, the important convolution exponential functional $exp^*\phi = \sum_{n\geq 0} \frac{\phi^{*n}}{n!}$, wich cannot be in general an element of the usual white noise distributions spaces introduced in [8], is well defined in the $\mathcal{F}'_\theta(N')$-spaces. This permits to solve some stochastic differential equations in the distributions spaces of type $\mathcal{F}'_\theta(N')$. Moreover, this solutions as elements of the $\mathcal{F}'_\theta(N')$-spaces have more regularity and properties than those of the bigger distributions space $(N)^{-1}$ of Kondratiev-Streit type, systematically used for example by Oksendal in [13].

2 Convolution of distributions

In infinite dimension complex analysis [2], a convolution operator on the test space $\mathcal{F}_\theta(N')$ is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself which commutes with translation operators.
Let $x \in N'$, we define the translation operator τ_{-x} on $\mathcal{F}_\theta(N')$ by

$$\tau_{-x}\varphi(y) = \varphi(x+y)\ ,\ \ y \in N'\ ,\ \varphi \in \mathcal{F}_\theta(N').$$

It is easy to see that τ_{-x} is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself. Now, we define the convolution product of a distribution $\phi \in \mathcal{F}'_\theta(N')$ with a test function $\varphi \in \mathcal{F}_\theta(N')$ as follows

$$\phi * \varphi(x) \ = \ll \phi, \tau_{-x}\varphi \gg\ ,\ \ x \in N'.$$

If ϕ is represented by $\vec{\phi} \ = (\phi_n)_{n\geq 0} \in G_\theta(N')$, then

$$\phi * \varphi(x) \ = \ \sum_{n\geq 0} \langle x^{\otimes n}, \psi^{(n)}\rangle,$$

where for every integer $n \in I\!N$

$$\psi^{(n)} = \ \sum_{k\geq 0} k! C_{n+k}^n \langle \phi_k, \varphi^{(n+k)}\rangle.$$

A direct calculation shows that the sequence $(\psi^{(n)})_{n\geq 0}$ is an element of $F_\theta(N)$ and consequently $\phi * \varphi \in \mathcal{F}_\theta(N')$. It was proved in [4] that T is a convolution operator on $\mathcal{F}_\theta(N')$ if and only if there exists $\phi \in \mathcal{F}'_\theta(N')$ such that

$$T(\varphi) \ = \ \phi * \varphi\ ,\ \ \forall\, \varphi \in \mathcal{F}_\theta(N'). \tag{2}$$

We denote the convolution operator T by T_ϕ. Moreover for every $\varphi \in \mathcal{F}_\theta(N')$ we have

$$T_\phi(\varphi)(0) = \ll \phi, \varphi \gg .$$

Let $\phi_1, \phi_2 \in \mathcal{F}'_\theta(N')$ and T_{ϕ_1}, T_{ϕ_2} be the associated convolution operators respectively. It is clear that the composition $T_{\phi_1} \circ T_{\phi_2}$ is also a convolution operator. Consequently there exists a unique element of $\mathcal{F}'_\theta(N')$ denoted by $\phi_1 * \phi_2$ such that

$$T_{\phi_1} \circ T_{\phi_2} = T_{\phi_1 * \phi_2}. \tag{3}$$

The distribution $\phi_1 * \phi_2$, defined by (3) is called the convolution product of ϕ_1 and ϕ_2.

Proposition 1 *For every* $\varphi \in \mathcal{F}_\theta(N)$ *we have*

$$\begin{aligned} \ll \phi_1 * \phi_2, \varphi \gg \; &:= \; [(\phi_1 * \phi_2) * \varphi](0) \\ &= \; [\phi_1 * (\phi_2 * \varphi)](0). \end{aligned}$$

Moreover, $\forall\ \phi_1, \phi_2 \in \mathcal{F}'_\theta(N')$ *it holds that*

$$\widehat{\phi_1 * \phi_2} = \widehat{\phi}_1\ \widehat{\phi}_2\ . \tag{4}$$

Proof
Let $\varphi \in \mathcal{F}_\theta(N')$, in view of (2) and (3) we obtain

$$[(\phi_1 * \phi_2) * \varphi](x) = [\phi_1 * (\phi_2 * \varphi)](x)\ ,\ \forall\ x \in N'.$$

In particular if we put $x = 0$ then we get

$$\ll \phi_1 * \phi_2, \varphi \gg = [\phi_1 * (\phi_2 * \varphi)](0),$$

from wich follows (4) by taking $\varphi(x) = e^{\langle x, \xi \rangle}\ ,\ \xi \in N.$ ■

Let $\mathcal{L}^c_\theta$ be the space of convolution operators on $\mathcal{F}_\theta(N')$. Taking (3) into consideration, we immediatly obtain

Lemma 1

$$\begin{aligned} (\mathcal{F}'_\theta(N'), *) \quad &\longrightarrow (\mathcal{L}^c_\theta, \circ) \\ \phi \qquad &\longmapsto T_\phi \end{aligned}$$

is an isomorphism of algebra.

It follows from (4) that $(\mathcal{F}'_\theta(N'), *)$ is a commutative algebra. Hence we deduce from lemma 1 that so is $(\mathcal{L}^c_\theta, \circ)$.

Theorem 1 *Let* γ *be a Young function on* $I\!R_+$ *wich does not necessaerily satisfy* $\lim_{x\to+\infty} \frac{\gamma(x)}{x} = +\infty$ *and* $f \in Exp(\mathbf{C}, \gamma, m)$ *for some* $m > 0$. *Then for every distribution* $\phi \in \mathcal{F}'_\theta(N')$, *the functional* $f^*(\phi)$ *defined by*

$$\widehat{f^*(\phi)} = f(\widehat{\phi}) \tag{5}$$

belongs to $\mathcal{F}'_\lambda(N')$, *where* $\lambda = (\gamma \circ e^{\theta^*})^*$.

Proof
By the duality theorem, it is sufficient to prove that $f(\widehat{\phi}) \in \mathcal{G}_{\lambda^*}(N)$. In fact let $\phi \in \mathcal{F}'_\theta(N')$, then there exist $p \in I\!N, m' > 0$ and $c' > 0$ such that

$$|\widehat{\phi}(\xi)| \leq c' e^{\theta^*(m'|\xi|_p)} \ , \ \xi \in N.$$

On the other hand there exists $c > 0$ such that

$$|f(z)| \leq c \ e^{\gamma(m|z|)} \ , \quad z \in \mathbf{C}.$$

Then combining the last inequality we get

$$\begin{aligned} |f(\widehat{\phi}(\xi))| &\leq c \ e^{\gamma(mc' e^{\theta^*(m'|\xi|_p)})} \ , \ \ \xi \in N \\ &\leq \begin{cases} c \ e^{\gamma(e^{\theta^*(m'|\xi|_p)})} & \text{if} \ \ mc' \leq 1 \\ c \ e^{\gamma(e^{\theta^*(cmm'|\xi|_p)})} & \text{if} \ \ mc' > 1. \end{cases} \end{aligned}$$

This inequality with the holomorphy of $f(\widehat{\phi})$ on N show that $f(\widehat{\phi}) \in \mathcal{G}_{\lambda^*}(N)$. ■

If we take $\gamma(x) = x \ , \ x \in I\!R_+$ and $f(z) = e^z \ , \ z \in \mathbf{C}$ in theorem 1, we get the following result

Corollary 1 *Let $\phi \in \mathcal{F}'_\theta(N')$, then the convolution exponential function of ϕ , denoted by $e^{*\phi}$, is an element of $\mathcal{F}'_{(e^{\theta^*})^*}(N')$. If in addition $\widehat{\phi}(\xi)$ is a polynomial in ξ of degree $k \in I\!N$, $k \geq 2$ then $e^{*\phi} \in \mathcal{F}'_\lambda(N')$, where $\lambda(x) = x^{\frac{k}{k-1}}$, $x \geq 0$.*

A similar result of corollary 1, in the particular case where $\widehat{\phi}$ is a polynomial, was proved in [12] with Wick product.

3 Applications to stochastic differential equations

A one parameter generalized stochastic process with values in $\mathcal{F}'_\theta(N')$ is a family of distributions $\{\phi_t, \ t \in I\} \subset \mathcal{F}'_\theta(N')$, where I is an interval, without loss generality we can assume that $0 \in I$. The process ϕ_t is said to be continuous if the map $t \longmapsto \phi_t$ is continuous. In order to introduce generalized stochastic integrals, we need the following result proved in [17].

Proposition 2 *[17] Let $(\phi_n)_{n\geq 0}$ be a sequence in $\mathcal{F}'_\theta(N')$. Then (ϕ_n) converges in $\mathcal{F}'_\theta(N')$ if and only if the following conditions hold :*

(D1) There exist $p \geq 0, m > 0$ and $c \geq 0$ such that for every integer n

$$|\widehat{\phi}_n(\xi)| \leq c\, e^{\theta^*(m|\xi|_p)}\ ,\ \ \forall\, \xi \in N.$$

(D2) The sequence $\widehat{\phi}_n(\xi)$ converges in $\mathbf{C}$ for each $\xi \in N$.

Let $\{\phi_t\}_{t\in I}$ be a continuous $\mathcal{F}'_\theta(N')$-process and put

$$\phi_n = \frac{t}{n}\sum_{k=0}^{n-1}\phi_{\frac{tk}{n}}\ \ n \in I\!N^*\ ,\ t \in I.$$

It is easy to prove that the sequence $(\widehat{\phi}_n)$ is bounded in $\mathcal{G}_{\theta^*}(N')$ and for every $\xi \in N$, $(\widehat{\phi}_n(\xi))_n$ converges to $\int_0^t \widehat{\phi}_s(\xi)ds$. Thus we conclude by proposition 2 that (ϕ_n) converges in $\mathcal{F}'_\theta(N')$. We denote its limit by

$$\int_0^t \phi_s ds := \lim_{n\to+\infty}\phi_n\ \ in\ \mathcal{F}'_\theta(N').$$

Proposition 3 *$E_t = \int_0^t \phi_s ds$, $t \in I$ is a continuous $\mathcal{F}'_\theta(N')$-process which satisfies*

$$\widehat{\int_0^t \phi_s ds} = \int_0^t \widehat{\phi}_s ds.$$

Moreover, The process E_t is differentiable in $\mathcal{F}'_\theta(N')$ i.e. $\frac{\partial E_t}{\partial t} = \phi_t$, $t \in I$.

Proof

Since the map $s \longmapsto \widehat{\phi}_s \in \mathcal{G}_{\theta^*}(N)$ is continuous, $\{\widehat{\phi}_s\ , s \in [0,t]\}$ becomes a compact set, in particular it is bounded in $\mathcal{G}_{\theta^*}(N)$ *i.e.* there exist $p \in I\!N$, $m > 0$ and $C_t > 0$ such that for every $\xi \in N_p$ we have

$$|\widehat{\phi}_s(\xi)| \leq C_t\, e^{\theta^*(m|\xi|_p)}\ ,\ \forall\ s \in\ [0,t]. \tag{6}$$

Then inequality (6) show that the function $\xi \longmapsto \int_0^t \widehat{\phi}_s(\xi)ds$ belongs to $\mathcal{G}_{\theta^*}(N)$. Consequently the pointwise convergence of the sequence of functions $(\widehat{\phi}_n)$ to $\int_0^t \widehat{\phi}_s ds$ becomes a convergence in $\mathcal{G}_{\theta^*}(N)$ and we get

$$\widehat{\int_0^t \phi_s ds} = \int_0^t \widehat{\phi}_s ds.$$

Let $t_0 \in I$ and let $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset I$. It then follows from (6) that

$$\begin{aligned} \|\widehat{E}_t - \widehat{E}_{t_0}\|_{\theta^*,p,m} &\leq \int_{t_0}^{t} \|\widehat{\phi}_s\|_{\theta^*,p,m} ds \\ &\leq |t - t_0| C_{t_0+\varepsilon}. \end{aligned}$$

This proves the continuity of the map $t \in I \longmapsto \widehat{E}_t \in \mathcal{G}_{\theta^*}(N)$ which is equivalent to the continuity of the process E_t. By the same argument we prove the differentiability of E_t. ■

3.1 Stochastic Volterra equation

Let $J : [0,T] \longrightarrow \mathcal{F}'_\theta(N'), K : [0,T] \times [0,T] \longrightarrow \mathcal{F}'_\theta(N')$ be two continuous generalized processes. We consider the stochastic Volterra equation

$$E(t) = J(t) + \int_0^t K(t,s) * E(s) ds \ , \quad 0 \leq t \leq T. \tag{7}$$

Theorem 2 *Suppose that there exist $p \in I\!N$, $m > 0$ and $M > 0$ such that*

$$||\widehat{K}(t,s)||_{\theta^*,p,m} \leq M \ , \quad \forall \ 0 \leq s \leq t \leq T,$$

then there exists a unique continuous $\mathcal{F}'_{(e^{\theta^*})^*}(N')$-process that solves (7). The solution $E(t)$ is given by

$$E(t) = J(t) + \int_0^t H(t,s) * J(s) ds \tag{8}$$

where $H(t,s) = \sum_{n \geq 1} K_n(t,s)$ with K_n given inductively by

$$K_{n+1}(t,s) = \int_s^t K_n(t,u) * K(u,s) du, \quad n \geq 1$$

and $K_1(t,s) = K(t,s)$.

Proof
The solution is given by Picard iteration. In fact, put $E_0(t) = J(t)$ and consider

$$E_{n+1}(t) = J(t) + \int_0^t K(t,s) * E_n(s) ds, \quad n \geq 0 \tag{9}$$

By iteration we get

$$E_n(t) = J(t) + \int_0^t H_n(t,s) * J(s)ds, \ \ n \geq 1$$

where $H_n(t,s) = \sum_{l=1}^n K_l(t,s)$. Now, we use proposition 2 to prove that for every $t, s \in [0,T]$ the sequence $H_n(t,s)$ converges in $\mathcal{F}'_{(e^{\theta^*})^*}(N')$. By assumption we have

$$|\widehat{K}(t,s)(\xi)| \leq M e^{\theta^*(m|\xi|_p)}\ , \ \ \xi \in N_p.$$

Thus by induction we get

$$|\widehat{K_l}(t,s)(\xi)| \leq M^l \frac{(t-s)^{l-1}}{(l-1)!} (e^{\theta^*(m|\xi|_p)})^l. \tag{10}$$

Then, summing up both sides of (10) we come to

$$\begin{aligned} |\widehat{H}_n(t,s)(\xi)| &\leq \sum_{l=1}^n M^l \frac{(t-s)^{l-1}}{(l-1)!} (e^{\theta^*(m|\xi|_p)})^l \\ &\leq M e^{\theta^*(m|\xi|_p)} exp[M(t-s)e^{\theta^*(m|\xi|_p)}] \\ &\leq M e^{\theta^*(m|\xi|_p)} exp[\frac{M^2(t-s)^2}{2} + e^{2\theta^*(m|\xi|_p)}] \\ &\leq M e^{M^2(t-s)^2} exp(e^{\theta^*(3m|\xi|_p)}). \end{aligned}$$

Hence we get the first condition (D1) of proposition 2. For the second condition (D2) we just note that for every $0 \leq s \leq t \leq T$ and $\xi \in N$, $(\widehat{H}_n(t,s)(\xi))_{n\geq 0}$ is a Cauchy sequence in $\mathbf{C}$. We have thus proved that the infinite series $H(t,s) = \sum_{l\geq 1} K_l(t,s)$ converges in $\mathcal{F}'_{(e^{\theta^*})^*}(N')$. Consequently, the sequence $(E_n(t))_{n\geq 0}$ converges also in $\mathcal{F}_{(e^{\theta^*})^*}(N')$ to $E(t) = J(t) + \int_0^t H(t,s) * J(s)ds$. By equation (9) , $E(t)$ is a solution of the stochastic Volterra equation. Finally, we use the Granwall inequality to prove the uniqueness. ■

3.2 Differential equations associated with convolution operators

Let θ_1 and θ_2 be two fixed Young functions, and let $\{\phi_t\}_{t\in I}$ be a continuous $\mathcal{F}'_{\theta_1}(N')$-process. Consider the Cauchy problem

$$\begin{cases} \frac{\partial U}{\partial t}= & \phi_t * U \ , \ \ t \in I \\ U(0)= & f \ \in \mathcal{F}_{\theta_2}(N'). \end{cases} \tag{11}$$

Theorem 3 *If there exists constant $C > 0$ such that $e^{\theta_1^*(r)} \leq C\ \theta_2^*(r)$ for r large enough, then the Cauchy problem (11) has a unique solution given by*

$$U(t,x) = (e^{* \int_0^t \phi_s ds} * f)(x)\ ,\ x \in N'\ ,\ t \in I. \tag{12}$$

Moreover, $U(t) \in \mathcal{F}_{\theta_2}(N')\ \ \forall\, t \in I$. If in addition $\widehat{\phi}_t(\xi)$ is a polynomial in ξ of degree $k \geq 2$, $\forall\, t \in I$, then $U(t)$ given by (12) is also the unique solution of equation (11) with values in $\mathcal{F}_{\theta_2}(N')$ whenever $\lim_{r \mapsto +\infty} \frac{r^k}{\theta_2^*(r)}$.

Proof
The solution $U(t)$ is obtained by Picard iteration as in the proof of theorem 2. ■

As an application of theorem 3 we give the heat equation associated with Gross Laplacian. In fact,
let $\varphi(x) = \sum_{n \geq 0} \langle x^{\otimes n}, \varphi^{(n)} \rangle \in \mathcal{F}_\theta(N)$. The Gross Laplacian [5], [10] of φ at $x \in N'$ is given by

$$\Delta_G \varphi(x) = \sum_{n \geq 0} (n+2)(n+1) \langle x^{\otimes n}, \langle \tau, \varphi^{(n+2)} \rangle \rangle$$

where τ is the trace operator defined by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \ \ \xi, \eta \in N.$$

Let γ be the standard gaussian measure on X' defined by its characteristic function $\int_{X'} e^{i\langle y,\xi \rangle} d\gamma(y) = e^{-\frac{|\xi|^2}{2}}$, see [6],[7],[11] [14].

Corollary 2 *Let θ be a Young function satisfying $\lim_{r \to +\infty} \frac{\theta(r)}{r^2} < +\infty$ and $f \in \mathcal{F}_\theta(N')$. Then the heat equation associated with the Gross Laplacian*

$$\frac{\partial U}{\partial t} = \frac{1}{2}\, \Delta_G U\ ,\ \ t \geq 0\ ,\ \ U(0) = f, \tag{13}$$

has a unique solution in $\mathcal{F}_\theta(N')$ given by

$$U(t,x) = \int_{X'} f(x + \sqrt{t} y) d\gamma(y)\ .$$

Proof
In fact, the Gross Laplacian Δ_G is a convolution operator. The distribution associated to Δ_G is $\vec{\phi}_\tau = (0, 0, \tau, 0, \cdots)$, then it follows from equality (2) that

$$\Delta_G(\varphi) = \phi_\tau * \varphi \ , \ \forall \ \varphi \in \mathcal{F}_\theta(N').$$

Thus the heat equation (13) is equivalent to

$$\frac{\partial U}{\partial t} = \phi_{\frac{\tau}{2}} * U \ , \quad t \geq 0 \ , \quad U(0) = f.$$

Since $\widehat{\phi}_{\frac{\tau}{2}}(\xi) = \frac{\langle \xi, \xi \rangle}{2}$, $\xi \in N$ is a polynomial of degree 2, then it follows from theorem 3 that the equation (13) has a unique solution in $\mathcal{F}_\theta(N')$ given by

$$U(t, x) = (e^{*t\phi_{\frac{\tau}{2}}} * f)(x) \ , \quad t \geq 0.$$

On the other hand, since $\widehat{e^{*t\phi_{\frac{\tau}{2}}}}(\xi) = e^{\frac{t\langle \xi, \xi \rangle}{2}} = \widehat{\gamma}_{\sqrt{t}}(\xi)$, $\xi \in N$, $t \geq 0$ where $\gamma_{\sqrt{t}}$ is a gaussian measure on X' [10], then the solution $U(t)$ can be expressed as

$$U(t, x) = (\gamma_{\sqrt{t}} * f)(x) = \int_{X'} f(x + \sqrt{t}y) d\gamma(y) \ , \quad t \geq 0 \ , \ x \in N'.$$

■

Let $\{\phi_t\}$ and $\{M_t\}$ be two continuous $\mathcal{F}'_\theta(N')$-processes. Consider the initial value problem

$$\frac{dX_t}{dt} = \phi_t * X_t + M_t \ , \quad X(0) = X_0 \in \mathcal{F}'_\theta(N'). \tag{14}$$

Then using the Laplace transform we prove the following theorem

Theorem 4 *The stochastic differential equation (14) has a unique solution in* $\mathcal{F}'_{(e^{\theta^*})^*}(N')$*, given by*

$$X_t = X_0 * e^{* \int_0^t \phi_s ds} + \int_0^t e^{* \int_s^t \phi_u du} * M_s ds$$

The next example is an application of theorem 4 :
In fact, let $\phi(t)$, $t \geq 0$ and $F(x)$, $x \in I\!\!R^d$ be two continuous $\mathcal{F}'_\theta(N')$-processes. Suppose that there exist $p \in I\!\!N$, $m > 0$ and a positive function

$\beta \in L^1(I\!R, d\lambda)$ such that $|\widehat{F}(x,\xi)| \leq \beta(x)\, e^{\theta^*(m|\xi|_p)}$. Then the heat equation with stochastic potential

$$\begin{cases} \frac{\partial U(t,x,\omega)}{\partial t} = & \frac{\sigma^2}{2}\Delta_x u(t,x,\omega) + \phi(t,\omega) * U(t,x,\omega),\ t>0, x\in I\!R^d \\ u(0,x,\omega) = & F(x,\omega),\ x \in I\!R^d \end{cases}$$

has a unique solution given by

$$U(t,x) = exp^*\Big(\int_0^t \phi(s)ds\Big) * \int_{I\!R^d} F(y)\frac{e^{-\frac{|x-y|^2}{2\sigma^2 t}}}{\sqrt{2\pi t}\sigma}dy.$$

Moreover, U(t,x) is a continuous $\mathcal{F}'_{(e^{\theta^*})^*}(N')$-process. In particular if $\phi(t) = W(t)$ the white noise, then U(t,x) becomes a continuous $\mathcal{F}'_\theta(N')$-process. See [15] in the case $\theta(x) = x^k$.

Now, we give an example of non-linear stochastic differential equation: Let $\{\phi_t\}$ be a continuous $\mathcal{F}'_\theta(N')$-process and consider the Verhulst equation

$$\begin{cases} \frac{\partial X_t}{\partial t} & = X_t * (X_t - 1) * \phi_t\ ,\quad t \geq 0 \\ X(0) & = x_0 \in]0,1[\end{cases} \tag{15}$$

In an obvious manner we show that

$$\widehat{X_t} = \frac{1}{1 + (\frac{1}{x_0} - 1)e^{\int_0^t \widehat{\phi}_s ds}},\quad t \geq 0 \tag{16}$$

Lemma 2 *[4] Let $f \in \mathcal{G}_\varphi(N)$such that $f(z) \neq 0,\ \forall z \in N$, then $\frac{1}{f} \in \mathcal{G}_\varphi(N)$.*

Since the function $\xi \longmapsto exp(\int_0^t \widehat{\phi}_s(\xi)ds)$ is an element of $\mathcal{G}_{e^{\theta^*}}(N)$, the above lemma shows that $\widehat{X_t} \in \mathcal{G}_{e^{\theta^*}}(N)$. Then using the duality theorem, X_t given by (16) is the unique continuous $\mathcal{F}'_{(e^{\theta^*})^*}(N')$-process that solves equation (15). In particular if $\widehat{\phi}_t(\xi)$ is a polynomial in ξ of degree $k \geq 2$ then the solution X_t becomes a continuous $\mathcal{F}'_{x^{\frac{k}{k-1}}}(N')$-process.

Remark

If the Young function θ satisfies $\lim_{x\to+\infty} \frac{\theta(x)}{x^2} < +\infty$, we get [3]

$$\mathcal{F}_\theta(N') \hookrightarrow L^2(X',\gamma) \hookrightarrow \mathcal{F}'_\theta(N'), \tag{17}$$

where γ is the standard gaussian measure on X'. In this case the test space $\mathcal{F}_\theta(N')$ coincides with the space $(X)_\theta$ introduced in [1]. In addition, the function $\xi \longmapsto e^{\frac{\langle \xi,\xi \rangle}{2}}$, $\xi \in N$ becomes an element of $\mathcal{G}_{\theta^*}(N)$ and the usual S-transform, denoted by S, is obtained by

$$S(\phi)(\xi) = \widehat{\phi}(\xi)e^{-\frac{\langle \xi,\xi \rangle}{2}}, \ \ \xi \in N, \phi \in \mathcal{F}'_\theta(N'). \tag{18}$$

Unlike to the Laplace transform, we see here that the chaotic transform S can not be defined on all spaces of generalized functions $\mathcal{F}'_\theta(N')$, it is defined only on the space $\mathcal{F}'_\theta(N')$ with $\lim_{x\to+\infty} \frac{\theta(x)}{x^2} < +\infty$. Recall that in the gaussian analysis, the Wick product of two generalized functions ϕ and ψ in $\mathcal{F}'_\theta(N')$, denoted by $\phi \diamond \psi$, is the unique distribution in $\mathcal{F}'_\theta(N')$ such that $S(\phi \diamond \psi) = S\phi \, S\psi$, see [7] [10]. Then using (18) we can derive the following relationships between convolution and Wick product

$$\phi \diamond \psi = \phi * \psi * \nu \text{ and } \phi * \psi = \phi \diamond \psi \diamond \gamma_{\sqrt{2}}, \tag{19}$$

where ν and $\gamma_{\sqrt{2}}$ are two distrubitions in $\mathcal{F}'_{x^2}(N')$ given by there Laplace transforms $\widehat{\nu}(\xi) = e^{-\frac{1}{2}\langle \xi,\xi \rangle}$ and $\widehat{\gamma}_{\sqrt{2}}(\xi) = e^{\langle \xi,\xi \rangle}$, $\xi \in N$.

A similar convolution calculus can be developed if we replace the space $\mathcal{F}'_\theta(N)$ by a space of test functions with several variables introduced in [16]

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