Stochastic dynamics of fluctuations in classical continuous systems

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Abstract

Fluctuations in classical continuous systems are studied. In the low activity high temperature regime for these fluctuations a central limit theorem is proven and the space of macroscopic fluctuations is constructed. Furthermore, it is shown that the generator of the microscopic stochastic dynamics in the fluctuation limit converges to the generator of a stochastic dynamics in the space of macroscopic fluctuations.

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1 Introduction

The space of configurations (collection of point particles) in $\mathbb{R}^d$ of classical continuous systems is modeled by the configuration space

$$\Gamma := \{\gamma \subset \mathbb{R}^d \,|\, |\gamma \cap K| < \infty \,\text{for any compact}\, K \subset \mathbb{R}^d\},$$

see e.g. [AKR98a]. Equilibrium states of classical continuous system are given by measures on Γ. Usually such measures correspond to given interaction potentials ϕ between particles and belong to the class of so called Gibbs measures. In the LA–HT (low activity high temperature) regime the corresponding Gibbs measure, say μ, has good mixing properties. The latter is important for the validity of the central limit theorem below. If, in addition, the interactions are invariant w.r.t. the continuous translations $\Gamma \ni \gamma \mapsto \tau_x(\gamma) := \{y + x \,|\, y \in \gamma\} \in \Gamma$ then μ is translation invariant, too.

Let F be a local observable, i.e., a measurable function on Γ which depends only on points inside a bounded volume $\Lambda' \subset \mathbb{R}^d$. For any Borel measurable set $\Lambda \subset \mathbb{R}^d$ with compact closure we can introduce the fluctuation of F in Λ as

$$F^\Lambda := \frac{1}{\mathrm{vol}(\Lambda)^{1/2}} \int_\Lambda \left(\tau_x F - \int_\Gamma F \, d\mu \right) dx,$$

where $\mathrm{vol}(\Lambda)$ denotes the volume of $\Lambda \subset \mathbb{R}^d$ w.r.t. the Lebesgue measure dx on $\mathbb{R}^d$. In [KMRS99] the authors have proven an exponential decay of correlations for local observables w.r.t. the Gibbs measure μ in the LA–HT regime. With the help of this decay of correlations and the equivalent mixing property, respectively, in Theorem 3.7 we prove a central limit theorem for F^Λ as $\Lambda \nearrow \mathbb{R}^d$ (thermodynamic limit, sometimes also called fluctuation limit). Here we would like to note that the decay of correlations proved in [KMRS99] itself is an useful result. In [GKLR00] it was one of the tools used in order to prove a scaling limit for stochastic dynamics in classical continuous systems.

Following the corresponding considerations for lattice models, [GVV91], [ADKR99], we study not only the individual fluctuations F^Λ. We are rather interested in the collective fluctuation limit, that means, the consideration of all fluctuations of observables at the same time in the thermodynamic limit. Hence, we construct the space of macroscopic fluctuations $\mathcal{F}_\mu$ utilizing the fluctuations of observables in the thermodynamic limit. From the technical point of view $\mathcal{F}_\mu$ is a space of square-integrable functions w.r.t. a Gaussian measure ν_μ on an infinite dimensional co-nuclear space $\mathcal{N}'$.

The main subject of [GVV91] and [ADKR99] has been to construct the stochastic dynamics in the space of macroscopic fluctuations starting from a given microscopic time evolution. More precisely, in these papers it has been shown that in the lattice case the generator of the microscopic stochastic dynamics in the fluctuation limit converges to the generator of a stochastic dynamics in the space of macroscopic fluctuations. The lattice case is quite different from the continuous case considered in the present paper. For example, in the lattice case one considers other types of interactions (the equilibrium position of the particles is fixed), the reference measure (state without interaction) is given by a Gaussian measure where in the continuous case this is the Poisson measure, and in the lattice case one has discrete instead of continuous translations, finite sums instead of integrals. Hence, it is interesting to note that the same concepts and constructions work in both cases, the lattice and the continuous case.

In order to perform the same construction as in the lattice case in Section 4 we recall the analysis and geometry on the configuration space Γ introduced in [AKR98a] and [AKR98b]. With the help of the Dirichlet form approach there the authors have constructed the equilibrium stochastic dynamics corresponding to the Gibbs measure μ. Here we would like to stress that the Dirichlet form and the associated stochastic process exist under quite general assumptions on the interaction potential ϕ. For example the potential ϕ may have a singularity at the origin, may not be compactly supported and, in particular, the case of a Leonard Jones potential can be treated.

After having constructed the generator of the microscopic stochastic dynamics H_μ^Γ, in Lemma 4.3 we prove that the objects of interest, e.g. fluctuations of local observables and differentiable mappings of them, are in the domain of its closure. Such a lemma is not necessary in the lattice case, because in that case these objects are obviously in the domain of the generator. As we already mentioned above in the transition to the continuous case discrete translations are replaced by continuous ones and finite sums by integrals. This is the reason why in the continuous case the statement of Lemma 4.3 is not obvious.

The facts we derive in Lemma 4.3 are necessary in order to lift the generator of the microscopic stochastic dynamics H_μ^Γ to the space of macroscopic fluctuations $\mathcal{F}_\mu$. The first step of this lifting is performed in Theorem 5.1 where we lift H_μ^Γ to the tangent space $\mathcal{K}_\mu$ of $\mathcal{F}_\mu$. The Friedrichs' extension of this lifted operator will be denoted by A_μ. Then we introduce the gradient Dirichlet form in the space of macroscopic fluctuations with coefficient operator A_μ. The generator corresponding to it will be denoted by H_{ν_μ,A_μ}.

Finally, in Theorem 5.2 we prove that when starting with the generator of the microscopic stochastic dynamics H_μ^Γ in the fluctuation limit there appears the generator H_{ν_μ,A_μ} of the macroscopic stochastic dynamics. Different from the central limit theorem which we can prove for quite general interaction potentials ϕ, see Theorem 3.7, here we have to require that the interaction potential ϕ is compactly supported. This property gives us that the generator H_μ^Γ maps local functions into local functions. For technical reasons this is essential, because in the proof of Theorem 5.2 we again need a decay of correlations, and the decay of correlations, due to Theorem 3.3, we only have for local functions. From a physical point of view this assumption is not so restrictive. It is more important that our proof also works for potentials which have a singularity at the origin. Such a singularity may reflect a repulsion of interacting particles.

The operator H_{ν_μ,A_μ} generates an infinite dimensional Ornstein-Uhlenbeck semi-group

$$T_t^\mu := \exp(-tH_{\nu_\mu,A_\mu}), \quad t \geq 0,$$

in $L^2(\nu_\mu)$. This semi-group is associated to a generalized Ornstein-Uhlenbeck process $(\Xi_t)_{t\geq 0}$ on $\mathcal{N}'$ which is called macroscopic stochastic dynamics.

2 Gibbs states of classical continuous systems

2.1 Configuration space and Poisson measure

$\mathcal{O}(\mathbb{R}^d)$ is defined as the family of all open sets of $\mathbb{R}^d$ with norm $|\cdot|_{\mathbb{R}^d}$ given by the Euclidean scalar product $(\cdot,\cdot)_{\mathbb{R}^d}$. By $\mathcal{B}(\mathbb{R}^d)$ we denote the corresponding Borel σ-algebra. $\mathcal{O}_c(\mathbb{R}^d)$ and $\mathcal{B}_c(\mathbb{R}^d)$ denote the systems of all elements in $\mathcal{O}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d)$, respectively, which have compact closures. The Lebesgue measure on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ will be denoted by dx.

The *configuration space* $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$ is defined as the set of all locally finite subsets (configurations) in $\mathbb{R}^d$:

$$\Gamma := \{\gamma \subset \mathbb{R}^d \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\}.$$

Here $|A|$ denotes the cardinality of a set A.

Via the identification, which we shall use below at various occasions without further notice, of $\gamma \in \Gamma$ with

$$\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}_p(\mathbb{R}^d),$$

where ε_x denotes the Dirac measure in $x \in \mathbb{R}^d$, Γ can be considered as a subset of the set of all positive Radon measures $\mathcal{M}_p(\mathbb{R}^d)$ on $\mathbb{R}^d$. Hence Γ can be topologized by the vague topology, i.e., the weakest topology on Γ such that the maps

$$\gamma \mapsto <\varphi, \gamma> := \int_{\mathbb{R}^d} \varphi(x)\, d\gamma(x) = \sum_{x\in\gamma} \varphi(x)$$

are continuous. Here $\varphi \in C_0(\mathbb{R}^d)$, the set of continuous functions on $\mathbb{R}^d$ with compact support. We denote by $\mathcal{B}(\Gamma)$ the corresponding Borel σ-algebra. For $B \in \mathcal{B}(\mathbb{R}^d)$ we define the map $N_B : \Gamma \to \mathbb{N}_0 \cup \{\infty\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, as

$$N_B(\gamma) := \gamma(B).$$

Then we have

$$\mathcal{B}(\Gamma) = \sigma(\{N_B \mid B \in \mathcal{B}_c(\mathbb{R}^d)\}).$$

Furthermore, we introduce for any subset $A \in \mathcal{B}(\mathbb{R}^d)$ the sub σ-algebra $\mathcal{B}_A(\Gamma)$ which is generated by all functions N_B, $B \subset A$, $B \in \mathcal{B}_c(\mathbb{R}^d)$. A function F on Γ is called a *local function* if it is $\mathcal{B}_\Lambda(\Gamma)$-measurable for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. For a local function F we have $F(\gamma) = F(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and all $\gamma \in \Gamma$, where $\gamma_A := \gamma \cap A$, $A \in \mathcal{B}(\mathbb{R}^d)$.

For a given $z > 0$ (activity parameter) let π_z denote the Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity measure $z\,dx$. The Poisson measure is characterized via its Fourier transform

$$\int_\Gamma \exp(i <\varphi, \gamma>)\, d\pi_z(\gamma) = \exp\Big(z \int_{\mathbb{R}^d} (\exp(i\varphi(x)) - 1)\, dx\Big),$$

where $\varphi \in C_0(\mathbb{R}^d)$. For a construction of this measure as a measure on the configuration space we refer e.g. to [AKR98a].

2.2 Gibbs measures in the LA–HT regime

Let ϕ be a symmetric pair potential, i.e., a measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $\phi(x) = \phi(-x)$. For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the conditional energy $E_\Lambda^\phi : \Gamma \to \mathbb{R} \cup \{\infty\}$ is defined by

$$E_\Lambda^\phi(\gamma) := E_\Lambda^\phi(\gamma_\Lambda) + W(\gamma_\Lambda|\gamma_{\Lambda^c}),$$

where the term

$$W(\gamma_\Lambda|\gamma_{\Lambda^c}) := \sum_{x\in\gamma_\Lambda, y\in\gamma_{\Lambda^c}} \phi(x-y) \tag{1}$$

describes the interaction energy between γ_Λ and γ_{Λ^c} ($\Lambda^c := \mathbb{R}^d \backslash \Lambda$) and

$$E_\Lambda^\phi(\gamma_\Lambda) := \sum_{\{x,y\}\subset\gamma_\Lambda} \phi(x-y) \tag{2}$$

is the potential energy corresponding to $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$.

From know on we assume that the potential fulfills the following conditions:

(S) (*stability*) There exists $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and for all $\gamma \in \Gamma$ with $\gamma = \gamma_\Lambda$

$$E_\Lambda^\phi(\gamma) \ \geq\ -B|\gamma|.$$

(ED) (*exponential decay*) There exists $r_0, C_0 > 0$ and $\kappa > 0$ such that

$$|\phi(x)| < C_0 \exp(-\kappa|x|), \quad \text{for all } |x| > r_0.$$

(S) and (ED) imply that the potential is bounded from below and

(EI) (*exponential integrability*)

$$D(\beta) := \int_{\mathbb{R}^d} |\exp(-\beta\phi(x)) - 1| \exp\left(\frac{\kappa}{2}|x|\right) dx < \infty$$

for all $\beta \geq 0$.

(EI), obviously, is stronger than:

(I) (*integrability*)

$$C(\beta) := \int_{\mathbb{R}^d} |\exp(-\beta\phi(x)) - 1|\, dx < \infty$$

for all $\beta \geq 0$. This condition sometimes is called *regularity*, see e.g. [Rue69]).

On $(\Gamma, \mathcal{B}(\Gamma))$ we consider the finite volume Gibbs measures μ_Λ in $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ with empty boundary condition:

$$d\mu_\Lambda(\gamma) = \frac{1}{Z_\Lambda} \exp(-\beta E^\phi_\Lambda(\gamma_\Lambda)) d\pi_z(\gamma),$$

where $\beta \geq 0$ is the inverse temperature and

$$Z_\Lambda = \int_\Gamma \exp(-\beta E^\phi_\Lambda(\gamma_\Lambda)) d\pi_z(\gamma)$$

is the partition function. Using (S) one easily proves that it is finite. In e.g. [Min67] and [MM91] it has been proved that in the LA–HT (low activity high temperature) regime, i.e., for

$$z < \Big(2\exp(2\beta B + 1) C(\beta)\Big)^{-1},$$

the weak limit

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \mu_\Lambda = \mu \tag{3}$$

exists in the sense that

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \int_\Gamma F(\gamma_\Lambda)\, d\mu_\Lambda(\gamma) = \int_\Gamma F(\gamma)\, d\mu(\gamma),$$

for all bounded local functions F. Furthermore, it can be shown that μ is a Gibbs measure and that

$$\int_\Gamma | < \varphi, \gamma > |^p \, d\mu(\gamma) < \infty, \quad \forall \varphi \in C_0(\mathbb{R}^d), \quad p \geq 1, \tag{4}$$

see [Rue70] and [Kun99].

3 Mixing properties and the space of macroscopic fluctuations

Our aim is to prove a central limit theorem for local fluctuations of observables. The *observables* of our consideration are elements from the space

$$\mathcal{L}_\mu^{2+\epsilon} := \cup_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} L_\Lambda^{2+\epsilon}(\mu),$$

where

$$L_\Lambda^{2+\epsilon}(\mu) := L^{2+\epsilon}(\Gamma, \mathcal{B}_\Lambda(\Gamma), \mu), \quad \epsilon > 0,$$

is the space of $L^{2+\epsilon}$-integrable Λ-local (i.e., $\mathcal{B}_\Lambda(\Gamma)$-measurable) functions.

Definition 3.1 *For observables F, i.e., elements from $\mathcal{L}_\mu^{2+\epsilon}$, we define the local fluctuation in $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ w.r.t. μ by*

$$F^\Lambda := \frac{1}{\mathrm{vol}(\Lambda)^{1/2}} \int_\Lambda \left(\tau_x F - \int_\Gamma F \, d\mu \right) dx, \tag{5}$$

where $\tau_x F(\gamma) := F(\tau_x(\gamma))$ with

$$\Gamma \ni \gamma \mapsto \tau_x(\gamma) := \{y + x \,|\, y \in \gamma\} \in \Gamma$$

for $x \in \mathbb{R}^d$.

Remark 3.2 *(i) The construction of μ, see (3), together with the translation invariance of the potential energy, see (2), implies the translation invariance of the measure μ, i.e.,*

$$\int_\Gamma \tau_x F \, d\mu = \int_\Gamma F \, d\mu$$

for all $F \in \mathcal{L}_\mu^{2+\epsilon}$.
(ii) Since the measure μ is translation invariant the norm of $\tau_x F$ in the Banach space $L^{2+\epsilon}(\Gamma, \mathcal{B}(\Gamma), \mu)$ is independent of $x \in \mathbb{R}^d$. Hence the integral in (5) exists as a Bochner integral in $L^{2+\epsilon}(\Gamma, \mathcal{B}(\Gamma), \mu)$. Obviously, F^Λ is again a local function.

The following exponential decay of correlations proved in [KMRS99] is essential for the proof of our limit theorem.

Theorem 3.3 *Let the potential ϕ satisfy (S), (ED) and let us assume that $z < (2\exp(2\beta B+1)D(\beta))^{-1}$. Furthermore, we assume that $F \in L^{2+\epsilon}_{\Lambda}(\mu)$ and $G \in L^{2+\epsilon}_{\Lambda'}(\mu)$, where $\Lambda \cap \Lambda' = \emptyset$, $\Lambda, \Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$. Then the covariance*

$$\mathrm{cov}_\mu(F,G) := \int_\Gamma F(\gamma)G(\gamma)\,d\mu(\gamma) - \int_\Gamma F(\gamma)\,d\mu(\gamma)\int_\Gamma G(\gamma)\,d\mu(\gamma)$$

satisfies the following estimate:

$$|\mathrm{cov}_\mu(F,G)| \le C_1 \parallel F \parallel_{2+\epsilon} \parallel G \parallel_{2+\epsilon} \exp\Big(-\frac{\epsilon\kappa}{2(2+\epsilon)}\mathrm{dist}(\Lambda,\Lambda')\Big), \qquad (6)$$

where

$$\begin{aligned} \mathrm{dist}(\Lambda,\Lambda') &:= \inf\{|x-y|_{\mathbb{R}^d} \mid x\in\Lambda, y\in\Lambda'\}, \\ C_1 &= C_2\exp(\delta(\mathrm{vol}(\Lambda)+\mathrm{vol}(\Lambda')))(\mathrm{vol}(\Lambda)+\mathrm{vol}(\Lambda'))^{\epsilon/(2+\epsilon)}, \end{aligned}$$

and $C_2 = C_2(\beta, z, \epsilon, \kappa, B), \delta = \delta(\beta, z, \epsilon, \kappa, B)$ are constants.

Remark 3.4 *The decay of correlations has also been studied by H. Spohn, see [Spo86]. He has proven an L^2-exponential decay of correlations under the assumption that the potential ϕ is positive and $\phi \in C^3_0(\mathbb{R}^d)$, the space of 3-times continuously differentiable and compactly supported functions on $\mathbb{R}^d$ (an assumption which is quite restrictive from a physical point of view).*

Definition 3.5 *Let $\mathcal{A}_1, \mathcal{A}_2$ be sub-σ-algebras of $\mathcal{B}(\Gamma)$. Then their α-mixing w.r.t. μ is defined by*

$$\alpha_{\mathcal{A}_1,\mathcal{A}_2} := \sup_{A_1\in\mathcal{A}_1, A_2\in\mathcal{A}_2} |\mathrm{cov}_\mu(\mathbf{1}_{A_1}, \mathbf{1}_{A_2})|,$$

where $\mathbf{1}_A$ is the indicator function of $A \in \mathcal{B}(\Gamma)$.

As consequence of Theorem 3.3 we have the following estimate for the α-mixing.

Corollary 3.6 *Under the same assumptions on the potential ϕ and the activity z as in Theorem 3.3 we have the following estimate for the α-mixing $\alpha_{\mathcal{B}_\Lambda,\mathcal{B}_{\Lambda'}}$ of the σ-algebras $\mathcal{B}_\Lambda = \mathcal{B}_\Lambda(\Gamma)$ and $\mathcal{B}_{\Lambda'} = \mathcal{B}_{\Lambda'}(\Gamma)$, for $\Lambda, \Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$:*

$$\alpha_{\mathcal{B}_\Lambda,\mathcal{B}_{\Lambda'}} \le (C_1 \vee 1)\exp\Big(-c\,\mathrm{dist}(\Lambda,\Lambda')\Big), \quad c = \frac{\epsilon\kappa}{2(2+\epsilon)}. \qquad (7)$$

Our aim is to prove a central limit theorem for the local fluctuations as $\Lambda \nearrow \mathbb{R}^d$. Hence for any $n \in \mathbb{N}$ we introduce the cube

$$\Lambda_n := \Big\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \Big| \max_{i=1,\ldots,d} |x_i| \leq n - 1/2 \Big\}. \tag{8}$$

We observe, that the random variables F^{Λ_n} can be written as

$$F^{\Lambda_n} = \frac{1}{(2n-1)^d} \sum_{\substack{k \in \mathbb{Z}^d \\ \max_{i=1,\ldots,d} |k_i| \leq n-1}} \int_{k+\Lambda_1} \left(\tau_x F - \int_\Gamma F\, d\mu \right) dx, \tag{9}$$

where $k + \Lambda_1 := \{k + x |\, x \in \Lambda_1\}$. Of course,

$$\xi_k(F) := \int_{k+\Lambda_1} \left(\tau_x F - \int_\Gamma F\, d\mu \right) dx, \quad k \in \mathbb{Z}^d,$$

is a stationary random field on the lattice $\mathbb{Z}^d$.

The following central limit theorem for local fluctuations is the main result of the first part of this note.

Theorem 3.7 *Let the potential ϕ satisfy (S), (ED) and let us assume that $z < (2\exp(2\beta B + 1) D(\beta))^{-1}$. Then for all $F, G \in \mathcal{L}_\mu^{2+\epsilon}$ we have:*

$$\int_{\mathbb{R}^d} \left| \int_\Gamma F \tau_x G d\mu - \int_\Gamma F\, d\mu \int_\Gamma G\, d\mu \right| dx < \infty. \tag{10}$$

Furthermore, for any $F \in \mathcal{L}_\mu^{2+\epsilon}$ the random variables $F^{\Lambda_n}, n \in \mathbb{N}$, converge in distribution as $n \to \infty$ to a centered Gaussian random variable with covariance $\langle F, F \rangle_\mu$, where the positive semi-definite bilinear form $\langle F, G \rangle_\mu, F, G \in \mathcal{L}_\mu^{2+\epsilon}$, is defined by

$$\begin{aligned} \langle F, G \rangle_\mu &:= \lim_{\Lambda \nearrow \mathbb{R}^d} \int_\Gamma F^\Lambda G^\Lambda\, d\mu \\ &= \int_{\mathbb{R}^d} \left(\int_\Gamma F \tau_x G d\mu - \int_\Gamma F\, d\mu \int_\Gamma G\, d\mu \right) dx. \end{aligned} \tag{11}$$

Proof: Since all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ have a compact closure there exist constants $a(\Lambda), b(\Lambda) > 0$ such that

$$\exp(-\operatorname{dist}(\Lambda, \Lambda + x)) \leq a \exp(-b|x|_{\mathbb{R}^d}), \quad x \in \mathbb{R}^d. \tag{12}$$

Now Theorem 3.3, together with Remark 3.2(i) and (12), implies the integrability of (10).

Next let us consider the stationary random field $(\xi_k(F))_{k\in\mathbb{Z}^d}$, where $F \in L^{2+\epsilon}(\Gamma, \mathcal{B}_\Lambda(\Gamma), \mu)$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. We are interested in the α-mixing of the σ-algebras $\mathcal{A}_{A_i}$ generated by the random variables $(\xi_k(F))_{k\in A_i}$, where $A_i \subset \mathbb{Z}^d$, $i = 1, 2$, are finite sets. The σ-algebras $\mathcal{A}_{A_i}$ are sub-σ-algebras of the σ-algebras $\mathcal{B}_{\Lambda^i}$ with $\Lambda^i = \{x + k|\, x \in \Lambda,\, k \in A_i\}$, respectively. With the help of the mixing condition in Corollary 3.6 and (12) we find the following estimate for the α-mixing of the σ-algebras $\mathcal{A}_{A_1}$ and $\mathcal{A}_{A_2}$:

$$\begin{aligned}\alpha_{\mathcal{A}_{A_1},\mathcal{A}_{A_2}} &\leq (C_1 \vee 1) \exp\Big(- c \,\mathrm{dist}(\Lambda^1, \Lambda^2)\Big)\\ &\leq a^c (C_1 \vee 1) \exp\Big(- c\, b \,\mathrm{dist}(A_1, A_2)\Big). \end{aligned} \tag{13}$$

From estimate (13) we deduce that the stationary random field $(\xi_k(F))_{k\in\mathbb{Z}^d}$ fulfills all assumptions of Theorem 2 in [Nah88], see also [Nah91], Theorem 7.2.1, which in turn implies that $(\xi_k(F))_{k\in\mathbb{Z}^d}$ satisfies the central limit theorem, i.e.,

$$F^{\Lambda_n} = \frac{1}{(2n-1)^d} \sum_{\substack{k\in\mathbb{Z}^d \\ \max_{i=1,\ldots,d} |k_i| \leq n-1}} \xi_k(F)$$

converges in distribution as $n \to \infty$ to a centered Gaussian random variable with covariance $\langle F, F\rangle_\mu$. ■

Remark 3.8 *(i) The form $\langle \cdot, \cdot \rangle_\mu$ is degenerate on $\mathcal{L}_\mu^{2+\epsilon}$. Indeed, for any $F, G \in \mathcal{L}_\mu^{2+\epsilon}$ and each $x \in \mathbb{R}^d$ it follows from Remark 3.2(i) that*

$$\langle F - \tau_x F, G\rangle_\mu = 0.$$

(ii) Theorem 3.7 implies that for all $F \in \mathcal{L}_\mu^{2+\epsilon}$

$$\lim_{n\to\infty} \int_\Gamma \exp(itF^{\Lambda_n})\, d\mu = \exp\Big(- \frac{1}{2} \langle F, F\rangle_\mu\Big), \quad t \in \mathbb{R}.$$

Having Theorem 3.7 in hands now we can construct the space of macroscopic fluctuations. We start with introducing the Hilbert space $\mathcal{K}_\mu$ obtained from $\mathcal{L}_\mu := \cup_{\epsilon>0} \mathcal{L}_\mu^{2+\epsilon}$ together with the bilinear positive semi-definite form

$\langle\cdot,\cdot\rangle_\mu$ via factorization and completion. Its scalar product will again be denoted by $\langle\cdot,\cdot\rangle_\mu$. In this way each element $F \in \mathcal{L}_\mu$ is mapped to an element $\widehat{F} \in \mathcal{K}_\mu$.

Furthermore, we consider a nuclear space $\mathcal{N}$ densely and continuously embedded into $\mathcal{K}_\mu$ and introduce the nuclear triple

$$\mathcal{N} \subset \mathcal{K}_\mu \subset \mathcal{N}',$$

where $\mathcal{N}'$ is the topological dual space to $\mathcal{N}$ w.r.t. $\mathcal{K}_\mu$ (that is, the dualization between $\mathcal{N}$ and $\mathcal{N}'$ restricted to $\mathcal{N} \times \mathcal{K}_\mu$ is given by $\langle\cdot,\cdot\rangle_\mu$).

In order to introduce a probability measure on the vector space $\mathcal{N}'$ we consider the σ-algebra $\mathcal{C}_\sigma(\mathcal{N}')$ generated by cylinder sets

$$C^{\varphi_1,\ldots,\varphi_n}_{B_1,\ldots,B_N} = \Big\{\omega \in \mathcal{N}' \Big| \langle\varphi_1,\omega\rangle_\mu \in B_1, \ldots, \langle\varphi_N,\omega\rangle_\mu \in B_N\Big\},$$

where $\varphi_i \in \mathcal{N}, B_i \in \mathcal{B}(\mathbb{R})$, $1 \leq i \leq N$, $N \in \mathbb{N}$. The standard Gaussian measure ν_μ on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ is given by its characteristic function

$$C(\varphi) = \int_{\mathcal{N}'} \exp(i\langle\varphi,\omega\rangle_\mu)\, d\nu_\mu(\omega) = \exp\Big(-\frac{1}{2}|\varphi|^2_\mu\Big), \quad \varphi \in \mathcal{N},$$

via Minlos' theorem, see e.g. [BK95]. Any $\varphi \in \mathcal{N}$ gives us a measurable linear functional $\langle\varphi,\cdot\rangle_\mu$ on $\mathcal{N}'$. Let $h \in \mathcal{K}_\mu$ be approximated by a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{N}$. Then the sequence of linear functionals $\langle\varphi_n,\cdot\rangle_\mu$ converges in $L^2(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \nu_\mu)$. Its limit defines a measurable linear functional

$$L^2(\nu_\mu) \ni \langle h,\cdot\rangle_\mu := \lim_{n\to\infty} \langle\varphi_n,\cdot\rangle_\mu \tag{14}$$

on $\mathcal{N}'$ which does not depend on the choice of the approximating sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{N}$.

Now we can define a mapping $L_\mu : \mathcal{L}_\mu \to \mathcal{F}_\mu := L^2(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \nu_\mu)$ as

$$\mathcal{L}_\mu \ni F \mapsto L_\mu(F) := \langle\widehat{F},\cdot\rangle_\mu \in \mathcal{F}_\mu.$$

Remark 3.9 *(i) Using this notation Remark 3.8(ii) reads as:*

$$\lim_{n\to\infty} \int_\Gamma \exp(itF^{\Lambda_n})\, d\mu = \int_{\mathcal{N}'} \exp(itL_\mu(F))\, d\nu_\mu, \quad t \in \mathbb{R}.$$

(ii) The set

$$\Big\{\exp(iL_\mu(F)) \Big| F \in \mathcal{L}_\mu\Big\}$$

is total in $\mathcal{F}_\mu$.
(iii) From (i) and (ii) we can conclude that for each $N \in \mathbb{N}$, *all* $g \in C_b(\mathbb{R}^N)$ *(the set of continuous bounded functions on* $\mathbb{R}^N$*), and any* $F_1, \dots F_N \in \mathcal{L}_\mu$ *we have*

$$\lim_{n\to\infty} \int_\Gamma g(F_1^{\Lambda_n}, \dots, F_N^{\Lambda_n})\, d\mu = \int_{\mathcal{N}'} g(L_\mu(F_1), \dots, L_\mu(F_N))\, d\nu_\mu.$$

(iv) The map L_μ *is not injective since it is a composition of the mappings*

$$\begin{array}{ccc} \mathcal{L}_\mu \to & \mathcal{K}_\mu & \to \mathcal{F}_\mu \\ F \to & \widehat{F} & \to \langle \widehat{F}, \cdot \rangle_\mu \end{array}$$

where the first mapping, from $\mathcal{L}_\mu$ *to* $\mathcal{K}_\mu$, *is not injective.*
(v) The space $\mathcal{F}_\mu$ *is called space of macroscopic fluctuations, see e.g. [GVV91] for a related discussion in lattice models. In physics the states of this space are called coarse grained. Mathematically this fact is reflected by the non-injectivity of the mapping* L_μ.

4 Analysis and geometry on the configuration space Γ

Here we recall the analysis and geometry on the configuration space Γ developed in [AKR98a] and [AKR98b].

4.1 Flows, directional derivatives, and gradient

By $V_0(\mathbb{R}^d)$ let us denote the set of all C^∞-vector fields on $\mathbb{R}^d$ with compact support. Any vector field $v \in V_0(\mathbb{R}^d)$ defines (via the exponential mapping) a one-parameter group $\psi_t^v \in \mathrm{Diff}_0(\mathbb{R}^d)$, $t \in \mathbb{R}$, where $\mathrm{Diff}_0(\mathbb{R}^d)$ denotes the group of all diffeomorphisms $\psi : \mathbb{R}^d \to \mathbb{R}^d$ which are trivial outside a compact. Any such ψ defines a transformation of Γ by

$$\Gamma \ni \gamma \mapsto \psi(\gamma) = \{\psi(x) \,|\, x \in \gamma\} \in \Gamma.$$

For $F : \Gamma \to \mathbb{R}$ we define the directional derivative along v as

$$(\nabla_v^\Gamma F)(\gamma) = \frac{d}{dt} F(\psi_t^v(\gamma))\Big|_{t=0}$$

(provided the right hand side exists).

This definition applies to F in the following class of so-called smooth cylinder functions. Let $\mathcal{D} := C_0^\infty(\mathbb{R}^d)$, the set of smooth functions on $\mathbb{R}^d$ with compact support. We define $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ as the set of all functions on Γ of the form

$$\gamma \mapsto F(\gamma) = g_F(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle), \tag{15}$$

where $\varphi_1,\ldots,\varphi_N \in \mathcal{D}$ and $g_F \in C_b^\infty(\mathbb{R}^N)$. Clearly, $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ is dense in $L^p(\mu), p \geq 1$. For any $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ we have

$$(\nabla_v^\Gamma F)(\gamma) = \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle)\langle\nabla_v\varphi_j,\gamma\rangle, \tag{16}$$

where $x \mapsto (\nabla_v\varphi)(x) = (\nabla\varphi(x), v(x))_{\mathbb{R}^d}$ is the usual directional derivative of φ on $\mathbb{R}^d$ along v and $\nabla = \nabla^{\mathbb{R}^d}$ denotes the gradient on $\mathbb{R}^d$. We obtain a differential operator

$$\mathcal{F}C_b^\infty(\mathcal{D},\Gamma) \ni F \mapsto \nabla_v^\Gamma F \in L^2(\mu)$$

on $L^2(\mu)$.

The tangent space $T_\gamma(\Gamma)$ to the configuration space Γ at the point γ is defined as the space of measurable γ-square-integrable vector fields $V_\gamma : \mathbb{R}^d \to \mathbb{R}^d$ with scalar product

$$(V_\gamma^1, V_\gamma^2)_{T_\gamma(\Gamma)} = \int (V_\gamma^1(x), V_\gamma^2(x))_{\mathbb{R}^d}\, d\gamma(x),$$

$V_\gamma^1, V_\gamma^2 \in T_\gamma(\Gamma)$. The corresponding tangent bundle is

$$T_\gamma = \bigcup_{\gamma\in\Gamma} T_\gamma(\Gamma).$$

Having a tangent space we can define the intrinsic gradient of a function $F : \Gamma \to \mathbb{R}$ as the mapping

$$\Gamma \ni \gamma \mapsto (\nabla^\Gamma F)(\gamma) \in T_\gamma(\Gamma)$$

such that for any $v \in V_0(\mathbb{R}^d)$

$$(\nabla_v^\Gamma F)(\gamma) = (\nabla^\Gamma F, v)_{T_\gamma(\Gamma)}.$$

By (16) for any $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ of the form (15) we have

$$(\nabla^\Gamma F)(\gamma,x) = \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle)\nabla\varphi_j(x), \quad \gamma\in\Gamma,\, x\in\mathbb{R}^d. \tag{17}$$

4.2 Dirichlet forms, their generators, and corresponding microscopic stochastic dynamics

First we introduce further conditions on the potential ϕ. For every $r = (r_1, \dots, r_d) \in \mathbb{Z}^d$ we define a cube

$$Q_r = \left\{ x \in \mathbb{R}^d \,\middle|\, r_i - 1/2 \leq x_i < r_i + 1/2 \right\}.$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$ we set $\gamma_r := \gamma_{Q_r}$, $r \in \mathbb{Z}^d$. A condition stronger than stability is the following.

(SS) (*superstability*) There exist $A > 0$, $B \geq 0$ such that if $\gamma = \gamma_{\Lambda_n}$ for some $n \in \mathbb{N}$ then

$$E^{\phi}_{\Lambda_n}(\gamma) \geq \sum_{r \in \mathbb{Z}^d} \Big(A|\gamma_r|^2 - B|\gamma_r| \Big).$$

(LR) (*lower regularity*) There exists a decreasing positive function $a : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r \in \mathbb{Z}^d} a(\| r \|) < \infty$$

and for any Λ', Λ'', which are finite unions of cubes of the form Q_r and disjoint,

$$W(\gamma'|\gamma'') \geq - \sum_{r', r'' \in \mathbb{Z}^d} a(\| r' - r'' \|) |\gamma'_{r'}| \cdot |\gamma''_{r''}|,$$

provided $\gamma' = \gamma'_{\Lambda'}$, $\gamma'' = \gamma''_{\Lambda''}$. Here W is the interaction energy, see (1), extended to arbitrary disjoint configurations and $\| \cdot \|$ denotes the maximum norm on $\mathbb{R}^d$.

(D) (*differentiability*) The function $\exp(-\phi)$ is weakly differentiable on $\mathbb{R}^d$, ϕ is weakly differentiable on $\mathbb{R}^d \setminus \{0\}$ and the weak gradient $\nabla \phi$ (which is a locally dx-integrable function on $\mathbb{R}^d \setminus \{0\}$) considered as an dx-a.e. defined function on $\mathbb{R}^d$ satisfies

$$\nabla \phi \in L^p(\mathbb{R}^d, \exp(-\phi)\, dx), \quad \forall p \geq 1. \tag{18}$$

Note that for many typical potentials in Statistical Physics we have $\phi \in C^\infty(\mathbb{R}^d \backslash \{0\})$. For such "regular outside the origin" potentials condition (D) nevertheless does not exclude a singularity at the point $0 \in \mathbb{R}^d$.

Lemma 4.1 *Let ϕ satisfying (SS), (ED), (LR), and (D). Furthermore, we assume the LA–HT regime. For $n \in \mathbb{N}$ let Λ_n as before denote the cube with side $2n-1$ centered at the origin in $\mathbb{R}^d$. For any vector field $v \in V_0(\mathbb{R}^d)$ we consider the function*

$$\Gamma \ni \gamma \mapsto L^\phi_{v,n}(\gamma) := - \sum_{\{x,y\} \subset \gamma_{\Lambda_n}} (\nabla \phi(x-y), v(x) - v(y))_{\mathbb{R}^d}.$$

Then for any Gibbs measure μ and all $v \in V_0(\mathbb{R}^d)$ we have that

$$L^\phi_v = \lim_{n \to \infty} L^\phi_{v,n} \tag{19}$$

exists in $L^q(\mu)$ for all $q \geq 1$.

The proof of Lemma 4.1 essentially is the same as the proof of Lemma 4.1 in [AKR98b]. In (D) we assume a stronger integrability assumption on $\nabla\phi$, see (18), as it has been done in [AKR98b]. There the authors only have assumed $\nabla \phi \in L^1(\mathbb{R}^d, \exp(-\phi)\, dx) \cap L^2(\mathbb{R}^d, \exp(-\phi)\, dx)$, and in Lemma 4.1 they have proven that $L^\phi_v \in L^2(\mu)$. Analyzing their proof one easily finds that our stronger assumption gives a stronger integrability property of L^ϕ_v, see (19). This stronger integrability condition is essential for the proof of Lemma 4.3 below.

For $F, G \in \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)$ we define

$$\mathcal{E}^\Gamma_\mu(F,G) := \int_\Gamma (\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{T_\gamma(\Gamma)}\, d\mu(\gamma).$$

Since we only consider Gibbs measures μ in the LA–HT regime they have all moments, see (4). Thus, by (17) we have $(\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{T_\gamma(\Gamma)} \in L^1(\mu)$. Furthermore, the gradient respects μ-classes $\mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)^\mu$ determined by $\mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)$. Hence, $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma))$ is a densely defined positive definite symmetric bilinear form on $L^2(\mu)$.

The assumptions in Lemma 4.1 are sufficient to prove an integration by parts formula for the gradient ∇^Γ, see [AKR98b], Theorem 4.3. The *local summability* (LS) assumed there is a consequence of (ED) together with (D), see [AKR98b], Example 4.1. Utilizing this formula we obtain for $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$:

$$\mathcal{E}_\mu^\Gamma(F,G) = -\int_\Gamma \Delta_\mu F G \, d\mu, \tag{20}$$

where the Laplacian w.r.t. μ for F as in (15) is given by

$$\begin{aligned}\Delta_\mu F(\gamma) \;&=\; \sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle)\langle (\nabla\varphi_i, \nabla\varphi_j)_{\mathbb{R}^d}, \gamma\rangle \\ &+\; \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle)\Big(\langle \Delta\varphi_j, \gamma\rangle + L^\phi_{\nabla\varphi_j}(\gamma)\Big), \quad \gamma \in \Gamma.\end{aligned} \tag{21}$$

Since the Gibbs measure μ has all moments and due to Lemma 4.1 the operator $(\Delta_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ is densely defined in $L^2(\mu)$. On the basis of (20) in [AKR98b], Proposition 5.1, the following statement has been proven.

Proposition 4.2 *Let the potential ϕ have the same properties as assumed in Lemma 4.1 and still we consider μ in the LA–HT regime. Then the bilinear form $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ is closable on $L^2(\mu)$ and its closure $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ is a symmetric Dirichlet form which is conservative and local. Its generator, denoted by $-H_\mu^\Gamma$, is the Friedrichs' extension of Δ_μ.*

In order to prove the main theorem of the final section we need that $H_\mu^\Gamma F$, $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, is a local function, because there we again use the exponential decay of correlations for local functions, see Theorem 3.3. Hence, we have to assume:

(CS) (*compactly supported*) There exists $r_0 > 0$ such that

$$\phi(x) = 0, \quad \text{for all } |x| > r_0.$$

(CS) implies that the potential fulfills the property (LR), see e.g. [Kun99], Proposition 2.2.17. The following lemma provides us with the essential functional analytic properties required for proving the results of the final section.

Lemma 4.3 *Let ϕ satisfying (SS), (CS), and (D). Furthermore, we assume the LA–HT regime. Then for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we have:*

(i) $\int_\Gamma H_\mu^\Gamma F\,d\mu = 0$ *and* H_μ^Γ *is translation invariant, i.e.,* $\tau_x H_\mu^\Gamma F = H_\mu^\Gamma \tau_x F$ *for all* $x \in \mathbb{R}^d$*;*
(ii) $H_\mu^\Gamma F$ *is a local function and in* $L^q(\mu)$ *for all* $q \geq 1$*; and*
(iii) F^Λ, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, *is in the domain of the closure of* Δ_μ*, hence in particular in* $D(H_\mu^\Gamma)$*, the domain of* H_μ^Γ*. Furthermore,*

$$
\begin{aligned}
H_\mu^\Gamma F^\Lambda &= \frac{1}{\mathrm{vol}(\Lambda)^{1/2}} H_\mu^\Gamma \int_\Lambda \left(\tau_x F - \int_\Gamma F\,d\mu \right) dx \qquad (22)\\
&= \frac{1}{\mathrm{vol}(\Lambda)^{1/2}} \int_\Lambda \left(\tau_x H_\mu^\Gamma F - \int_\Gamma H_\mu^\Gamma F\,d\mu \right) dx = (H_\mu^\Gamma F)^\Lambda .
\end{aligned}
$$

(iv) For all $F_1, \dots F_N \in \mathcal{F}C_b^2(\mathcal{D},\Gamma)$ *and any* $g \in C_b^2(\mathbb{R}^N)$, $N \in \mathbb{N}$*, the function* $g(F_1^\Lambda, \dots, F_N^\Lambda)$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$*, is in the domain of the closure of* Δ_μ *and*

$$
\begin{aligned}
H_\mu^\Gamma g(F_1^\Lambda, \dots, F_N^\Lambda) &= \sum_{j=1}^N \frac{\partial g}{\partial s_j}(F_1^\Lambda, \dots, F_N^\Lambda) H_\mu^\Gamma F_j^\Lambda \\
&+ \sum_{i,j=1}^N \frac{\partial^2 g}{\partial s_i s_j}(F_1^\Lambda, \dots, F_N^\Lambda)(\nabla^\Gamma F_i^\Lambda, \nabla^\Gamma F_j^\Lambda)_{T(\Gamma)}.
\end{aligned}
$$

Proof: (i) Since the constant function $1 \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, obviously, $\int_\Gamma H_\mu^\Gamma F\,d\mu = 0$. For any $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ of the form (15) we have

$$
(\tau_x F)(\gamma) = \sum_{j=1}^N g(\langle \varphi_1^x, \gamma\rangle, \dots, \langle \varphi_N^x, \gamma\rangle), \quad \gamma \in \Gamma,\ x \in \mathbb{R}^d,
$$

where $\varphi^x(y) = \varphi(y-x)$, $x, y \in \mathbb{R}^d$. Thus, $\tau_x F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ and $H_\mu^\Gamma \tau_x F$ is well-defined. Now, together with the explicit formula for H_μ^Γ on $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, see (21), the second statement of (i) is easy to verify.
(ii) Again, we consider the explicit formula for H_μ^Γ on $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ given in (21). Since $\nabla\varphi \in V_0(\mathbb{R}^d)$ for all $\varphi \in \mathcal{D}$, we know from Lemma 4.1 that $L_{\nabla\varphi}^\phi \in L^q(\mu)$ for all $q \geq 1$. This together with the fact that the measure μ has all moments implies $H_\mu^\Gamma F \in L^p(\mu)$ for all $p \geq 1$ and any $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$. The function $H_\mu^\Gamma F$ is local, because the functions $\varphi_i \in \mathcal{D}$ and the potential ϕ have compact supports.
(iii) For every $r = (r_1, \dots, r_d) \in \mathbb{Z}^d$ and any $n \in \mathbb{N}$ we define the cube

$$
Q_{r,n} = \left\{ x \in \mathbb{R}^d \,\middle|\, r_i - 1/2 \leq 2^{n-1} x_i < r_i + 1/2 \right\}.
$$

These cubes for fixed $n \in \mathbb{N}$ form a partition of $\mathbb{R}^d$. Furthermore, we define $Q_{\Lambda,r,n} := Q_{r,n} \cap \Lambda$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Clearly, the functions

$$\begin{aligned} F_n^\Lambda &:= \frac{1}{\mathrm{vol}(\Lambda)^{1/2}} \int_\Lambda \sum_{r\in\mathbb{Z}^d} \mathbf{1}_{Q_{\Lambda,r,n}}(x) \left(\tau_{(2^{1-n}r)} F - \int_\Gamma F\, d\mu \right) dx \\ &= \sum_{r\in\mathbb{Z}^d} \frac{\mathrm{vol(Q_{\Lambda,r,n})}}{\mathrm{vol}(\Lambda)^{1/2}} \left(\tau_{(2^{1-n}r)} F - \int_\Gamma F\, d\mu \right) \end{aligned} \tag{23}$$

converge to F^Λ in $L^2(\mu)$ as $n \to \infty$. Note that the sum in (23) is finite and therefore $F_n^\Lambda \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$. We have

$$H_\mu^\Gamma F_n^\Lambda = \frac{1}{\mathrm{vol}(\Lambda)^{1/2}} \int_\Lambda \sum_{r\in\mathbb{Z}^d} \mathbf{1}_{Q_{\Lambda,r,n}}(x) \tau_{(2^{1-n}r)} H_\mu^\Gamma F\, dx.$$

Since

$$\left\| \sum_{r\in\mathbb{Z}^d} \mathbf{1}_{Q_{\Lambda,r,n}}(x) \tau_{(2^{1-n}r)} H_\mu^\Gamma F \right\|_{L^2(\mu)} \le \| H_\mu^\Gamma F \|_{L^2(\mu)} < \infty, \quad \forall x \in \Lambda,$$

and

$$\lim_{n\to\infty} \sum_{r\in\mathbb{Z}^d} \mathbf{1}_{Q_{\Lambda,r,n}}(x) \tau_{(2^{1-n}r)} H_\mu^\Gamma F = \tau_x H_\mu^\Gamma F, \quad \forall x \in \Lambda,$$

we have

$$\lim_{n\to\infty} H_\mu^\Gamma F_n^\Lambda = \frac{1}{\mathrm{vol}(\Lambda)^{1/2}} \int_\Lambda \tau_x H_\mu^\Gamma F\, dx \tag{24}$$

in $L^2(\mu)$ by the Lebesgue dominated convergence theorem for Bochner integrals. Thus, F^Λ is in the domain of the closure of Δ_μ. Additionally, (24) implies (22) having in mind that $H_\mu^\Gamma \int_\Gamma F\, d\mu = \int_\Gamma H_\mu^\Gamma F\, d\mu = 0$.
(iv) For clarity of the proof we consider only the case $N = 1$, the idea easily generalizes to arbitrary $N \in \mathbb{N}$. Since $(F_n^\Lambda)_{n\in\mathbb{N}}$ converges to F^Λ in $L^2(\mu)$ there exists a subsequence $(F_{n_k}^\Lambda)_{k\in\mathbb{N}}$ which converges μ-a.e. to F^Λ. We denote $F_{n_k}^\Lambda = F_k$, $k \in \mathbb{N}$, and have

$$g(F_k) \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma), \quad k \in \mathbb{N}.$$

Continuity and boundedness of g imply

$$\lim_{k\to\infty} g(F_k) = g(F^\Lambda), \quad \mu\text{-a.e.}$$

and also in $L^2(\mu)$. An elementary calculation gives

$$H_\mu^\Gamma g(F_k) = g'(F_k) H_\mu^\Gamma F_k + g''(F_k)(\nabla^\Gamma F_k, \nabla^\Gamma F_k)_{T(\Gamma)}.$$

Utilizing the triangle and Hölder inequality one shows that

$$\lim_{k\to\infty} H_\mu^\Gamma g(F_k) = g'(F^\Lambda) H_\mu^\Gamma F^\Lambda + g''(F^\Lambda)(\nabla^\Gamma F^\Lambda, \nabla^\Gamma F^\Lambda)_{T(\Gamma)},$$

in $L^2(\mu)$. ■

The diffusion process corresponding to $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ has been constructed in [AKR98b], see Theorem 5.2. It lives on the bigger state space $\ddot{\Gamma}$ consisting of all integer valued Radon measures on $\mathbb{R}^d$ which is Polish, see e.g. [Kal75]. In [RS98], Corollary 1, the authors have proven that the set $\ddot{\Gamma}/\Gamma$ is $\mathcal{E}_\mu^\Gamma$-exceptional. Thus, the associated diffusion process can be restricted to a process on Γ.

Theorem 4.4 *Under the same assumptions as in Lemma 4.1 there exists a conservative diffusion process*

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F_t})_{t\geq 0}, (\mathbf{\Theta_t})_{t\geq 0}, (\mathbf{X_t})_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma})$$

on Γ which is properly associated with $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$, i.e., for all ($\mu$-versions) of $F \in L^2(\Gamma, \mu)$ and all $t > 0$ the function

$$\gamma \mapsto p_t F(\gamma) := \int_{\mathbf{\Omega}} F(\mathbf{X}_t)\, dP_\gamma, \quad \gamma \in \Gamma,$$

is an $\mathcal{E}_\mu^\Gamma$-quasi-continuous version of $\exp(tH_\mu^\Gamma)F$. The process $\mathbf{M}$ is up to μ-equivalence unique, has μ as an invariant measure and is called microscopic stochastic dynamics.

In the above theorem $\mathbf{M}$ is canonical, i.e., $\mathbf{\Omega} = C([0,\infty) \to \Gamma)$, $\mathbf{X_t}(\xi) = \xi(t), \xi \in \mathbf{\Omega}$, $(\mathbf{F_t})_{t\geq 0}$ together with $\mathbf{F}$ is the corresponding minimum completed admissible family and $(\mathbf{\Theta_t})_{t\geq 0}$ are the corresponding natural time shifts. For a detailed discussions of these objects and the notion of $\mathcal{E}_\mu^\Gamma$-quasi-continuity we refer to [MR92].

5 Stochastic dynamics of fluctuations

In this section first we transport the generator H_μ^Γ of the microscopic stochastic dynamics onto $\mathcal{K}_\mu$ and then onto the space of macroscopic fluctuations $\mathcal{F}_\mu$. The resulting operator is the generator of the macroscopic stochastic dynamics and will be denoted by H_{ν_μ,A_μ}. Finally, in Theorem 5.2, the main result of the second part of this note, we will show that when starting with the generator H_μ^Γ of the microscopic stochastic dynamics in the fluctuation limit there appears the generator H_{ν_μ,A_μ} of the macroscopic stochastic dynamics.

We start by transporting H_μ^Γ onto $\mathcal{K}_\mu$ by

$$\widehat{H_\mu^\Gamma}\widehat{F} := \widehat{H_\mu^\Gamma F}, \quad F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma).$$

Theorem 5.1 *Let the potential ϕ satisfy (SS), (CS), (D) and let us assume that $z < (2\exp(2\beta B+1)D(\beta))^{-1}$. Then the operator $\widehat{H_\mu^\Gamma}$ is a well and densely defined, symmetric, positive semi-definite operator on $\mathcal{K}_\mu$.*

Proof: From Lemma 4.3(ii) we know that for all $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ the function $H_\mu^\Gamma F$ is in $\mathcal{L}_\mu$. Since $\int_\Gamma H_\mu^\Gamma F\, d\mu = 0$ and H_μ^Γ is translation invariant, see Lemma 4.3(i), we have

$$\begin{aligned}\left\langle \widehat{H_\mu^\Gamma}\widehat{F}, \widehat{G}\right\rangle_\mu &= \int_{\mathbb{R}^d}\int_\Gamma H_\mu^\Gamma F \tau_x G\, d\mu\, dx \\ &= \int_{\mathbb{R}^d}\int_\Gamma F \tau_x H_\mu^\Gamma G\, d\mu\, dx, \quad F,G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma).\end{aligned}$$

Thus, $\widehat{H_\mu^\Gamma}$ is symmetric. The explicit formula for the scalar product $\langle\cdot,\cdot\rangle_\mu$, see (11), together with the fact that $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ is dense in $L^p(\mu), p \geq 1$, implies that the set

$$\left\{\widehat{F}\,\middle|\, F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)\right\}$$

is dense in $\mathcal{K}_\mu$. For $\widehat{F_1} = \widehat{F_2}$, $F_1, F_2 \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, we have

$$\left\langle \widehat{H_\mu^\Gamma}\widehat{F_1} - \widehat{H_\mu^\Gamma}\widehat{F_2}, \widehat{G}\right\rangle_\mu = \left\langle \widehat{F_1} - \widehat{F_2}, \widehat{H_\mu^\Gamma}\widehat{G}\right\rangle_\mu = 0, \quad \forall G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma),$$

which implies that $\widehat{H_\mu^\Gamma}\widehat{F_1} = \widehat{H_\mu^\Gamma}\widehat{F_2}$.

Applying Theorem 3.7 and Lemma 4.3(iii), we obtain

$$\begin{aligned}\left\langle \widehat{H^\Gamma_\mu}\widehat{F},\widehat{F}\right\rangle_\mu &= \lim_{n\to\infty}\int_\Gamma (H^\Gamma_\mu F)^{\Lambda_n}F^{\Lambda_n}\,d\mu\\ &= \lim_{n\to\infty}\int_\Gamma H^\Gamma_\mu F^{\Lambda_n}F^{\Lambda_n}\,d\mu \geq 0,\end{aligned}$$

because of the positive semi-definiteness of H^Γ_μ. ■

Next we transport H^Γ_μ onto $\mathcal{F}_\mu$ via a second quantization of the Friedrichs' extension of $\widehat{H^\Gamma_\mu}$. Before, however, we have to introduce further concepts from Gaussian analysis. Analogously to the space of smooth cylinder functions $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ we define the space

$$\begin{aligned}\mathcal{F}C_b^\infty(\mathcal{N},\mathcal{N}') &:= \Big\{\Psi\,\Big|\,\Psi(\omega)=g_\Psi(\langle\varphi_1,\omega\rangle_\mu,\dots,\langle\varphi_N,\omega\rangle_\mu),\\ &\qquad \omega\in\mathcal{N}',\varphi_1,\dots,\varphi_N\in\mathcal{N},g_\Psi\in C_b^\infty(\mathbb{R}^N),N\in\mathbb{N}\Big\}\end{aligned}$$

of smooth cylinder functions on $\mathcal{N}'$. Similarly, we can define the space $\mathcal{F}C_b^\infty(\mathcal{U},\mathcal{N}')$ for any linear subset $\mathcal{U}\subset\mathcal{K}_\mu$ in the sense of measurable functions, see (14).

For any $\Psi\in\mathcal{F}C_b^\infty(\mathcal{N},\mathcal{N}')$ the gradient $\nabla^{\mathcal{K}_\mu}\Psi(\omega)=\Psi'(\omega)$ at $\omega\in\mathcal{N}'$, which is defined by the equality

$$\langle\nabla^{\mathcal{K}_\mu}\Psi(\omega),\vartheta\rangle_\mu := \frac{\partial\Psi}{\partial t}(\omega+t\vartheta)\Big|_{t=0},\quad t\in\mathbb{R},\quad \vartheta\in\mathcal{N}',$$

exists, and

$$\nabla^{\mathcal{K}_\mu}\Psi(\omega)=\sum_{k=1}^N\frac{\partial g_\Psi}{\partial s_k}(\langle\varphi_1,\omega\rangle_\mu,\dots,\langle\varphi_N,\omega\rangle_\mu)\varphi_k\in\mathcal{N}.$$

Thus, $\mathcal{K}_\mu$ can be interpreted as the tangent space.

Let us remark that this definition easily can be extended to the case of $\Psi\in\mathcal{F}C_b^\infty(\mathcal{U},\mathcal{N}')$. In this case we have $\nabla^{\mathcal{K}_\mu}\Psi(\omega)\in\mathcal{U}$.

We identify the second derivative $\Psi''(\omega)$ of $\Psi\in\mathcal{F}C_b^\infty(\mathcal{N},\mathcal{N}')$ with the finite range operator on $\mathcal{K}_\mu$ defined by

$$\Psi''(\omega)h := \nabla^{\mathcal{K}_\mu}\langle\nabla^{\mathcal{K}_\mu}\Psi(\omega),h\rangle_\mu,\quad h\in\mathcal{K}_\mu.$$

By A_μ we denote the Friedrichs' extension of the operator $\widehat{H^\Gamma_\mu}$ and denote its domain by $D(A_\mu)$. In what follows, we choose $\mathcal{N}$ being a domain of essential self-adjointness of A_μ and such that both A_μ and $\exp(-tA_\mu), t \geq 0$, leave $\mathcal{N}$ invariant and act on $\mathcal{N}$ continuously, such a choice is always possible due to Theorem 1.2 and Example 1.1 in Chapter 4 of [BK95]. Now we can define the classical pre-Dirichlet form $\mathcal{E}_{\nu_\mu,A_\mu}$ associated with the measure ν_μ and the coefficient operator A_μ given on $\mathcal{F}C_b^\infty(\mathcal{N},\mathcal{N}')$ by the formula

$$\mathcal{E}_{\nu_\mu,A_\mu}(\Psi,\Phi) := \int_{\mathcal{N}'} \langle \nabla^{\mathcal{K}_\mu}\Psi(\omega), A_\mu \nabla^{\mathcal{K}_\mu}\Phi(\omega)\rangle_\mu \, d\nu_\mu(\omega).$$

This form is associated with the operator H_{ν_μ,A_μ} in $L^2(\nu_\mu)$ given on functions in $\mathcal{F}C_b^\infty(\mathcal{N},\mathcal{N}')$ by the expression

$$H_{\nu_\mu,A_\mu}\Psi(\omega) = -\mathrm{Tr}_{\mathcal{K}_\mu}(A_\mu \Psi''(\omega)) + \langle \omega, A_\mu \nabla^{\mathcal{K}_\mu}\Psi(\omega)\rangle_\mu \tag{25}$$

in the sense that

$$\mathcal{E}_{\nu_\mu,A_\mu}(\Psi,\Phi) = \int_{\mathcal{N}'} H_{\nu_\mu,A_\mu}\Psi(\omega)\Phi(\omega)\, d\nu_\mu(\omega), \quad \Psi,\Phi \in \mathcal{F}C_b^\infty(\mathcal{N},\mathcal{N}'),$$

see [BK95], Chapter 6. The differential expression (25) is also well-defined for $\Psi \in \mathcal{F}C_b^2(D(A_\mu),\mathcal{N}')$.

Let us note that the space $L^2(\nu_\mu)$ is isomorphic to the symmetric Fock space $\mathrm{EXP}(\mathcal{K}_\mu)$ associated with the one particle space $\mathcal{K}_\mu$. In this framework the operator H_{ν_μ,A_μ} coincides with the second quantization $d\mathrm{EXP}(A_\mu)$ of the operator A_μ, see e.g. Chapter 6 of [BK95].

Finally, in the next theorem we can show that when starting with the generator H_μ^Γ of the microscopic stochastic dynamics in the fluctuation limit there appears the generator H_{ν_μ,A_μ}.

Theorem 5.2 *Let the potential ϕ satisfy (SS), (CS), (D) and let us assume that $z < (2\exp(2\beta B+1)D(\beta))^{-1}$. Then for each $N \in \mathbb{N}$, all $f,g \in C_b^2(\mathbb{R}^N)$, and any $F_1,\dots F_N, G_1,\dots,G_N \in \mathcal{F}C_b^2(\mathcal{D},\Gamma)$ we have*

$$\begin{aligned}&\lim_{n\to\infty}\int_\Gamma f(F_1^{\Lambda_n},\dots,F_N^{\Lambda_n}) H_\mu^\Gamma\, g(G_1^{\Lambda_n},\dots,G_N^{\Lambda_n})\, d\mu \\ =&\int_{\mathcal{N}'} f(L_\mu(F_1),\dots,L_\mu(F_N)) H_{\nu_\mu,A_\mu}\, g(L_\mu(G_1),\dots,L_\mu(G_N))\, d\nu_\mu.\end{aligned} \tag{26}$$

Proof: For simplicity, we give the proof only for $N=1$, the idea easily generalizes to arbitrary $N \in \mathbb{N}$.

From Lemma 4.1(iv) we know that the first integral in (26) is well-defined. Furthermore, we know that $f(L_\mu F) \in \mathcal{F}C_b^2(\mathcal{D}, \Gamma)$, because $\widehat{F} \in D(A_\mu)$, thus also the second integral in (26) is well defined. Then $\nabla^{\mathcal{K}_\mu} f(\langle \widehat{F}, \cdot \rangle_\mu) \in D(A_\mu)$, and

$$
\begin{aligned}
&\int_{\mathcal{N}'} f(L_\mu(F)) H_{\nu_\mu, A_\mu}\, g(L_\mu(G))\, d\nu_\mu \\
= &\int_{\mathcal{N}'} \langle \nabla^{\mathcal{K}_\mu} f(\langle \widehat{F}, \cdot \rangle_\mu), A_\mu \nabla^{\mathcal{K}_\mu} g(\langle \widehat{G}, \cdot \rangle_\mu) \rangle_\mu\, d\nu_\mu \\
= &\langle \widehat{F}, \widehat{H_\mu^\Gamma} \widehat{G} \rangle_\mu \int_{\mathcal{N}'} f'(\langle \widehat{F}, \cdot \rangle_\mu) g'(\langle \widehat{G}, \cdot \rangle_\mu)\, d\nu_\mu.
\end{aligned}
$$

For the other integral we have:

$$
\begin{aligned}
&\int_\Gamma f(F^{\Lambda_n}) H_\mu\, g(G^{\Lambda_n})\, d\mu = \int_\Gamma (\nabla^\Gamma f(F^{\Lambda_n}), \nabla^\Gamma g(G^{\Lambda_n}))_{T(\Gamma)}\, d\mu \\
= &\frac{1}{\mathrm{vol}(\Lambda_\mathrm{n})} \int_\Gamma \Bigg(\int_{\Lambda_n} \int_{\Lambda_n} (\tau_x \nabla^\Gamma F, \tau_y \nabla^\Gamma G)_{T(\Gamma)}\, dx\, dy \qquad (27) \\
- &\int_\Gamma \int_{\Lambda_n} \int_{\Lambda_n} (\tau_x \nabla^\Gamma F, \tau_y \nabla^\Gamma G)_{T(\Gamma)}\, dx\, dy\, d\mu \Bigg) f'(F^{\Lambda_n}) g'(G^{\Lambda_n})\, d\mu \\
+ &\frac{1}{\mathrm{vol}(\Lambda_\mathrm{n})} \int_\Gamma \int_{\Lambda_n} \int_{\Lambda_n} (\tau_x \nabla^\Gamma F, \tau_y \nabla^\Gamma G)_{T(\Gamma)}\, dx\, dy\, d\mu \\
\cdot &\int_\Gamma f'(F^{\Lambda_n}) g'(G^{\Lambda_n})\, d\mu.
\end{aligned}
$$

Let us prove that the first term of (27) tends to zero as $n \to \infty$. Before we note that by assumption on F and G there exists $s \in \mathbb{N}$ such that F and G are Λ_s-local, i.e., measurable w.r.t. $\mathcal{B}_{\Lambda_s}(\mathbb{R}^d)$. We have:

$$
\begin{aligned}
&\Bigg| \frac{1}{\mathrm{vol}(\Lambda_\mathrm{n})} \int_\Gamma \Bigg(\int_{\Lambda_n} \int_{\Lambda_n} (\tau_x \nabla^\Gamma F, \tau_y \nabla^\Gamma G)_{T(\Gamma)}\, dx\, dy \\
- &\int_\Gamma \int_{\Lambda_n} \int_{\Lambda_n} (\tau_x \nabla^\Gamma F, \tau_y \nabla^\Gamma G)_{T(\Gamma)}\, dx\, dy\, d\mu \Bigg) f'(F^{\Lambda_n}) g'(G^{\Lambda_n})\, d\mu \Bigg|^2 \\
\leq &\frac{c}{\mathrm{vol}(\Lambda_\mathrm{n})^2} \int_\Gamma \Bigg| \int_{\Lambda_n} \int_{\Lambda_n} (\tau_x \nabla^\Gamma F, \tau_y \nabla^\Gamma G)_{T(\Gamma)}\, dx\, dy \\
- &\int_{\Lambda_n} \int_{\Lambda_n} \int_\Gamma (\tau_x \nabla^\Gamma F, \tau_y \nabla^\Gamma G)_{T(\Gamma)}\, d\mu\, dx\, dy \Bigg|^2 d\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{c}{\mathrm{vol}(\Lambda_\mathrm{n})^2}\int_\Gamma \Big| \int_{\Lambda_n} \tau_x \int_{\Lambda_n} (\nabla^\Gamma F, \tau_{y-x}\nabla^\Gamma G)_{T(\Gamma)}\, dy\, dx \\
&- \int_{\Lambda_n}\int_{\Lambda_n}\int_\Gamma (\nabla^\Gamma F, \tau_{y-x}\nabla^\Gamma G)_{T(\Gamma)}\, d\mu\, dy\, dx \Big|^2\, d\mu \\
&= \frac{c}{\mathrm{vol}(\Lambda_\mathrm{n})^2}\int_\Gamma \Big| \int_{\Lambda_n} \left(\tau_x b_{x,n} - \int_\Gamma b_{x,n}\, d\mu\right) dx \Big|^2\, d\mu,
\end{aligned}
$$

where $c := \sup_{t\in\mathbb{R}}\{|f'(t)g'(t)|\} < \infty$ and we used the translation invariance of μ. Additionally, $b_{x,n} := \int_{\Lambda_n} (\nabla^\Gamma F, \tau_{y-x}\nabla^\Gamma G)_{T(\Gamma)}\, dy \in L^q_{\Lambda^{3s}}(\mu)$, for all $q \geq 1$. With the help of Theorem 3.3 and (12) we have the estimate

$$
\left| \int_\Gamma b_{x,n}\tau_q b_{y,n}\, d\mu - \int_\Gamma b_{x,n}\, d\mu \int_\Gamma b_{y,n}\, d\mu \right| \leq c_1 \exp(-c_2 \parallel q \parallel), \quad q \in \mathbb{R}^d,
$$

$c_1, c_2 > 0$, which is uniform in $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}$. Let us continue the estimation:

$$
\begin{aligned}
& \frac{c}{\mathrm{vol}(\Lambda_\mathrm{n})^2}\int_\Gamma \Big| \int_{\Lambda_n} \left(\tau_x b_{x,n} - \int_\Gamma b_{x,n}\, d\mu\right) dx \Big|^2\, d\mu \\
&= \frac{c}{\mathrm{vol}(\Lambda_\mathrm{n})^2}\int_{\Lambda_n}\int_{\Lambda_n}\int_\Gamma \tau_x b_{x,n}\tau_{y-x} b_{y,n} - \int_\Gamma b_{x,n}\, d\mu \int_\Gamma b_{y,n}\, d\mu\, d\mu\, dy\, dx \\
&\leq \frac{c}{\mathrm{vol}(\Lambda_\mathrm{n})^2}\int_{\Lambda_n}\int_{\mathbb{R}^d} \Big| \int_\Gamma \tau_x b_{x,n}\tau_q b_{q+x,n} - \int_\Gamma b_{x,n}\, d\mu \\
&\cdot \int_\Gamma b_{q+x,n}\, d\mu\, d\mu \Big|\, dq\, dx \leq \frac{c\, c_1}{\mathrm{vol}(\Lambda_\mathrm{n})}\int_{\mathbb{R}^d} \exp(-c_2 \parallel q \parallel)\, dq. \qquad (28)
\end{aligned}
$$

Of course, (28) converges to zero as $n \to \infty$. Then we have:

$$
\begin{aligned}
& \lim_{n\to\infty} \int_\Gamma f(F^{\Lambda_n}) H_\mu\, g(G^{\Lambda_n})\, d\mu \\
&= \lim_{n\to\infty} \frac{1}{\mathrm{vol}(\Lambda_\mathrm{n})}\int_\Gamma\int_{\Lambda_n}\int_{\Lambda_n} (\tau_x\nabla^\Gamma F, \tau_y\nabla^\Gamma G)_{T(\Gamma)}\, dx\, dy\, d\mu \\
&\cdot \lim_{n\to\infty}\int_\Gamma f'(F^{\Lambda_n}) g'(G^{\Lambda_n})\, d\mu \\
&= \lim_{n\to\infty}\int_\Gamma F^{\Lambda_n}(H^\Gamma_\mu G)^{\Lambda_n}\, d\mu \int_{\mathcal{N}'} f'(\langle \widehat{F}, \cdot\rangle_\mu) g'(\langle \widehat{G}, \cdot\rangle_\mu)\, d\nu_\mu \\
&= \langle \widehat{F}, \widehat{H^\Gamma_\mu} \widehat{G}\rangle_\mu \int_{\mathcal{N}'} f'(\langle \widehat{F}, \cdot\rangle_\mu) g'(\langle \widehat{G}, \cdot\rangle_\mu)\, d\nu_\mu,
\end{aligned}
$$

because of Theorem 3.7, see also Remark 3.9(iii), Lemma 4.3(i), (ii), and again Theorem 3.7. ■

It is well-known [BK95] that the operator H_{ν_μ,A_μ} is essentially self-adjoint on $\mathcal{F}C_b^2(\mathcal{N},\mathcal{N}')$. We preserve the same notation for its closure. We note that the space $\mathcal{F}C_b^2(D(A_\mu),\mathcal{N}')$ is included in the domain of this operator, see [BK95], Chapter 6.

The operator H_{ν_μ,A_μ} generates an infinite dimensional Ornstein-Uhlenbeck semi-group

$$T_t^\mu := \exp(-tH_{\nu_\mu,A_\mu}), \quad t \geq 0,$$

in $L^2(\nu_\mu)$. This semi-group is associated to a generalized Ornstein-Uhlenbeck process $(\Xi_t)_{t\geq 0}$ on $\mathcal{N}'$, see [BK95], Chapter 6, Section 1.5. This process is called macroscopic stochastic dynamics.

Let us summarize this paper with a concluding remark on the two main results.

Remark 5.3 *(i) The construction of the space of macroscopic fluctuations, via the central limit theorem for local fluctuations proved in Theorem 3.7, and of the stochastic dynamics in it is quite general. The main property we need is the mixing condition (7) or, equivalently, the decay of correlations (6).*
(ii) Theorem 5.2 proves that when starting with the generator H_μ^Γ of the microscopic stochastic dynamics in the fluctuation limit there appears the generator H_{ν_μ,A_μ} of the macroscopic stochastic dynamics.

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