SOME STOCHASTIC DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL DRIFT

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Summary: In dimension 1 we study a martingale problem related to a parabolic PDE operator L with continuous (non-degenerate) diffusion term and with drift being the derivative of a continuous function. In several situations, this problem turns out to correspond to a true SDE. We study existence and uniqueness and other properties of the solution. We state a necessary and sufficient characterization for the solution X (or $f(X)$) to be a semimartingale. When X is a semimartingale, we also establish an Itô formula under very weak conditions for $f(X)$.

Key words: Martingale problems, Lyons-Zheng processes, time reversal, distributional drift

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Introduction

The aim of this paper is to study stochastic differential equations of the type

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b'(X_s)\, ds, \tag{0.1}$$

where b, σ are continuous functions such that $\sigma \geq c > 0$. In such a case the formal operator associated with X is given by $Lf = \frac{\sigma^2}{2} f'' + b' f'$.

Equation (0.1) will be considered as a martingale problem and sometimes in the weak (in law) sense.

Diffusions in a generalized sense were studied by several authors. First, we mention a classical book by N.I. Portenko ([39]) which, however, remains in the framework of semimartingales. The point of view adopted in this book is different from ours; his aim is to start from a Markov semigroup in order to construct solutions to some stochastic differential equations in a generalized sense. We choose to adopt a direct stochastic analysis perspective without using Markov properties. At this stage, comparing the two approaches appears to be a delicate challenge.

Relevant work in this area was done by H.J. Engelbert and W.M. Schmidt ([14]) who investigated solutions to stochastic differential equations with generalized drift remaining however in the class of semimartingales. More recently, H.J. Engelbert and J. Wolf ([15]) considered special cases of processes solving stochastic differential equations with generalized drift; those cases include examples coming from Bessel processes. Those solutions are no longer semimartingales but Dirichlet processes. A special case of equation (0.1) with $\sigma = 1$ and continuous b was treated by P. Seignourel ([47]) without defining the stochastic analysis framework in relation with long time behaviour. This is the case of irregular medium; the case of b being a Brownian path appears also in the literature with the denomination "random medium"; for recent results we refer to [31, 32]

The literature on Dirichlet processes in the framework of Dirichlet forms is huge and it is impossible to list it completely. We only want to mention some very useful monographies such as [22, 23]. The subject has shown a large development in infinite dimension starting from [2]. A later monography is

[28]. Recently, the case of time-dependent Dirichlet forms has attracted a lot of interest, see [36, 48].

Our point of view of Dirichlet processes is pathwise, following [19, 4]. A (continuous) Dirichlet process is the sum of a local martingale M and a zero quadratic variation process A. A special class of Dirichlet processes which contains the class of reversible semimartingales is the set of Lyons-Zheng (LZ) processes. Those processes were defined in [46] after inspiration from [27, 26]. A (strong) LZ process essentially is a Dirichlet process $M+A$ whose zero quadratic variation part can be expressed as

$$A = \frac{1}{2}(M^2 - M^1) + V, \tag{0.2}$$

where $M^1 = M$, M^2 is a backward local martingale and V is a bounded variation process.

Examples of such processes are first of all C^1-functions of reversible semimartingales; furthermore, Bessel processes of arbitrary dimension belong to this class, see [46]. In this paper, we will also see that, generally, solutions to stochastic differential equations with distributional drift are LZ processes.

For a LZ process X, it is quite natural to define a stochastic integral (of symmetric type) and an Itô formula for $f(X)$, $f \in C^1$, see [27, 46]. An Itô formula under weak conditions is also the object of this paper. We recall that two papers were simultaneously written by Föllmer, Protter, Shiryaev ([21]) and Russo, Vallois ([44]) for $f(B)$ and $f(S)$, respectively; in the first case $f \in W^{1,2}_{loc}$ and B is a classical Brownian motion; in the second case $f \in C^1$ and S is a reversible (multidimensional) semimartingale. Subsequently, generalizations of the first case were treated in [5], [34], where S is first an elliptic diffusion with smooth coefficients and a non-degenerate martingale in the sense of Malliavin calculus. Errami, Russo and Vallois ([16]) generalized the paper [44] to the case of processes with jumps. The multidimensional situation for $f \in W^{1,2}_{loc}$ was treated in [20] for Brownian motion B and in [34] (resp. [35]) when X is one-dimensional (resp. multidimensional) Brownian martingale which is non-degenerate regarding Malliavin calculus.

The paper is organized as follows. First we introduce the concept of a C^1-generalized solution to $Lf = \dot{l}$, where $\dot{l} \in C^0$, $f \in C^1$. Under the assumption that there exists $h \in C^1$ with $Lh = 0$, $h'(x) \neq 0$ for every x, we can show that $Lf = \dot{l}$ admits a solution for any $\dot{l} \in C^0$. $\mathcal{D}_L$ will be the subset of C^1-functions f such that $Lf = \dot{l}$ for some $\dot{l} \in C^0$. Significant examples arise when $b = \alpha\sigma^2/2 + \beta$, where $\alpha \in [0,1]$ and β is a function of bounded variation. A particular situation arises when L is close to divergence type which means that

$$b = \frac{\sigma^2}{2} + \beta. \tag{0.3}$$

In Section 3, we present a martingale problem related to L. This problem will have a unique solution provided that a condition of non-explosion is fulfilled. Moreover, we show that the occupation measure always admits a density. If L is close to divergence type then it is possible to show that the martingale problem is equivalent to a stochastic differential equation in the weak sense (0.1); more precisely, the solution X to the martingale problem associated with L will solve

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + A(b), \tag{0.4}$$

where $A : C^0(\mathbb{R}) \to \mathcal{C}$ is the unique extension of the map

$$l \mapsto \int_0^{\cdot} l'(X_s)\, ds$$

defined on $C^1(\mathbb{R})$; $\mathcal{C}$ denotes the metric space of continuous processes endowed with the ucp topology. The existence of such an extension is explained by the fact that the map $\mathcal{L} : \mathcal{D}_L \to C^0$, defined by

$$\mathcal{L}f(x) = \int_0^x Lf(y)\, dy$$

can be extended uniquely to $C^1(\mathbb{R})$.

In Section 3, we also prove that L is truely the infinitesimal generator associated with the solution a martingale problem. Moreover, we treat a suitable Kolmogorov equation which allows to deduce that the law of X_t admits a density p_t, $t \geq 0$, and to examine some properties.

In Section 4, we reveal that the solution to a martingale problem is in fact a LZ process, at least under some weak technical assumption on the coefficients. For this, we show that it is the C^1-transformation of a time reversible semimartingale.

Time reversal tools are essential for this task. Similar calculations have been performed by several authors in other situations, see for instance [19, 33, 7, 8, 29].

The second significant result of Section 4 is an Itô formula under weak conditions. It applies to $f \in W^{1,2}_{loc}$ and solutions X to the equation

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t \gamma(X_s)\, ds, \tag{0.5}$$

where $\sigma > 0$ is continuous and $\gamma \in L^\infty_{loc}$. Previous results in this direction were obtained in [21, 44, 34, 35, 20, 16].

Section 5 is finally devoted to characterize the class of $f \in W^{1,2}_{loc}$ such that $f(X)$ is a semimartingale. In particular, we obtain a condition on b, σ ensuring that X is a semimartingale.

Semimartingale characterizations for functions of the Brownian motion were first considered by [10] with extensions to the Markov case. Extensions for time inhomogeneous functions of the Brownian motion have been obtained by [9, 30].

We conclude the paper in Section 6 by examining the case when L is of divergence type

$$Lf = \frac{1}{2}(\sigma^2 f')'. \tag{0.6}$$

A first interpretation of X was given in [48] and in [40] related to a LZ type decomposition. In [40], however, no stochastic differential equation appears. If $\sigma \in C^1$ then it is immediate to show

$$X_t = x_0 + \int_0^t \sigma(X_s)\, d^+W_s, \tag{0.7}$$

where d^+W denotes the backward integral.

If X solves a martingale problem associated with L then it follows from Section 5 that X is a semimartingale if and only if σ has bounded variation. In this case we show that X solves also (0.6). In the general case (σ continuous) this is no longer true.

1 Notations and recalls

If I is a real open interval then $C(I)$ will be the F-type space (according to the notations of [12, Chapter 2]) of continuous functions on I endowed with the topology of uniform convergence on compacts. For $k \geq 0$, $C^k(I)$ will be a similar space equipped with the topology of uniform convergence of the first k derivatives. If $I = \mathbb{R}$ we will simply write C, C^k instead of $C(\mathbb{R})$, $C^k(\mathbb{R})$.

We also need to introduce the following subspaces of C^1:

$$\begin{aligned} C_0^1 &:= \{f \in C^1 : f(0) = 0\}, \\ C_{0,0}^1 &:= \{f \in C^1 : f(0) = f'(0) = 0\}. \end{aligned}$$

Furthermore, we will work with the following F-type spaces. L^2_{loc} denotes the space of all Borel functions which are square integrable when restricted to compact subsets. $W^{1,2}_{loc}$ is the space of all absolutely continuous functions f admitting a density $f' \in L^2_{loc}$. It is equipped with the distance which sums $|f(0)|$ and the distance of f' in L^2_{loc}. A subspace of $W^{1,2}_{loc}$ will be

$$W^{1,2}_{0,loc} := \{f \in W^{1,2}_{loc} : f(0) = 0\}.$$

Similarly, we can consider L^p_{loc} for $p \geq 1$. We denote the set of C^k real functions with compact support by C^k_c, $k \geq 0$. *const* will denote a generic positive constant.

T will be a fixed real number. We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. All processes will be considered with index in $\mathbb{R}$. The F-type space of continuous processes equipped with the ucp topology is denoted by $\mathcal{C}$. We recall that a sequence of processes (H_n) in $\mathcal{C}$ converges ucp to H if, for every $T > 0$, $\sup_{t\in[0,T]} |(H_n - H)(t)|$ converges to zero in probability. Note that H belongs automatically to $\mathcal{C}$.

For convenience, we follow the framework of stochastic calculus introduced in [42] and continued in [43, 44, 45], [51, 52, 53] and [46]. Let $X = (X_t, t \in [0,T])$ be a continuous process and $Y = (Y_t, t \in [0,T])$ be a process with paths in L^1_{loc}. We recall in the sequel the most useful rules of calculus.

The forward, backward and symmetric integrals and the covariation process are defined by the following limits in the ucp (uniform convergence in probability) sense whenever they exist

$$\int_0^t Y_s\, d^- X_s \quad := \quad \lim_{\varepsilon \to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\, ds \tag{1.1}$$

$$\int_0^t Y_s\, d^+ X_s \quad := \quad \lim_{\varepsilon \to 0+} \int_0^t Y_s \frac{X_s - X_{(s-\varepsilon)\vee 0}}{\varepsilon}\, ds \tag{1.2}$$

$$\int_0^t Y_s\, d^0 X_s \quad := \quad \lim_{\varepsilon \to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{(s-\varepsilon)\vee 0}}{2\varepsilon}\, ds \tag{1.3}$$

$$[X,Y]_t \quad := \quad \lim_{\varepsilon \to 0+} C_\varepsilon(X,Y)_t, \tag{1.4}$$

where

$$C_\varepsilon(X,Y)_t := \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)\, ds.$$

For $[X,X]$ we shortly write $[X]$.

All stochastic integrals and covariation processes will be of course elements of $\mathcal{C}$.

For a given process $Z = (Z_t, t \in [0,T])$ we set $\hat{Z}_t := Z(T-t)$, $t \in [0,T]$.

Remark 1.1

a) $\int_0^t Y_s\, d^0 X_s = \frac{1}{2} \int_0^t Y_s\, d^- X_s + \frac{1}{2} \int_0^t Y_s\, d^+ X_s$,

b) $[X,Y]_t = \int_0^t Y_s\, d^+ X_s - \int_0^t Y_s\, d^- X_s$, provided that two of the three terms in a) and b) exist.

c) $X_t Y_t = X_0 Y_0 + \int_0^t Y_s\, d^- X_s + \int_0^t X_s\, d^- Y_s + [X,Y]_t$ with similar conventions as in a) and b).

d) $[X,Y]_t = [\hat{X}, \hat{Y}]_T - [\hat{X}, \hat{Y}]_{T-t}$.

e) If one of the two following members exists then

$$\int_0^t Y_s\, d^+ X_s = - \int_{T-t}^T \hat{Y}_s\, d^- \hat{X}_s$$

holds, where the integrals from a to b ($a, b \in \mathbb{R}$) are analogously defined as in (1.1), ..., (1.4).

Remark 1.2

a) If $[X,X]$ exists then it is always an increasing process and X is called a *finite quadratic variation process*. If $[X,X] \equiv 0$ then X is said to be a *zero quadratic variation process* (or a zero energy process).

b) If X, Y are continuous processes such that $[X,Y]$, $[X,X]$, $[Y,Y]$ exist then $[X,Y]$ has bounded (total) variation. If $f, g \in C^1$ then

$$[f(X), g(Y)]_t = \int_0^t f'(X) g'(Y) \, d[X,Y].$$

c) If A is a zero quadratic variation process and X is a finite quadratic variation process then $[X,A] \equiv 0$.

d) A bounded variation process is a zero quadratic variation process.

e) (*Classical Itô formula*) If $f \in C^2$ then $\int_0^\cdot f'(X) \, d^- X$ exists and is equal to

$$f(X) - f(X_0) - \frac{1}{2} \int_0^\cdot f''(X) \, d[X].$$

f) If $f \in C^1$ and $g \in C^2$ then the forward integral $\int_0^\cdot f(X) d^- g(X)$ is well defined.

In this paper all filtrations are supposed to fulfill the usual conditions. If $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a filtration, X an $\mathbb{F}$-semimartingale, Y is $\mathbb{F}$-adapted with the suitable square integrability conditions, then $\int_0^\cdot Y \, d^- X$ is the usual Itô integral. If Y is an $\mathbb{F}$-semimartingale then $\int_0^\cdot Y \, d^0 X$ is the classical Fisk-Stratonovich integral and $[X,Y]$ the usual covariation process $\langle X, Y \rangle$.

A semimartingale X such that $\hat{X}$ is again a semimartingale is said to be a *time reversible semimartingale*.

A *Dirichlet process* is the sum of an $\mathbb{F}$-local continuous martingale M and an $\mathbb{F}$-adapted zero quadratic variation process A, see [19], [4].

Remark 1.3 ([46]) If $X = M + A$ is a Dirichlet process and $f \in C^1$ then $f(X) = M^f + A^f$ is a Dirichlet process, where

$$M^f = \int_0^\cdot f'(X_s) \, dM_s$$

and $A^f := f(X) - M^f$ has zero quadratic variation. □

A sequence (τ^N) of $\mathbb{F}$- stopping times will be said to be "suitable" if

$$\bigcup_N \{\tau^N \leq T\}$$

has probability one. We will use the notation of stopped process as usually X^τ.

Remark 1.4 Let X a $\mathbb{F}$- adapted continuous process.

X is a semimartingale (resp. Dirichlet processes) if and only if the stopped processes X^{τ^N} are also semimartingales (resp. Dirichlet processes). □

At this stage, we recall the concept of a LZ process, see [46].

Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ be two filtrations. A process $Y = (Y_t, t \in [0,T])$ is said to be $(\mathbb{F},\mathbb{H})$-*adapted* if Y is $\mathbb{F}$-adapted and $\hat{Y}$ is $\mathbb{H}$-adapted.

A continuous $(\mathbb{F},\mathbb{H})$-adapted process $(X_t)_{t\in[0,T]}$ is called a (strong) $(\mathbb{F},\mathbb{H})$-*Lyons-Zheng process* (or simply *LZ process*) if there are $M^i = (M^i_t,\ t \in [0,T])$, $i = 1, 2$, $V = (V_t,\ t \in [0,T])$, such that

$$X = \frac{1}{2}M^1 + \frac{1}{2}M^2 + V \tag{1.5}$$

and the following conditions are satisfied:

a) M^1 is a local $\mathbb{F}$-martingale with $M^1_0 = 0$.

b) $\hat{M}^2$ is a local $\mathbb{H}$-martingale with $M^2_T = 0$.

c) V is a bounded variation process.

d) $M^1 - M^2$ is a zero quadratic variation process.

Remark 1.5 Let $X = (X_t, t \in [0,T])$ be a $(\mathbb{F},\mathbb{H})$-LZ process.

a) $[X,X] = \frac{1}{2}([M^1,M^1]+[M^2,M^2])$. In particular, X is a finite quadratic variation process.

b) If X is a $(\mathbb{F},\mathbb{H})$-LZ process then $\hat{X}$ is a $(\mathbb{H},\mathbb{F})$-LZ process.

c) The decomposition (1.5) is unique.

d) A time reversible semimartingale is a LZ process with respect to the natural filtrations.

e) If X is a LZ process with $X = \frac{1}{2}(M^1 + M^2) + V$ then $Y = f(X)$, where $f \in C^1$, is again a LZ process with decomposition

$$M_f^1 = \int_0^{\cdot} f'(X)\, dM^1, \quad M_f^2 = -\int_{\cdot}^{T} f'(X)\, d^+ M^2.$$

f) A LZ process which is also a semimartingale is a time reversible semimartingale.

h) A LZ process is a Dirichlet process.

i) Let X be a $(\mathbb{F}, \mathbb{H})$-adapted process admitting a decomposition of type (1.5) satisfying the conditions a), b), c). If X is a Dirichlet process with M^1 as martingale part, then it is truely a LZ process; that means that also condition d) is realized.

j) If X is a stationary symmetric Markov process associated with a Dirichlet form (see f. ex. [22]) and u belongs to the domain of the form then $u(X)$ is a LZ process (see [26]). In this case we have $V = 0$.

We recall briefly the notion of LZ stochastic integration in a specific framework. Let Y be a $(\mathbb{F}, \mathbb{H})$-adapted process and X a $(\mathbb{F}, \mathbb{H})$-Lyons Zheng process with decomposition (1.5). The LZ-symmetric integral is then defined by

$$\int_0^t Y \circ dX = \frac{1}{2} \int_0^t Y\, d^- M^1 - \frac{1}{2} \int_{T-t}^{T} \hat{Y}\, d^- \hat{M}^2 + \int_0^t Y\, dV. \tag{1.6}$$

We recall that

$$\begin{aligned} \int_0^t Y\, d^- M^1 &= \int_0^t (Y_s - Y_0)\, dM_s^1 + Y_0 M_t^1 \\ \int_0^t Y\, d^+ M^2 &= -\int_{T-t}^{T} (\hat{Y}_s - Y_T)\, d\hat{M}_s^2 + Y_T M_t^2. \end{aligned}$$

Remark 1.6 If $[Y, M^i]$, $i = 1, 2$, exist then

$$\int_0^t Y\, d^0 X = \int_0^t Y \circ dX + \frac{1}{4}[Y, M^1 - M^2]_t.$$

In particular, if Y is a zero quadratic variation process then $[Y, M^1 - M^2] = 0$ and so

$$\int_0^\cdot Y\, d^0X = \int_0^\cdot Y \circ dX.$$

Remark 1.7 Let X be a $(\mathbb{F}, \mathbb{H})$-LZ process, H, R be $(\mathbb{F}, \mathbb{H})$-adapted processes. We define $Y_t := \int_0^t R \circ dX$, $0 \le t \le T$. Then we have

$$\int_0^t H \circ dY = \int_0^t HR \circ dX.$$

Remark 1.8 Let $f \in C^1(\mathbb{R}^n)$, $\underline{X} = (X_1, \ldots, X_n)$ be a vector of $(\mathbb{F}, \mathbb{H})$-LZ processes. The following Itô formula holds:

$$f(\underline{X}_t) = f(\underline{X}_0) + \int_0^t \Big(\sum_{i=1}^n \partial_i f(\underline{X}_s) \Big) \circ dX_s^i$$

(see [46, 4.4]).

2 A particular type of solution

Let σ, $b \in C^0(\mathbb{R})$ such that $\sigma > 0$. We consider formally a PDE operator of the following type:

$$Lg = \frac{\sigma^2}{2} g'' + b' g'. \tag{2.1}$$

By a mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x)\, dx = 1$. We denote

$$\Phi_n(x) := n\Phi(nx), \quad \sigma_n^2 := \sigma^2 * \Phi_n, \quad b_n := b * \Phi_n.$$

We then consider

$$L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'. \tag{2.2}$$

A priori, σ_n^2, b_n and the operator L_n depend on the mollifier Φ.

Definition 2.1 A function $f \in C^1(\mathbb{R})$ is said to be a solution to

$$Lf = \dot{l}, \tag{2.3}$$

where $\dot{l} \in C^0$, (in the C^1-generalized sense) if, for any mollifier Φ, there are sequences (f_n) in C^2, $(\dot{l}_n)$ in C^0 such that

$$L_n f_n = \dot{l}_n, \quad f_n \to f \text{ in } C^1, \quad \dot{l}_n \to \dot{l} \text{ in } C^0. \tag{2.4}$$

Remark 2.2 The previous definition and notations can be adapted when $\mathbb{R}$ is replaced by a real interval $I =]a, b[$, $-\infty \leq a < b \leq +\infty$, $\sigma, b \in C^0(I)$ and (2.1) is defined on I. We extend σ, b by zero on I^c and, for $g \in C^2(I)$, we define

$$L_n g = \frac{(\sigma_n^2)_{|I}}{2} g'' + (b_n')_{|I} g'.$$

Then $f' \in C^1(I)$ is a C^1-generalized solution to $Lf = \dot{l}$ if (2.3) and (2.4) hold when C^1 and C^0 are replaced by $C^1(I)$ and $C^0(I)$, respectively. □

Remark 2.3 Let I be as above. If $b' \in C^0(I)$ and $f \in C^2(I)$ is a cassical solution to $Lf = \dot{l}$ then f is immediately seen to be a C^1-generalized solution.

We go on stating results for $I = \mathbb{R}$.

Proposition 2.4 *There is a solution $h \in C^1$ to $Lh = 0$ such that $h'(x) \neq 0$ for every $x \in \mathbb{R}$ if and only if*

$$\Sigma(x) := \lim_{n \to \infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy$$

exists in C^0, independently from the mollifier. Moreover, in this case, any solution f to $Lf = 0$ fulfills

$$.f'(x) = e^{-\Sigma(x)} f'(0). \tag{2.5}$$

Proof. Let $h \in C^1$ be a solution to $Lh = 0$ with $h' \neq 0$ for every $x \in \mathbb{R}$. Then there are sequences $(\dot{a}_n)$ in C^0 and (h_n) in C^1 such that $\dot{a}_n \to 0$ in C^0, $h_n \to h$ in C^1 and $L_n h_n = \dot{a}_n$. Setting $g_n := h_n'$ we have $g_n' \frac{\sigma_n^2}{2} + g_n b_n' = \dot{a}_n$ and $g_n \to g = h'$ in C^0. Dividing by $(g_n \sigma_n^2)/2$, we get

$$(\log g_n)' + 2\frac{b_n'}{\sigma_n^2} = 2\frac{\dot{a}_n}{\sigma_n^2 g_n}. \tag{2.6}$$

Since $g = h' > 0$ and $g_n^{-1} \to g^{-1}$ in C^0, by integrating (2.6), Σ is well-defined and we have

$$(\log) g(x) = -\Sigma(x) + const. \tag{2.7}$$

This proves the direct sense of the implication; it also proves that h is of the type $h'(x) = h'(0) \exp(-\Sigma(x))$. The converse is clear.

It remains to prove that any other solution to $Lf = 0$ fulfills (2.5). Let $f \in C^1$ be a solution and $x_0 \in \mathbb{R}$ with $f'(x_0) > 0$. By continuity, there is a neighbourhood I_0 of x_0 such that $f'(x) > 0$ holds for every $x \in I_0$. By the same reasoning as before, we easily verify

$$\log g(x) - \log g(x_0) = -\Sigma(x) + \Sigma(x_0)$$

for every $x \in I_0$. This establishes (2.5) on I_0.

Since f' is continuous, (2.7) holds for every x belonging to the closure of $J = \{x : f'(x) \neq 0\}$. This implies that

$$f'(x) = f'(0)\exp(-\Sigma(x)) \tag{2.8}$$

for every $x \in J$. At this point, we have two possibilities.

a) Either $f'(0) = 0$ so that $J = \emptyset$ holds according to (2.8). Thus, $f' \equiv 0$.

b) Or we have $f'(0) \neq 0$. Then J is non empty. Since J^c is open, ∂J is not empty except when $J = \mathbb{R}$. Let $a \in \partial J$. By continuity of Σ, we have

$$f'(a) = \lim_{x \to a} f'(x) = f'(0)\exp(-\Sigma(a)) \neq 0.$$

On the other hand, we observe

$$f'(a) = \lim_{x \to a, x \notin J} f'(x) = 0.$$

This contradiction implies $J = \mathbb{R}$. □

From now on, throughout the whole paper, we will suppose the existence of this function Σ. We will set

$$h'(x) := \exp(-\Sigma(x)), h(0) = 0.$$

Thus, $h'(0) = 1$ holds.

Remark 2.5 In particular, this proves the uniqueness of the problem

$$Lf = \dot{l}, \quad f \in C^1, \quad f(0) = x_0, \quad f'(0) = x_1 \tag{2.9}$$

for every $\dot{l} \in C^0$, $x_0, x_1 \in \mathbb{R}$.

Remark 2.6 We present three examples.

a) If $b = \alpha \frac{\sigma^2}{2}$ for some $\alpha \in]0,1]$ then

$$\Sigma(x) = \log(\sigma^{+2\alpha}(x))$$

and

$$h'(x) = \sigma^{-2\alpha}(x).$$

b) Suppose that b is of bounded variation. Then we get

$$\int_0^x \frac{b'_n}{\sigma_n^2}(y)\,dy = \int_0^x \frac{db_n(y)}{\sigma_n^2(y)} \to \int_0^x \frac{db}{\sigma^2},$$

since $db_n \to db$ weakly-$*$ and $\frac{1}{\sigma^2}$ is continuous.

c) If σ has bounded variation then we have

$$\Sigma(x) = -2\int_0^x b\,d(\frac{1}{\sigma^2}) + \frac{2b}{\sigma^2}(x) - \frac{2b}{\sigma^2}(0).$$

In particular, this example contains the cas where $\sigma = 1$ for any b.

Lemma 2.7 *If $Lh = 0$ for some $h \in C^1$ with $h'(x) \neq 0$ for every $x \in \mathbb{R}$ then a solution to problem (2.9) is given by*

$$\begin{aligned} f(0) &= x_0, \\ f'(x) &= h'(x)\left(2\int_0^x \frac{\imath(y)}{(\sigma^2 h')(y)}\,dy + \frac{x_1}{h'(0)}\right). \end{aligned}$$

Proof. We define $f_n \in C^1$ such that

$$\begin{aligned} f_n(0) &= x_0, \\ f'_n(x) &= \left(2\int_0^x \frac{\imath}{\sigma_n^2 h'_n}(y)\,dy + \frac{x_1}{h'_n(0)}\right) h'_n(x), \end{aligned}$$

where

$$h_n(0) = 0 \quad \text{and} \quad h'_n(x) = \exp\left(-\int_0^x \frac{2b'_n}{\sigma_n^2}(y)\,dy\right).$$

Clearly, we have $L_n h_n \equiv 0$ and $h_n \to h$ in C^1. So, we observe

$$L_n f_n = \frac{\sigma_n^2}{2} f''_n + b'_n f'_n = \imath$$

and

$$f_n' \to \left(2\int_0^x \frac{\dot{l}}{\sigma^2 h'}(y)\,dy + \frac{x_1}{h'(0)}\right) h'(x)$$

in C^0. □

Remark 2.8 Let $\dot{l} \in C^0$ and $x_0, x_1, c \in \mathbb{R}$. Then there is a unique solution in the C^1-generalized sense to

$$\begin{aligned} Lu &= \dot{l} \\ u(c) &= x_0, \quad u'(c) = x_1. \end{aligned} \tag{2.10}$$

The solution satisfies

$$u'(c) = h_c'(x)\left(2\int_c^x \frac{\dot{l}}{\sigma^2 h_c'}(y)\,dy + \frac{x_1}{h_c'(c)}\right),$$

where $h_c'(x) = \exp(\Sigma(c) - \Sigma(x))$.

In the case $c = 0$ this is a consequence of Lemma 2.7 and Remark 2.5. In the general case the justification is analogous. □

We will denote by $\mathcal{D}_L$ (resp. $\mathcal{D}_L(I)$) the set of all $f \in C^1(\mathbb{R})$ (resp. $C^1(I)$) such that there exists some $\dot{l} \in C^0$ with $Lf = \dot{l}$ in the C^1-generalized sense.

Proposition 2.9 *$\mathcal{D}_L$ is dense in C^1.*

Proof. It suffices to show that every C^2-function is the C^1-limit of a sequence of functions in $\mathcal{D}_L$. Let $f \in C^2$. We define

$$\dot{l}_n := L_n f, \quad g := f'. \tag{2.11}$$

By uniqueness, we have

$$f'(x) = h_n'(x)\left(f'(0) + \int_0^x \frac{2\dot{l}_n}{\sigma_n^2}\frac{1}{h_n'}(y)\,dy\right),$$

where

$$h_n'(x) = \exp\left(-2\int_0^x \frac{b_n'}{\sigma_n^2}(y)\,dy\right), \quad h_n(0) = 0.$$

So, we see that

$$f'(x) = h_n'(x)\left(f'(0) + \int_0^x 2\frac{\dot{\lambda}_n}{\sigma^2 h'}(y)\,dy\right),$$

where $h'(x) = \exp(-\Sigma(x))$, $h(0) = 0$ and $\dot{\lambda}_n(y) = \dot{l}_n(\sigma/\sigma_n)^2(h'/h'_n)(y)$.

We define $f_n \in C^1$ by $f_n(0) = 0$ and

$$f'_n(x) = h'(x)\left(f'(0) + 2\int_0^x \frac{\dot{\lambda}_n}{\sigma^2 h'}(y)\,dy\right).$$

We then observe $Lf_n = \dot{\lambda}_n$ and therefore $f_n \in \mathcal{D}_L$ as well as $\lim_{n\to\infty} f'_n(x) = f'(x)$ in C^0. This finishes the proof of the proposition. □

Corollary 2.10 *$\mathcal{D}_L$ is dense in $W^{1,2}_{loc}$.*

Remark 2.11 Let us consider again the case of example a) with $b = \alpha\frac{\sigma^2}{2}, \alpha \in]0,1]$. Setting

$$f_n(x) = \int_0^x \sigma_n^{-2\alpha}(y)\,dy, \quad f(x) = \int_0^x \sigma^{-2\alpha}(y)\,dy,$$

we obtain $L_n f_n = 0$ and thus $Lf = 0$ in the generalized sense. Now Lf_n is a well defined distribution for each n. However, Lf_n does not converge to zero when $n \to \infty$, excepted for the case $\alpha = 1$ (divergence case).

This shows in particular that Lf cannot be defined using simply distributions theory. □

We need now to discuss technical aspects of the way L and its domain $\mathcal{D}_L$ are transformed by h. We recall that $Lh = 0$ and h' is strictly positive so that we may denote the image of h by $I = Im\,h =]a,b[$ and the inverse function by $h^{-1} : I \to \mathbb{R}$.

Let L^0 be the classical PDE operator

$$L^0\phi = \frac{\sigma_0^2}{2}\phi'', \tag{2.12}$$

where

$$\sigma_0(y) = \begin{cases} (\sigma h')(h^{-1}(y)) & : \; y \in I \\ 0 & : \; y \notin I. \end{cases}$$

L^0 is a classical PDE map; however we can also consider it at the formal level and introduce $\mathcal{D}_{L^0}$.

Proposition 2.12 *a)* $h^2 \in \mathcal{D}_L$, $Lh^2 = h'^2\sigma^2$,

b) $\mathcal{D}_{L^0}(I) = C^2(I)$,

c) $\phi \in \mathcal{D}_{L^0}(I)$ *holds if and only if* $\phi \circ h \in \mathcal{D}_L$. *Moreover, we have*

$$L(\phi \circ h) = (L^0\phi) \circ h \tag{2.13}$$

for every $\phi \in C^2(I)$.

Proof. a) can be easily justified by approximations. Here we only give the formal calculations:

$$\begin{aligned} Lh^2 &= \frac{\sigma^2}{2}(h^2)'' + b'(h^2)' \\ &= \sigma^2 h'' h + \sigma^2 h'^2 + 2h'b'h \\ &= 2hLh + \sigma^2 h'^2 \\ &= \sigma^2 h'^2. \end{aligned}$$

b) Remark 2.3 says that $C^2(I) \subset \mathcal{D}_{L^0}(I)$. Conversely, if $\phi \in \mathcal{D}_{L^0}$ then, for any mollifier Φ, there are sequences (ϕ_n) in C^2, $(\dot{l}_n)$ in $C^0(I)$ such that $L^0_n(\phi_n) = \dot{l}_n$, $\phi_n \to \phi$ in $C^1(I)$ and $\dot{l}_n \to \dot{l}$ in $C^0(I)$. Furthermore, we have $L^0_n g = (\sigma^2_{0,n}/2)g''$ and $\sigma^2_{0,n} = \sigma^2_0 * (\Phi_n)_{|I}$. In order to prove $\phi \in C^2(I)$ we only need to show that (ϕ''_n) is a Cauchy sequence. But this is verified by observing

$$\phi''_n = \frac{2}{\sigma^2_{0,n}} L^0_n \phi_n = \frac{2}{\sigma^2_{0,n}} \dot{l}_n \to \frac{2}{\sigma^2_0}\dot{l}$$

in $C^0(I)$.

c) We choose a $C^2(\mathbb{R})$-sequence (h_n) such that $L_n h_n = \dot{a}_n \to 0$ in C^0 and $L_n g = (\sigma^2_n/2)g'' + b'_n g$. We recall that $\sigma^2_0(y) = (\sigma^2 h'^2)(h^{-1}(y))$ for every $y \in I$.

If $f \in \mathcal{D}_L$ then there is a sequence (f_n) in C^1 such that $L_n f_n$ converges in C^0 to some $\dot{l}$ on C^0. We are going to prove $f \circ h^{-1} \in C^2(I)$. We evaluate

$$\begin{aligned} (f_n \circ h_n^{-1})'' &= f''_n \circ h_n^{-1} (h_n^{-1})'^2 + f'_n \circ h_n^{-1} (h_n^{-1})'' \\ &= \frac{f''_n}{h'^2_n} \circ h_n^{-1} - \frac{f'_n h''_n}{(h'_n)^3} \circ h_n^{-1} \\ &= \frac{2}{\sigma^2_n}\frac{L_n f_n}{h'^2_n} \circ h_n^{-1} - \frac{2\dot{a}_n}{\sigma^2_n}\frac{f'_n}{h'^3_n} \circ h_n^{-1}. \end{aligned}$$

Since $Lh = 0$ in the C^1 sense, $\dot{a}_n \to 0$ holds in C^0. Thus, the previous term converges in C^0 to

$$\frac{2\dot{l}}{\sigma^2 h'^2}(h^{-1}) = \frac{2Lf}{\sigma^2 h'^2} \circ h^{-1}.$$

Consequently, $(f_n \circ h_n^{-1})''$ is a Cauchy sequence in $C^0(I)$ and thus $f \circ h^{-1} \in C^2(I)$. Moreover, we have shown that

$$(f \circ h^{-1})'' = \frac{2}{\sigma_0^2}(Lf) \circ h^{-1}.$$

This entails

$$L^0(f \circ h^{-1}) = (Lf) \circ h^{-1}. \tag{2.14}$$

So, using b) we have proven the converse part of c).

In order to prove the direct implication of c) we have to show that $f = \phi \circ h \in \mathcal{D}_L$ holds for $\phi \in C^2(I)$. Then (2.13) will follow from (2.14). It is enough to find (f_n) in C^2 such that $f_n \to f$ in C^1 and $L_n f_n$ converge in C^0. We set $\dot{\lambda} := L^0 \phi$ and choose a sequence of regularizations $(\dot{\lambda}_n)$ of $\dot{\lambda}$. Let f_n be the C^2-function solving $L_n f_n = \dot{\lambda}_n$, $f_n(0) = f_n'(0) = 0$. By uniqueness, we have

$$f_n'(x) = 2 \exp(-\Sigma_n(x)) \int_0^x \frac{\dot{\lambda}_n}{\sigma_n^2}(y) \exp(\Sigma_n(y))\, dy,$$

where $\Sigma_n(x) = 2\int_0^x (b_n'/\sigma_n^2)(y)\, dy$. We observe

$$\lim_{n\to\infty} f_n'(x) = f'(x) = \exp(-\Sigma(x)) \int_0^x \frac{2\dot{\lambda}(y)}{\sigma^2(y)\exp(-\Sigma(y))}\, dy.$$

Clearly, we now see $f \in \mathcal{D}_L$ and $Lf = \lambda$. This finishes the proof of b). □

We introduce now another operation which is obtained through integration of Lf. We define

$$\mathcal{L} : \mathcal{D}_{\mathcal{L}} \subset C^1 \to C^0$$

by

$$\mathcal{L}f := \lim_{n\to\infty} \int_0^{\cdot} L_n f_n(y)\, dy,$$

whenever this limit exists in C^0 for every mollifier Φ.

Lemma 2.13 *We have*

(i) $\mathcal{D}_L \cup C^2 \subseteq \mathcal{D}_{\mathcal{L}}$,

(ii) $\mathcal{L}f(x) = \int_0^x (Lf)(y)\, dy$ *for every* $f \in \mathcal{D}_L$,

(iii) $\mathcal{L}f(x) = \int_0^x \left(\frac{\sigma^2}{2} - b\right) f''(y)\, dy + (bf')(x)$ *for every* $f \in C^2$.

Proof. The statement is clear for $f \in \mathcal{D}_L$.

Let $f \in C^2$ and (f_n) be a sequence in C^2 converging to f in C^2. Integrating by parts, we observe

$$\begin{aligned} & \int_0^{\cdot} L_n f_n(y)\, dy \\ = & \int_0^{\cdot} \left(\frac{\sigma_n^2}{2} f_n''\right)(y)\, dy + \int_0^{\cdot} (b_n' f_n')(y)\, dy \\ = & \int_0^{\cdot} \left(\frac{\sigma_n^2}{2} - b_n\right) f_n''(y)\, dy + b_n f_n' \end{aligned}$$

This converges to $\int_0^{\cdot} \left(\frac{\sigma^2}{2} - b\right) f''(y)\, dy + bf'$, when n goes to ∞. □

The next question concerns the closability of $\mathcal{L}$ into C^1 with values in C^0. This does not seem to be true in general. However, we are able to prove some closability of the operator with values in the space of locally bounded variation functions, as it will follow. This will be useful in Section 5 for studying semimartingale characterizations.

We denote by BV the space of continuous functions which have locally bounded variation. We equip BV with the metrizable topology that is associated with the following convergence. A sequence v_n in BV converges to v if and only if $v_n(0) \to v(0)$ and $dv_n \to dv$ holds with respect to the weak $*$-topology.

Remark 2.14

a) The sequence (dv_n) converges to dv if and only if, for every $\alpha \in C^0(\mathbb{R})$,

$$\int_0^t \alpha\, dv_n \to_{n\to\infty} \int_0^t \alpha\, dv$$

holds at every point of continuity t of v.

b) If (v_n) is a sequence converging in BV then the Banach-Steinhaus theorem implies that the total variations are uniformly bounded on every compact set K, i.e.,

$$\sup_n \int_K d|v_n| < \infty. \tag{2.15}$$

c) We have $v_n \to v$ in BV if and only if $v_n(x) \to v(x)$ at every point of continuity x of v and (2.15) holds.

d) Let (v_n) be a sequence in BV such that (2.15) holds. Let v_n^+, v_n^- be increasing functions such that

$$v_n = v_n^+ - v_n^- \quad \text{and} \quad |v_n| = v_n^+ + v_n^-.$$

Then there is a subsequence (n_k) such that $(v_{n_k}^+)$ and $(v_{n_k}^-)$ converge in BV. In particular, the subsequence $(|v_{n_k}|)$ of the total variations converges in BV.

In fact, (2.15) implies

$$\sup_n \int_K dv_n^\pm < \infty \tag{2.16}$$

for each compact interval K. By the Helly extraction argument, there is a subsequence (n_k) such that $(v_{n_k}^+)$ and $(v_{n_k}^-)$ converge respectively to some v^1, v^2 at each continuity point.

e) C^1 is dense in BV.

f) We have $BV \subset L^2_{loc}$, because a locally bounded variation function is locally bounded.

Moreover, using point d), it is not difficult to prove that the BV convergence implies the one in L^2_{loc}.

In fact, let v_n be a sequence of BV functions. We then have $v_n = v_n^+ - v_n^-$, where v_n^+, v_n^-, v^+, v^- are increasing functions vanishing at zero. We suppose that v_n converge to $v = v^+ - v^-$ in BV. Since L^2_{loc} is a metric space, it is enough to show that some subsequence (v_{n_k}) converges to v in L^2_{loc}. From point d), we learn the existence of a subsequence (n_k) such that $v_{n_k}^\pm(x) \to v^\pm(x)$, for each continuity point x therefore a.e. with respect to Lebesgue measure. (2.16) implies that $(v_{n_k}^\pm)$ are uniformly bounded on each compact interval K. Now, the theorem of dominated convergence yields $v_{n_k}^\pm \to v^\pm$ in L^2_{loc}.

We denote by BV^1 the set of all absolutely continuous functions whose derivative f' satisfies $f'/h' \in BV$. In particular, $BV^1 \subset W^{1,2}_{loc}$ holds. BV^1 becomes a Polish space of F-type when equipped with the following metrizable topology. A sequence (f_n) is defined to converge in BV^1 if $f_n \to f$ in C^0 and $(f'_n/h') \to (f'/h')$ in BV.

On $\mathcal{D}_L$, the operator $\mathcal{L}$ takes values in BV. We denote this restriction by $\mathcal{L}^{BV}$.

Lemma 2.15 *(i) The convergence in BV^1 implies the one in L^2_{loc}.*

(ii) $\mathcal{D}_L \subset BV^1$

(iii) $\mathcal{D}_L$ is dense in BV^1.

(iv) The mapping $f \mapsto \mathcal{L}^{BV} f$ admits a continuous extension from $\mathcal{D}_L$ to BV. It will be denoted by $\hat{\mathcal{L}}^{BV}$.

Proof.

(i) It is a consequence of the embedding $BV \subset L^2_{loc}$ given by Remark 2.14 f).

(ii) If $f \in \mathcal{D}_L$, we set $\dot{l} = Lf$. Lemma 2.7 implies that

$$\frac{f'}{h'}(x) = 2\int_0^x \frac{\dot{l}}{\sigma^2 h'}(y)dy + \frac{f'}{h'}(0).$$

This shows $f \in BV^1$.

(iii) Let $f \in BV^1$ and (ϕ_n) a sequence in C^2 such that $\phi'_n \to \frac{f'}{h'}$ in BV when n goes to ∞. Clearly we can define $f_n \in C^1$ such that $f'_n = \phi'_n h'$. Obviously $f_n \in \mathcal{D}_L$ and $Lf_n = \frac{\phi''_n \sigma^2 h'}{2}$.

(iv) Let (f_n) be a sequence in $\mathcal{D}_L$ converging to zero in BV^1. We have to show that $l_n := \mathcal{L} f_n$ converges to zero in BV.

By assumption, we have $(f'_n/h') \to 0$ in BV. Again Lemma 2.7 says that

$$f'_n(x) = h'(x)\left(2\int_0^x \frac{\dot{l}_n}{\sigma^2 h'^2}(y)\,dy + \frac{f'_n}{h'}(0)\right),$$

where $h'(x) = \exp(-\Sigma(x))$ and $\dot{l}_n = Lf_n$. This implies that

$$\frac{\dot{l}_n}{\sigma^2 h'^2}(y)\,dy$$

converges in the weak-$*$-topology to zero. Since $\sigma^2 h'^2$ is a continuous function, l_n converges to zero in BV. □

Proposition 2.16 *(i) The operator $\mathcal{L}^{BV}$ is closable in $W^{1,2}_{loc}$ with values in BV.*

(ii) The domain of the smallest closure is BV^1.

Proof. The proposition will be a consequence of the following lemma.

Lemma 2.17 *Let (f_n) be a sequence in $\mathcal{D}_L$ such that $l_n = \mathcal{L}f_n$ converge to some l in BV and $f_n \to f$ holds in $W^{1,2}_{loc}$. Then we have $f \in BV^1$ and $\hat{\mathcal{L}}^{BV} f = l$.*

Proof. Of course, we have $l_n \in C^1$, $Lf_n = \dot{l}_n$, $\dot{l}_n$ being the derivative of l_n. Independently of the convergence of (f_n), we have

$$\frac{2\dot{l}_n}{\sigma^2 h'^2}\,dy \to \frac{2dl}{\sigma^2 h'^2} \tag{2.17}$$

in the weak-$*$-topology because $1/(\sigma^2 h'^2)$ is a continuous function. The convergence of (f'_n) in L^2_{loc} and (2.17) force the convergence of the real sequence $(f_n/h')(0)$. Consequently, (f'_n/h') converges in BV and so, (f_n) converges in BV^1. Since the convergences in BV^1 and $W^{1,2}_{loc}$ must agree, we have $f = \lim_{n\to\infty} f_n$ in BV^1 so that $\hat{\mathcal{L}}^{BV} f = l$ holds by Lemma 2.15. □

So far, we have learnt how to eliminate the first order term in a PDE operator through a transformation which is called of Zvonkin type (see [54]). Now we would like to introduce a transformation which puts the PDE operator in a divergence form.

Remark 2.18 Let L be a PDE operator which is formally of type (2.1)

$$Lg = \frac{\sigma^2}{2} g'' + b'g'.$$

Of course always at a formal level, it can be written such that the second order part appears in a divergence form. This reads

$$L^d g = \left(\frac{\sigma^2}{2} g'\right)' + (b^d)' g', \tag{2.18}$$

where

$$b^d = b - \frac{\sigma^2}{2}. \tag{2.19}$$

Clearly, we can introduce the concept of a C^1-generalized solution for $L^d f = \dot{l}$ in a rigorous way. It is also clear that f is a C^1-generalized solution to $Lf = \dot{l}$ if and only if $L^d f = \dot{l}$.

Obviously, $\Sigma(x) = \lim_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma_n^2}(y)\,dy$ exists in C^0 if and only if $\Sigma^d(x) = \lim_{n\to\infty} 2\int_0^x \frac{(b_n^d)'}{\sigma_n^2}(y)\,dy$ exists. In that case we have

$$\Sigma^d = \Sigma(x) + \log(\sigma^{-2}(x)). \tag{2.20}$$

Thus, we actually may identify L and L^d and use the same notation L.

We consider a C^1-function $k : \mathbb{R} \to \mathbb{R}$ such that $\tilde{L}k = 0$, $k'(x) \neq 0$ for every $x \in \mathbb{R}$ and $\tilde{L}g = (\sigma^2/2)g'' - b^d g'$ in the C^1-generalized sense. Such a function exists since Σ^d exists. Clearly, we have $\tilde{\Sigma}(x) = -\Sigma(x) + \log\sigma^2(x)$. We can choose k such that

$$k(0) = 0 \quad \text{and} \quad k'(x) = \exp(-\tilde{\Sigma}(x)) = \sigma^{-2}(x)\exp(\Sigma(x)).$$

Remark 2.19 If there is no drift term then we have $k'(x) = \sigma^{-2}(x)$.

Lemma 2.20 *Under the usual assumptions we choose $k \in C^1$ such that $k'(x) = \sigma^{-2}(x)\exp(\Sigma(x))$. We consider the formal PDE operator given by*

$$L^1 g = (\frac{\sigma_1}{2} g')', \quad \sigma_1 = (\sigma k') \circ k^{-1}. \tag{2.21}$$

Let J be the image of k. Then

(i) $g \in \mathcal{D}_{L^1(J)}$ if and only if $g \circ k \in \mathcal{D}_L$,

(ii) for every $g \in \mathcal{D}_{L^1(J)}$ we have $L^1 g = L(g \circ k) \circ k^{-1}$.

Proof. Let σ_n^2, b_n be the usual regularizations of σ^2, b^2. We set

$$\begin{aligned} L_n f &:= \frac{\sigma_n^2}{2} f'' + b_n' f', \\ \tilde{L}_n f &:= \frac{\sigma_n^2}{2} f'' - (b_n^d)' f' \end{aligned}$$

for each $n \in \mathbb{N}$.

Let (k_n) be a sequence in C^1 such that $\tilde{L}_n k_n \to 0$ in C^0 and $k_n \to k$ in C^1. Let $g \in C^1(J)$ such that $g \circ k \in \mathcal{D}_L$. Let (g_n) be a sequence of functions in $C^1(J)$ converging to g and ensuring that the sequence $(\dot{\lambda}_n) \subset C^0(J)$, defined by

$$\dot{\lambda}_n \circ k_n^{-1} = \tilde{L}_n(g_n \circ k_n),$$

converges in $C^0(J)$ to some $\dot{\lambda}$.

We now calculate

$$\begin{aligned} \dot{\lambda}_n \circ k_n^{-1} &= \left(\frac{\sigma_n^2}{2} (g_n \circ k_n)' \right)' + (b_n^d)'(g_n \circ k)' \\ &= \left(\frac{\sigma_n^2}{2} g_n'(k_n) \frac{k_n'^2}{k_n'} \right)' + (b_n^d)'(g_n \circ k)' \\ &= \left(\frac{\sigma_{1,n}^2}{2} (k_n) \frac{g_n'(k_n)}{k_n'} \right)' + (b_n^d)' g_n'(k_n) k_n', \end{aligned}$$

where $\sigma_{1,n} = (\sigma_n k_n')(k_n^{-1})$. We continue to compute

$$\begin{aligned} \dot{\lambda}_n \circ k_n^{-1} &= \left(\frac{\sigma_{1,n}^2}{2} g_n' \right)' (k_n) - \frac{\sigma_{1,n}^2}{2}(k_n) \frac{k_n''}{k_n'^2} g_n'(k_n) + (b_n^d)' g_n'(k_n) k_n' \\ &= \left(\frac{\sigma_{1,n}^2}{2} g_n' \right)' (k_n) - g_n'(k_n) \left(\frac{\sigma_n^2}{2} k_n'' - (b_n^d)' k_n' \right) \\ &= \left(\frac{\sigma_{1,n}^2}{2} g_n' \right)' (k_n) - g_n'(k_n) \tilde{L}_n k_n. \end{aligned}$$

We have shown that

$$\dot{\lambda} \circ k^{-1} = \lim_{n\to\infty} \dot{\lambda}_n \circ k_n^{-1} = \lim_{n\to\infty} \left(\frac{\sigma_{1,n}^2}{2} g_n' \right)' (k_n)$$

in C^0, because $\tilde{L}k = 0$ holds in the generalized C^1-sense. Consequently, in $C^0(J)$ we have

$$\dot{\lambda} = \lim_{n\to\infty} \dot{\lambda}_n = \lim_{n\to\infty} \left(\frac{\sigma_{1,n}^2}{2} g_n' \right)'.$$

Setting $\dot{\mu}_n := ((\sigma_{1,n}^2/2)g_n')'$ and integrating, we get

$$g_n'(y) = \frac{2}{\sigma_{1,n}^2(y)} \left(\int_0^y \dot{\mu}_n(z)\, dz + g_n'(0)\sigma_{1,n}^2(0) \right).$$

Since $g_n \to g$ in $C^1(J)$, $\sigma_{1,n}^2 \to \sigma_1{}^2$ and $\dot{\mu}_n \to \dot{\lambda}$, we obtain

$$g'(y) = \frac{2}{\sigma_1{}^2(y)} \left(\int_0^y \lambda(z)\, dz + g'(0)\sigma_1{}^2(0) \right).$$

Using Lemma 2.7 and the uniqueness of the C^1-generalized solution (Remark 2.5), we conclude

$$L^1 g = \lambda$$

and so $g \in \mathcal{D}_{L^1}(J)$. On the other hand, we have also proven

$$L(g \circ k) = \dot{\lambda} \circ k. \tag{2.22}$$

This establishes the converse implication of i). The direct one follows by symmetric analogous arguments.

Statement (ii) follows from (2.22). □

We make still some comments on the operator L in situations related to divergence form.

In general, we do not even know if $L : \mathcal{D}_L \subset C^1 \to C^0$ is closable. We consider

$$\mathcal{L} : \mathcal{D}_L \subset C^1 \to C_0^0 := \{f \in C^0(\mathbb{R}) : f(0) = 0\},$$

defined by

$$\mathcal{L} : f \mapsto \int_0^{\cdot} Lf(y)\, dy.$$

A priori, $\mathcal{L}$ is not closable in this context. Under some particular assumptions we know more.

Proposition 2.21 *Suppose that we are given*

$$Lf = (\frac{\sigma^2}{2} f')' + \beta' f', \tag{2.23}$$

where β is a continuous function of bounded variation.

(i) $\mathcal{L}$ *admits a continuous extension from* $\mathcal{D}_L$ *to* C^1*, denoted by* $\hat{\mathcal{L}}$*.*

(ii) Let $T : C_0^1 \to C_{0,0}^1 := \{f \in C^1 : f(0) = f'(0) = 0\}$ *be defined by* $Tl = f$*, where* $f \in C_{0,0}^1$ *is the unique solution to* $Lf = l'$*. Then* T *admits a continuous extension to* C_0^0 *which we denote by* $\hat{T}$*.*

(iii) The restriction of $\hat{\mathcal{L}}$ *to* $C_{0,0}^1$ *is invertible on* C_0^0 *and* $\hat{\mathcal{L}}^{-1} = \hat{T}$*.*

(iv) The operator $\mathcal{L} : \mathcal{D}_L \subset W_{loc}^{1,2} \to L_{loc}^2$ *also admits a continuous extension* $\tilde{\mathcal{L}}$ *to the whole space* $W_{loc}^{1,2}$*.*

(v) The restriction of $\tilde{\mathcal{L}}$ *to*

$$W_{0,loc}^{1,2} := \{f \in W_{loc}^{1,2} : f(0) = 0\}$$

is also invertible; $\tilde{T} = \tilde{\mathcal{L}}^{-1}$ *extends* $\hat{T}$*.*

Remark 2.22

a) If L satisfies assumption (2.23) then we say that it *is close to the divergence type.*

b) $\hat{\mathcal{L}}$ coincides with the expression of $\mathcal{L}$ in C^2, see Lemma 2.13 (iii).

c) To avoid overcharge of notations, in the sequel we will denote the extension of $\mathcal{L}$ to $W_{loc}^{1,2}$ also by $\hat{\mathcal{L}}$.

Proof of Proposition 2.21. i) We first evaluate $\mathcal{L}f$ for $f \in \mathcal{D}_L$. In that case, we consider a sequence (f_n) of C^2-functions converging to f in C^1 such that, with the usual notations, $L_n f_n = ((\sigma_n^2/2) f_n')' + \beta_n' f_n'$ converge to Lf in C^0. Then we have

$$\begin{aligned} \mathcal{L}f(x) &= \lim_{n\to\infty} \int_0^x L_n f_n(y)\, dy \\ &= \lim_{n\to\infty} \left(\frac{\sigma_n^2}{2} f_n'(x) + \int_0^x f_n'\, d\beta_n \right) \\ &= \frac{\sigma^2}{2} f'(x) + \int_0^x f'\, d\beta. \end{aligned}$$

This shows that the linear map $\mathcal{L}$ is continuous on $\mathcal{D}_L$ with respect to the topology of C^1. Therefore, $\mathcal{L}$ can be extended to C^1. Thus, we get

$$\hat{\mathcal{L}}f(x) = \frac{\sigma^2}{2}f'(x) + \int_0^x f'\,d\beta. \tag{2.24}$$

ii) If $l \in C_0^1$ and $f = Tl$, using Lemma 2.7, we can write

$$f'(x) = \exp(-\Sigma(x))2\int_0^x \frac{l'(y)\exp(\Sigma(y))}{\sigma^2(y)}\,dy. \tag{2.25}$$

In particular, we have

$$\begin{aligned}\Sigma(x) &= \lim_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma_n^2}(y)\,dy \\ &= \log\sigma^2(x) - \log\sigma^2(0) + 2\int_0^x \frac{d\beta}{\sigma^2}.\end{aligned}$$

Therefore, we get

$$h(x) = \exp(-\Sigma(x)) = \frac{\sigma^2(0)}{\sigma^2(x)}\exp\left(-2\int_0^x \frac{d\beta}{\sigma^2}\right),$$

which solves in particular $Lh = 0$. Now (2.25) takes the form

$$\begin{aligned} f'(x) &= \frac{1}{\sigma^2(x)}\exp\Big(-2\int_0^x \frac{d\beta}{\sigma^2}\Big)\int_0^x l'(y)\exp\left(2\int_0^y \frac{d\beta}{\sigma^2}\right)dy \quad (2.26)\\ f'(x) &= \frac{1}{\sigma^2(x)}\Big(l(x) - \exp\Big(-2\int_0^x \frac{d\beta}{\sigma^2}\Big)\cdot \\ &\quad \cdot\int_0^x l(y)\exp\left(2\int_0^y \frac{d\beta}{\sigma^2}\right)\frac{1}{\sigma^2(y)}\,d\beta(y)\Big). \quad (2.27)\end{aligned}$$

The right term of (2.27) is continuous with respect to $l \in C_0^0$. This allows to define immediately the extension $\hat{T}l$.

iii) By construction, we know

$$\mathcal{L}Tl = l$$

for every $l \in C_0^1$ and

$$T\mathcal{L}f = f$$

for every $f \in \mathcal{D}_L \cap C_{0,0}^0$. Furthermore, $\hat{\mathcal{L}}\hat{T}$ can be extended from C_0^1 to C_0^0 with values in C_0^0 and $\hat{T}\hat{\mathcal{L}}$ admits a continuous extension from $\mathcal{D}_L \cap C_{0,0}^1$ to

$C^1_{0,0}$ with values in $C^1_{0,0}$. Therefore, we have $\hat{\mathcal{L}}\hat{T} = id$ on C^0_0 and $\hat{T}\hat{\mathcal{L}} = id$ on $C^1_{0,0}$. This establishes (iii).

iv) The expression (2.24) can be extended to $W^{1,2}_{loc}$ because the right member of (2.24) admits a continuous extension to $W^{1,2}_{loc}$.

v) The expression (2.27) can be extended to L^2_{loc}. We emphasize that C^0_0 is dense in L^2_{loc}. So, $C^1_{0,0}$ is dense in $W^{1,2}_{loc}$. Thus, (2.27) defines $\tilde{T} : W^{1,2}_{0,loc} \to L^2_{loc}$. A similar reasoning as in iii) now completes the proof. □

Corollary 2.23 *In particular, if $f(x) = x$ then*

$$\hat{\mathcal{L}}f(x) = \int_0^x f'\,d\beta + \frac{\sigma^2}{2}f'(x) = \frac{\sigma^2(x)}{2} + \beta(x) = b(x).$$

We need now to solve the equation $Lu = u$ in the C^1-generalized sense.

Proposition 2.24 *Let $c \in \mathbb{R}$ and consider the solution to*

$$Lv = 1, \quad v(c) = 0, \quad v'(c) = 0. \tag{2.28}$$

- *There is a unique solution to the equation*

$$Lu = u, \quad u(c) = 1, \quad u'(c) = 0. \tag{2.29}$$

- *u is non-negative and strictly decreasing (resp. increasing) on $]-\infty, c]$ (resp. $[c, +\infty[$).*

- $$1 + v(x) \le u(x) \le \exp(v(x)), \forall x \in \mathbb{R} \tag{2.30}$$

Proof. Without loss of generality, we may suppose $c = 0$. According to Lemma 2.7, we can write

$$v'(x) = 2\exp(-\Sigma(x)) \int_0^x \frac{\exp(\Sigma(y))}{\sigma^2(y)}\,dy.$$

We set $u_0 \equiv 1$ and, for $n \in \mathbb{N}$, we define recursively

$$\begin{aligned} u_n'(x) &= 2\exp(-\Sigma(x)) \int_0^x \frac{\exp(\Sigma(y))}{\sigma^2(y)} u_{n-1}(y)\,dy \\ u_n(0) &= 0 \end{aligned} \tag{2.31}$$

which means $Lu_n = u_{n-1}$, $u_n(0) = u_n'(0) = 0$. The u_n are easily seen to be non-negative, strictly increasing on $\mathbb{R}_+$ and strictly decreasing on $\mathbb{R}_-$. We can show by induction that

$$u_n(x) \leq \frac{v^n(x)}{n!} \tag{2.32}$$

for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Indeed, (2.32) is valid for $n = 0$. Assuming that it is true for $n - 1 \geq 0$ and using (2.31) we get for $x \geq 0$

$$\begin{aligned}
u_n(x) &\leq \int_0^x dy\, \exp(-\Sigma(y))2 \int_0^y dz\, \frac{\exp(\Sigma(z))}{\sigma^2(z)} u_{n-1}(z) \\
&\leq \frac{1}{(n-1)!} \int_0^x dy\, \exp(-\Sigma(y))2v^{n-1}(y) \int_0^y dz\, \frac{\exp(\Sigma(z))}{\sigma^2(z)} \\
&= \frac{1}{(n-1)!} \int_0^x dv(y)\, v^{n-1}(y) \\
&= \frac{v^n(x)}{n!}
\end{aligned}$$

for each $n \in \mathbb{N}$. This implies that

$$\sum_{n=0}^{\infty} u_n'(x)$$

converges absolutely and uniformly on compact real intervals. Another consequence is that so does

$$\sum_{n=0}^{\infty} u_n(x).$$

The function $u(x) := \sum_{n=0}^{\infty} u_n(x)$ clearly belongs to C^1 and we have

$$u'(x) = \sum_{n=0}^{\infty} u_n'(x).$$

Summing up (2.31), we get

$$u'(x) = \exp(-\Sigma(x))2 \int_0^x \frac{\exp(\Sigma(y))}{\sigma^2(y)} u(y)\, dy.$$

Since u is the sum of u_n , it is non-negative and strictly increasing (resp. decreasing) on $\mathbb{R}_+$ (resp. $\mathbb{R}_-$).

Lemma 2.7 now implies that $Lu = u$ holds in the C^1-generalized sense.

Given two solutions u^1 and u^2 of (2.28), it is possible to show $u^1 = u^2$ using similar arguments and Gronwall with Lemma 2.7.

The relation (2.30) obviously follows from

$$1 + v(x) = 1 + u_1(x) \leq \sum_{n=0}^{\infty} u_n(x) = u(x) \leq \sum_{n=0}^{\infty} \frac{v^n(x)}{n!} = \exp(v(x)).$$

□

Similarly to problem 5.27 and 5.28 of [24], we need the following result.

Lemma 2.25 *Let v_c be the solution to $Lv = 1$, $v_c(c) = v_c'(c) = 0$, $v \equiv v_0$.*

(i) If $h(+\infty) = +\infty$ then $v_c(+\infty) = +\infty$ holds for every $c \in \mathbb{R}$.

(ii) If $h(-\infty) = -\infty$ then $v_c(+\infty) = +\infty$ holds for every $c \in \mathbb{R}$.

(iii) $v_c(x) = v_c(a) + v_c'(a) \int_a^x \exp(-\Sigma(y))\, dy + v_a(x)$ holds for every $a, c \in \mathbb{R}$.

(iv) We have $v_c(\pm\infty) < \infty$ if and only if $v_0(\pm\infty) < \infty$.

Proof. i) For $x \geq c + 1$, we have

$$\begin{aligned} v_c(x) &= \int_c^x dy\, h'(y) \int_c^y \frac{2}{h'(z)\sigma^2(z)}\, dz \\ &\geq \int_{c+1}^x dy\, h'(y) \int_c^{c+1} \frac{2}{h'(z)\sigma^2(z)}\, dz \\ &= \int_c^{c+1} \frac{2}{h'(z)\sigma^2(z)}\, dz\, (h(x) - h(c+1)). \end{aligned}$$

If $h(+\infty) = +\infty$ then $v_c(+\infty) = +\infty$.

Statement ii) follows similarly to (i), whereas (iii) is a consequence of the explicit expression

$$v_c(x) = \int_c^x dy\, h'(y) \int_c^y \frac{2}{h'(z)\sigma^2(z)}\, dz.$$

For the proof of (iv), we rewrite (iii) as

$$v_c(x) - v_a(x) = v_c'(a)(h(x) - h(a)).$$

If $v_c(+\infty) < +\infty$ then $h(+\infty) < +\infty$ holds by i), thus showing $v_a(+\infty) < +\infty$.

□

3 A suitable martingale problem

In this section, we consider a PDE operator satisfying the same properties as in the previous section, i.e.

$$Lg = \frac{\sigma^2}{2} g'' + b' g', \tag{3.1}$$

where $\sigma > 0$ and b are continuous. In particular, we assume that

$$\Sigma(x) = \lim_{n\to\infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy \tag{3.2}$$

exists in C^0, independently from the chosen mollifier. Then $h(x) := \exp(-\Sigma(x))$ is a solution to $Lh = 0$ with $h' \neq 0$.

Definition 3.1 A process X is said to solve *the martingale problem* related to L with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)\, ds$$

is a local martingale for $f \in \mathcal{D}_L$ and $X_0 = x_0$.

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X_t^{s,x}, t \geq 0)$ solves the martingale problem related to L with initial value x at time s if

(i) $X_s^{s,x} = x$,

(ii) for every $f \in \mathcal{D}_L$,

$$f(X_t^{s,x}) - f(x) - \int_s^t Lf(X_r^{s,x})\, dr, \quad t \geq s$$

is a local martingale.

We remark that $X^{s,x}$ solves the martingale problem at time s if and only if $X_t := X_{t-s}^{s,x}$ solves the martingale problem at time 0.

Remark 3.2

(i) In general, $f(x) = x$ does not belong to $\mathcal{D}_L$. In fact in that case X would be a semimartingale; Corollary 5.8 will show that this rarely occurs.

(ii) We are interested in the operators

$$\mathcal{A} : \mathcal{D}_L \to \mathcal{C}, \text{ given by } \quad \mathcal{A}(f) = \int_0^{\cdot} Lf(X_s)\, ds$$

and

$$A : C^1 \to \mathcal{C}, \text{ given by } \quad A(f) = \int_0^{\cdot} l'(X_s)\, ds.$$

We may ask whether $\mathcal{A}$ and A are closable in C^1 and in C^0, respectively. We will see that $\mathcal{A}$ even admits a continuous extension to C^1. However, A can be extended to C^0 continuously when L is close to divergence type.

(iii) For the moment, we continue to work with the domains C^1 or C^0 because we do not need to examine in detail the fundamental solutions related to L which will be in fact the densities of the laws of the considered processes. Once, we will take into account the information of those densities. Then $\mathcal{A}$ will be extended to $W^{1,2}_{loc}$; if L is close to divergence type, A will be extended to L^2_{loc}.

The first result on solutions to the martingale problem related to L is the following

Proposition 3.3 *Let $I =]a, b[$ be the image of h, $-\infty \leq a < b \leq +\infty$. A process X solves the martingale problem related to L if and only if $Y = h(X)$ is a local martingale with values in I which solves weakly the stochastic differential equation*

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s)\, dW_s, \tag{3.3}$$

where $Y_0 = h(X_0)$ and $\sigma_0(y) = (\sigma h')(h^{-1}(y))$.

Remark 3.4

(i) Y always stays in the interval I.

(ii) Let $T > 0$ and $(Z_t)_{t \geq 0}$ be a process. We denote by $\mathbb{F} = \mathbb{F}_Z$ the natural forward filtration of Z, given by $\mathcal{F}_t = \sigma(Z_s : s \leq t)$, and by $\mathbb{H} = \mathbb{H}_Z$ the backward filtration, given by $\mathcal{H}_t = \sigma(\hat{Z}_s : s \leq t)$. Clearly, we have $\mathcal{F}_Y = \mathcal{F}_X$ and $\mathcal{H}_Y = \mathcal{H}_X$.

(iii) Since Y is a local martingale, we know from Remark 1.3 that $X = h^{-1}(Y)$ is a Dirichlet process with martingale part

$$M_t^X = \int_0^t (h^{-1})'(Y_s)\, dY_s = \int_0^t \sigma(X_s)\, dW_s.$$

In particular, X is a finite quadratic variation process with

$$[X, X] = [M^X, M^X]_t = \int_0^t \sigma^2(X_s)\, ds.$$

Proof of Propositon 3.3. First, let X be a solution to the martingale problem related to L. Since $h \in \mathcal{D}_L$ and $Lh = 0$, we know that $Y = h(X)$ is a local martingale. In order to calculate its bracket we recall that $h^2 \in \mathcal{D}_L$ and $Lh^2 = \sigma^2(h')^2$ hold by Proposition 2.12 a). Thus,

$$h^2(X_t) - \int_0^t (\sigma h')^2(X_s)\, ds$$

is a local martingale. This implies

$$[Y]_t = \int_0^t (\sigma h')^2(h^{-1}(Y_s))\, ds = \int_0^t \sigma_0^2(Y_s)\, ds.$$

Finally, Y solves weakly the SDE (3.3) with respect to the standard $\mathcal{F}_Y$-Brownian motion W given by

$$W_t = \int_0^t \frac{1}{\sigma_0(Y_s)}\, dY_s.$$

Now, let $Y = h(X)$ be a solution to (3.3) and $f \in \mathcal{D}_L$. Proposition 2.12 b) says that $\phi := f \circ h^{-1} \in \mathcal{D}_{L_0} \equiv C^2$, where

$$L_0\phi = \frac{\sigma_0^2}{2}\phi'' = (Lf) \circ h. \tag{3.4}$$

So we can apply Itô formula to evaluate $\phi(Y)$ which coincides with $f(X)$. This gives

$$\phi(Y_t) = \phi(Y_0) + \int_0^t \phi'(Y_s)\, dY_s + \frac{1}{2}\int_0^t \phi''(Y_s)\, d[Y_s].$$

Using $d[Y]_s = \sigma_0^2(Y_s)\, ds$ and taking into account (3.4), we conclude

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \int_0^t Lf(X_s)\, ds. \tag{3.5}$$

This establishes the proposition. □

Corollary 3.5 *The map $\mathcal{A}$ admits a continuous extension from $\mathcal{D}_L$ to C^1 with values in $\mathcal{C}$ which we will denote again by $\mathcal{A}$. Moreover, $\mathcal{A}(f)$ is a zero quadratic variation process for every $f \in C^1$.*

Proof. $\mathcal{A}$ has a continuous extension because of (3.5). $\mathcal{A}(f)$ is a zero quadratic variation process because X is a Dirichlet process with martingale part $\int_0^\cdot \sigma(X_s)\, dW_s$ and because of Remark 1.3. □

Remark 3.6 The extension of (3.5) to C^1 gives

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \mathcal{A}(f). \tag{3.6}$$

Choosing $f = id$ in (3.6), we get

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + \mathcal{A}(id).$$

We would now like to ask if $\mathcal{A}(f)$ corresponds to $A(l)$ for some $l \in C^0$. In that case, X would turn out to solve a stochastic differential equation with diffusion σ and generalized drift l'. Unfortunately, for the moment, we cannot answer the question in such a general framework. However, we will provide an answer if L is close to divergence type. Moreover, even if L is not of that type, we get results on local time.

Proposition 3.7 *If X solves the martingale problem with respect to L then it admits a local time (as a density of an occupation time measure).*

Proof. Let $l \in C^0$. For h defined as before, we have

$$\begin{aligned} \int_0^t l(X_s)\, ds &= \int_0^t l \circ h^{-1}(Y_s)\, ds \\ &= \int_0^t \Phi(Y(s))\, d[Y, Y]_s, \end{aligned}$$

where $\Phi(y) := \frac{l \circ h^{-1}}{(\sigma h')^2 \circ h^{-1}}(y)$. Using the occupation time density formula, we get

$$\int_0^t l(X_s)\, ds = \int L_t^Y(y) \Phi(y)\, dy, \tag{3.7}$$

where L^Y is the local time of Y (in the sense of Tanaka formula). Then, (3.7) becomes

$$\int L_t^Y(h(x))\frac{l(x)}{(\sigma h')^2(x)}h'(x)\,dx = \int \mathbb{L}_t(x)l(x)\,dx,$$

where $\mathbb{L}_t^X(x) = L_t^Y(h(x))/(\sigma^2(x)h'(x))$. □

Now the following question arises. Under which conditions on b is $\mathbb{L}^X$ a good Bouleau-Yor integrator? In other words, under which conditions does $d\mathbb{L}_t^X$ integrate continuous functions? For this, we would need to extend the operator A to the whole space $C^0(\mathbb{R})$.

Remark 3.8 If L is close to divergence type then $A : l \mapsto \int_0^\cdot l'(X_s)\,ds$ admits a continuous extension to C_0^0 and therefore to C^0 because of $A(l) \equiv A(l + const)$.

In fact, if $l \in C_0^1$ then $A(l) = \mathcal{A}(Tl)$, where T is defined in Proposition 2.21(i). Since T admits a continuous extension $\hat{T}$ to C_0^0 with values in $C_{0,0}^1$, the operator A can be extended to C_0^1 by $\mathcal{A}\circ\hat{T}$. We still denote this extension by A. □

An example of a process X solving a martingale problem with respect to L, where L is close to divergence type, is given by a solution of a stochastic differential equation of the following type,

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t \gamma(X_s)ds,$$

where σ is Lipschitz, positive and $\gamma \in L_{loc}^\infty$.

Let X be a stochastic process for which there is a mapping $A : C^0 \to \mathcal{C}$ which extends continuously $l \to \int_0^\cdot l'(X_s)ds$ from C^1. X is said to fulfill the **extended Bouleau-Yor property** if $\int_0^\cdot g(X)d^-A(l)$ exists for every $g \in C^2$ and every $l \in C^0$.

Lemma 3.9 *If X is a solution to a martingale problem related to a PDE operator which is close to divergence type, then it fulfills the extended Bouleau-Yor property.*

Proof. Let $l \in C^0$. There is $f \in C^1$ such that $\hat{\mathcal{L}} f = l$. Since $f(X)$ equals a local martingale plus $A(l)$, it remains to show that

$$\int_0^{\cdot} g(X) d^- f(X) \tag{3.8}$$

exists for any $g \in C^2$. Integrating by parts previous integral, (3.8) equals

$$(gf)(X_\cdot) - (gf)(X_0) - \int_0^{\cdot} f(X) d^- g(X) - [f(X), g(X)].$$

Remark 1.2 tells that the right member is well-defined. □.

Lemma 3.10 *Let X be a process having the extended Bouleau-Yor property. Then, for every $g \in C^2$ and every $l \in C^0$, we have*

$$\int_0^{\cdot} g(X) d^- A(l) = A(\Phi(g, l)) \tag{3.9}$$

where

$$\Phi(g, l)(x) = (gl)(x) - (gl)(0) - \int_0^x (lg')(y)\, dy \tag{3.10}$$

Proof. The Banach-Steinhaus theorem for F-spaces (see [12, ch. 2]) implies that, for every $g \in C^2$

$$l \mapsto \int_0^{\cdot} g(X)\, d^- A(l) \tag{3.11}$$

is continuous from C^0 to $\mathcal{C}$. Note that Φ is a continuous bilinear map from $C^1 \times C^0$ to C^0. Since $A : C^0 \to \mathcal{C}$ is continuous, the mapping $l \to A(\Phi(g, l))$ is also continuous from C^0 to $\mathcal{C}$. In order to conclude the proof, we need to check the identity (3.8) for $l \in C^1$. In that case, by differentiation of l and Φ both members of (3.9) equal

$$\int_0^{\cdot} (gl')(X_s) ds.$$

□

We are now going to investigate the relation between the martingale problem associated with L and stochastic differential equations with distributional drift.

Proposition 3.11 *Suppose that L is close to divergence form. If X solves the martingale problem with respect to L then it is a solution to the stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + A(b), \tag{3.12}$$

where $b = \sigma^2/2 + \beta$.

Proof. If X solves the martingale problem related to L then, by (3.5),

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + A(\mathcal{L}f) \tag{3.13}$$

holds for every $f \in \mathcal{D}_L$. Remark 3.8 and Proposition 2.21 allow us to extend (3.13) to any $f \in C^1$. Then $A(\mathcal{L}f)$ is replaced with $\mathcal{A}(\hat{\mathcal{L}}f)$.

If $f = id$ then $\hat{\mathcal{L}}f = b$ holds in view of Corollary 2.23. □

At this stage, it seems natural to ask whether the converse of Proposition 3.11 is true. In other words, if X solves (3.12), is it a solution to the martingale problem related to L? The answer is not immediate. We still suppose that L is close to divergence type. We know the answer only if X fulfills the extended Bouleau-Yor property. Let $f \in C^2$. By Corollary 3.5 and Proposition 2.21, we know that $A(b)$ has zero quadratic variation. Since X solves (3.12) and $\int_0^\cdot f'(X_s)\, d^- X_s$ always exists by the classical Itô formula (see Remark 1.4 e) of Chapter 1) we know that $\int_0^\cdot f'(X)\, d^- A(b)$ also exists and is equal to $\int_0^\cdot f'(X)\, d^- X - \int_0^\cdot (f'\sigma)(X)\, dW$. Therefore, this Itô formula says that

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\sigma(X_s)\, dW_s + \int_0^t f'(X)\, d^- A(b) \\ &\quad + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)\, ds \end{aligned}$$

holds.

Let L_n be the PDE operator defined in Section 2 by $L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'$. By Lemma 3.10, the linearity of mapping A and Lemma 2.13 we get

$$\textstyle\int_0^t f'(X) d^- A(b) + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s) ds$$

$$= \quad A\left(\Phi(f',b)\right)_t + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s)ds$$
$$= \quad \int_0^t (\sigma^2 - b)(X_s)f''(X_s)ds + A(bf') = A(\mathcal{L}f)$$

This shows

$$f(X_t) = f(X_0 + \int_0^t (f'\sigma)(X_s)dW_s + A(\mathcal{L}f). \tag{3.14}$$

Because of Remark 2.22 c), the previous expression can of course be prolongated to any $f \in C^1$. Taking $f \in \mathcal{D}_L$, it follows that X fulfills a martingale problem with respect to L.

Corollary 3.12 *Let the PDE operator L be close to divergence type. Then X solves the martingale problem related to L if and only if it solves the stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + A(b) \tag{3.15}$$

and X has the extended Bouleau-Yor property.

Proof. The statement follows from Lemma 3.9, Proposition 3.11, and from the considerations above. □

Now we are going to examine existence and uniqueness (and non-explosion).

Proposition 3.13 *Let v be the unique solution to $Lv = 1$, $v(0) = v'(0) = 0$. Then for any horizon $T > 0$, there exists a unique solution to the martingale problem related to L if and only if*

$$v(-\infty) = v(+\infty) = +\infty. \tag{3.16}$$

Remark 3.14 The previous result is a generalization of the Feller test for explosion stated for instance in [24, Theorem 5.29].

Proof of Proposition 3.13. Let X be a solution to the martingale problem related to L. Then Proposition 3.3 says that $Y := h(X)$ solves the stochastic differential equation

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s)\, dW_s \tag{3.17}$$

for $y_0 = h(x_0) \in I = Im\, h$.

At this level, we can apply the results of [14] stated also in [24, Theorem 5.7] (Engelbert-Schmidt theorem). According to their notations, we have $Z(\sigma_0) = I^c$, which means that the set of zeros of σ_0 is I^c. On the other hand, the set

$$I(\sigma_0) = \{x \in \mathbb{R} : \int_{(-\varepsilon,\varepsilon)} \frac{dy}{\sigma_0^2(x+y)} = +\infty\}$$

is equal to I^c. In fact, since σ is strictly positive and continuous and I is open, we have $I \subset I(\sigma_0)^c$. If $x \in I^c$ then σ_0 is zero in some neighbourhood of x and so x belongs to $I(\sigma_0)$. Thus, we have $I(\sigma_0) = Z(\sigma_0) = I^c$ so that the Engelbert-Schmidt theorem ensures that (3.17) has a unique solution.

Let Y be the solution to (3.17). We remark that this solution cannot explode, see [24, Problem 5.2]. So, if $I = Im\, h = \mathbb{R}$, Proposition 3.3 will yield existence for the martingale problem related to L. However, Y could reach I^c or equivalently ∂I. The following lemma now completes the proof. □

Lemma 3.15 *For $y_0 \in I$, the solution Y to (3.17) remains in I a.s. if and only if (3.16) holds.*

Proof. We recall that Y remains in I if and only if $X = h^{-1}(Y)$ is always finite, where h is extended to $\bar{\mathbb{R}}$ with values in $\bar{I}$.

For $m, n \in \mathbb{N}$, we define

$$X_t^{m,n} = X_{t\wedge\tau_m\wedge\phi_n} \quad \text{and} \quad Y_t^{m,n} = Y_{t\wedge\tau_m\wedge\phi_n},$$

where

$$\begin{aligned} \tau_m &:= \inf\{t \geq 0 : X_t \leq -m\}, \\ \phi_n &:= \inf\{t \geq 0 : X_t \geq n\} \end{aligned}$$

Let $u \in \mathcal{D}_L$; we know $\tilde{u} = u \circ h^{-1} \in \mathcal{D}_{L_0} \equiv C^2$ in view of Proposition 2.12. Then, by the classical Itô formula, we calculate

$$Z_t := \tilde{u}(Y_t) = \tilde{u}(y_0) + \int_0^t \sigma_0 \tilde{u}'(Y_s)\, dW_s + \int_0^t L^0 \tilde{u}(Y_s)\, ds.$$

Setting $Z_t^{m,n} := Z_{t\wedge\tau_m\wedge\phi_n}$, we get

$$Z_t^{m,n} = \tilde{u}(y_0) + \int_0^{t\wedge\tau_m\wedge\phi_n} \sigma_0\tilde{u}'(Y_s)\,dW_s + \int_0^{t\wedge\tau_m\wedge\phi_n} L^0\tilde{u}(Y_s)\,ds.$$

Using Proposition 2.12, for $Z_t^{m,n} = u(X_{t\wedge\tau_m\wedge\phi_n})$, we obtain

$$Z_t^{m,n} = u(x_0) + \int_0^{t\wedge\tau_m\wedge\phi_n} \sigma u'(X_s)\,dW_s + \int_0^{t\wedge\tau_m\wedge\phi_n} Lu(X_s)\,ds.$$

Let us now suppose $Lu = u$ according to Proposition 2.24. Integrating $M_t^{m,n} := \exp(-t\wedge\tau_m\wedge\phi_n)Z_t^{m,n}$ by parts yields

$$\begin{aligned} M_t^{m,n} &= M_0^{m,n} + \int_0^{t\wedge\tau_m\wedge\phi_n} \exp(-s)u'(X_s)\sigma(X_s)\,dW_s \\ &= M_0^{m,n} + \int_0^{t} \exp(-s)u'(X_s^{m,n})\sigma(X_s^{m,n})\,dW_s. \end{aligned}$$

Therefore $M^{m,n}$ is a local martingale which, by definition, is non-negative. Hence, $M^{m,n}$ is a supermartingale.

We consider the stopping times $\phi := \lim_{n\to\infty}\phi_n$ and $\tau := \lim_{m\to\infty}\tau_m$. We observe that the processes

$$\begin{aligned} M_t^m &:= \lim_{n\to\infty} M_t^{m,n} = \exp(-t\wedge\phi\wedge\tau_m)u(X_{t\wedge\phi\wedge\tau_m}), \\ M_t &:= \lim_{n\to\infty} M_t^{n,n} = \exp(-t\wedge\phi\wedge\tau)u(X_{t\wedge\phi\wedge\tau}) \end{aligned}$$

are also supermartingales. Therefore, for every $m \geq 0$,

$$M_\infty^m = \lim_{t\to\infty} M_t^m \text{ a.s.,} \tag{3.18}$$

$$M_\infty = \lim_{t\to\infty} M_t \text{ a.s.} \tag{3.19}$$

exist and are finite.

After these preliminaries, we suppose first that (3.16) holds. Then (2.30) in Proposition 2.24 implies $u(\pm\infty) = +\infty$. By (3.19), $M_\infty = +\infty$ holds on $\{\phi\wedge\tau = +\infty\}$. This entails $\mathbb{P}(\{\tau\wedge\phi < +\infty\}) = 0$. Hence, Y remains in I a.s.

Conversely, let us suppose that X does not explode and (3.16) fails, for instance suppose $v(+\infty) < +\infty$. Let $c \in \mathbb{Z}$ such that $x_0 > c$. By Lemma 2.25 (iv), $v_c(+\infty)$ is finite. This implies that the unique solution u to $Lu = u$,

$u(c) = 1$, $u'(c) = 0$ fulfills $u(+\infty) < \infty$, see (2.30) in Proposition 2.24. The continuous process

$$M_t^c = \exp(-t \wedge \tau_c \wedge \phi) u(X_{t\wedge\tau_c\wedge\phi}), \quad t \geq 0,$$

is a bounded supermartingale. But it is also a local martingale and hence, a martingale in L^1. The convergence (3.18) holds also in L^1. Consequently, we have

$$u(x_0) = \mathbb{E}(\exp(-\tau_c \wedge \phi) u(X_{\tau_c\wedge\phi})).$$

Since $\phi = +\infty$ a.s., X being always finite, we have

$$\begin{aligned} u(x_0) &= \mathbb{E}(\exp(-\tau_c) u(X_{\tau_c})) \\ &= \mathbb{E}(\mathbf{1}_{\{\tau_c<+\infty\}} \exp(-\tau_c) u(X_{\tau_c})) \leq u(c). \end{aligned}$$

This contradicts the fact that u is strictly increasing on $[c, x]$. Therefore, $v(+\infty) = +\infty$ holds. A similar reasoning works for $v(-\infty) = -\infty$. □

We would like to finish this section with two considerations. The first one asks in which sense L can be looked upon as the *extended infinitesimal generator* of a process X solving the martingale problem related to L. The second one concerns the *Kolmogorov equation* associated with the law of X.

a)**The extended infinitesimal generator.** We recall the notation C_c^k standing for the set of C^k-functions with compact support.

Lemma 3.16 *For every $f \in \mathcal{D}_L$ satisfying $Lf \in C_c^0$ there is a sequence (f_n) in $\mathcal{D}_L \cap C_c^1$ with $\lim_{n\to\infty} f_n = f$ in the sense of the graph norm.*

Proof. Let $f \in \mathcal{D}_L$. Then $f \circ h^{-1} \in C^2$ holds by Proposition 2.12. Since $C_0^2(I)$ is dense in $C^2(I)$, where $I = Im\, h$, we find a sequence $(\tilde{f}_n)$ in $C_0^2(I)$ such that $\tilde{f}_n \to f \circ h^{-1}$ in C^2. Clearly, $f_n = \tilde{f}_n \circ h$ is a sequence in $\mathcal{D}_L \cap C_c^1$ which tends to f in C^1. Moreover, $Lf_n = \frac{\sigma_0^2 \tilde{f}_n''}{2} \circ h$ are continuous functions with compact support and converge to $(L_0 f)(f) = Lf$. □

Let $(X_t^x, t \geq 0, x \in \mathbb{R})$ be a random field which is measurable in (t, x, ω) such that $X_0^x = x$. We say that L is its *infinitesimal generator* if

$$Lf(x) = \lim_{t\to 0} \frac{1}{t} \mathbb{E}(f(X_t^x) - f(x)) \tag{3.20}$$

holds for every $f \in \mathcal{D}_L \cap C_c^0$.

Remark 3.17 If $(X_t^x, t \geq 0)$ solves the martingale problem related to L with initial condition x then we know that $Y_t^x := h(X_t^x)$ solves the stochastic differential equation

$$Y_t^x = h(x) + \int_0^t (\sigma h') \circ h^{-1}(Y_s^x)\, dW_s. \tag{3.21}$$

By the classical theory of stochastic differential eqautions there is a version which is measurable with respect to (t, x, ω); so the same holds for X.

Remark 3.18 X^x solves the martingale problem related to L with initial condition x if and only if

$$f(X_t) - f(x) - \int_0^t (Lf)(X_s)\, ds \tag{3.22}$$

is a martingale for every $f \in C_c^1 \cap \mathcal{D}_L$.

This follows from the fact that, by Lemma 3.16, $\mathcal{D}_L \cap C_c^1$ is dense in $\mathcal{D}_L$ and from (3.5) in the proof of Proposition 3.3 which says that (3.22) equals $\int_0^t (f'\sigma)(X_s)\, dW_s$ and so it is truely a martingale if $f \in C_c^1$.

Proposition 3.19 *Let $(X_t^x, t \geq 0)$ be a random field as above such that X^x solves the martingale problem related to L for every initial condition x. Then L is its infinitesimal generator.*

Proof. Let $f \in C_c^1 \cap \mathcal{D}_L$. Taking the expectation in the martingale (3.22), we get

$$\frac{1}{t}\mathbb{E}(f(X_t) - f(x)) = \frac{1}{t}\int_0^t ds\, \mathbb{E}((Lf)(X_s)), \quad t > 0, \tag{3.23}$$

where $X = X^x$. We denote the law of X_t^x by ν_t. Now (3.23) can be rewritten as

$$\frac{1}{t}\int_0^t ds\, \nu_s(Lf). \tag{3.24}$$

But $t \mapsto \nu_t(g)$ is a continuous function for every $g \in C^0$ so that (3.24) equals $\nu_{\tilde{s}}(Lf)$, $\tilde{s} \in [0, t]$. Therefore, the previous term converges to $Lf(x)$. □

b) **The Kolmogorov equation**

Now we want to discuss the Kolmogorov equation corresponding to a random field $(X_t^{s,x}, t \geq s \geq 0, x \in \mathbb{R})$ such that $X^{s,x}$ solves the martingale problem related to L with initial condition x at time s. Again, L_n will be the same regularizing PDE operators as in Section 2.

We define the set $\mathcal{U}_L$ of bounded functions $u = (u(t,x), t \geq 0, x \in \mathbb{R})$ in $C^0([0,T] \times \mathbb{R})$ such that there are bounded functions v^a, $v^b \in C^0([0,T] \times \mathbb{R})$ and a sequence $u_n = (u_n(t,x), t \geq 0, x \in \mathbb{R})$ in $C^{1,2}([0,t] \times \mathbb{R})$ satisfying

(i) $u_n \to u$,

(ii) $\frac{\partial u_n}{\partial t} \to v^a$,

(iii) $L_n u_n \to v^b$

pointwise. In this case we say that $(\partial_t - L)(u) = v^a - v^b$ holds in the C^0-generalized sense.

Remark 3.20 If $u \in C^1(\mathbb{R}_+ \times \mathbb{R})$ with $u(t,\cdot) \in \mathcal{D}_L$ for every $t \geq 0$ then $u \in \mathcal{U}_L$ holds. □

It is also possible to consider the case of Dirichlet boundary conditions.

Given a bounded interval D, we define similarly to the above definition, the set $\mathcal{U}_L(D)$ of functions $u = (u(t,x), t \geq 0, x \in D)$ such that there are bounded functions v^a, $v^b \in C^0([0,T] \times \bar{D})$ and a sequence $u_n = (u_n(t,x), t \in [0,T], x \in \bar{D})$ in $C^{1,2}([0,T] \times \bar{D})$ with zero Dirichlet boundary conditions fulfilling points (i), (ii), (iii) above. In this case we say that $(\partial_t - L)(u) = v^a - v^b$ holds (in the C^0-generalized sense) with zero Dirichlet boundary conditions.

Theorem 3.21 *Let g be a bounded continuous real function and $X^{s,z}$ solve the martingale problem related to L with initial condition z at time s.*

(i) Let $u = (u(t,z), t \in [0,T], z \in \mathbb{R})$ in $\mathcal{U}_L$. Then we have

$$u(s,z) = \mathbb{E}(g(X_T^{s,z})) + \int_s^T \mathbb{E}\Big((\partial_t - L)(u(s, X_r^{s,z}))\Big)\, dr, \qquad (3.25)$$

(ii) Suppose that g vanishes at the boundary of D. Let $u = (u(t,z), t \in [0,T], z \in \mathbb{R})$ be in $\mathcal{U}_L(D)$ such that $(\partial_t - L)(u) = 0$ holds with zero

Dirichlet boundary conditions. Then we have

$$u(s,z) = \mathbb{E}\left(g(X_T^{s,z}) 1_{\{X_T^{s,z} \in D, \forall t \in [0,T]\}} \right).$$

Proof. In the case of (ii), we can prolongate u with zero outside D to get a function in $\mathcal{U}_L$.

We set $\tilde{u}_n(t,y) := u_n(t, h_n^{-1}(y))$, where $(h_n) \subset C^2$ satisfies $L_n h_n \to 0$ in C^0, $h_n \to h$ in C^1, $h_n(0) = 0$, $h_n'(0) = 1$. We can apply the classical Itô formula to $\tilde{u}_n(t, Y_t)$, where $Y = h(X)$ and $X = X^{s,z}$. We recall that

$$Y_t = h(z) + \int_s^t \sigma_0(Y_s)\, dW_s \tag{3.26}$$

holds, where $\sigma_0 = (\sigma h') \circ h^{-1}$. Therefore, we have

$$\begin{aligned} \tilde{u}_n(t,Y_t) &= \tilde{u}_n(s,h(z)) + \int_s^t \frac{\partial \tilde{u}_n}{\partial r}(r,Y_r)\, dr + \int_s^t \frac{\partial \tilde{u}_n}{\partial x}(r,Y_r)\sigma_0(Y_r)\, dW_r \\ &\quad + \frac{1}{2}\int_s^t \frac{\partial^2 \tilde{u}_n}{\partial x^2}(r, h_n(X_r))\sigma_0^2(Y_r)\, dr, \end{aligned}$$

where L^0 only acts on y. Coming back to X and setting $i_n := h_n \circ h^{-1}$, we calculate

$$\begin{aligned} u_n(t, i_n(X_t)) &= u_n(s, i_n(X_s)) + \int_s^t \frac{\partial u_n}{\partial r}(r, i_n(X_r))\, dr \\ &\quad + \int_s^t \frac{\partial \tilde{u}_n}{\partial x}(r, h_n(X_r))\sigma_0(h_n(X_r))\, dW_r \\ &\quad + \frac{1}{2}\int_s^t L_n u_n(r, i_n(X_r)) \frac{\sigma_0^2}{\sigma_{0,n}^2}(i_n(X_r))\, dr, \end{aligned} \tag{3.27}$$

where $\sigma_{0,n} = (\sigma_n h_n') \circ h_n^{-1}$. The last integral could be transformed using Proposition 2.12.

Given a bounded interval Δ containing z, we define the stopping time

$$\tau := \inf\{t \in [s,T] : X_t \notin \Delta\} \wedge (T+1).$$

Stopping the process X at time τ, we obtain

$$\begin{aligned} u_n(t\wedge\tau, i_n(X_{t\wedge\tau})) &= u_n(s, i_n(X_s^\tau)) + \int_s^{t\wedge\tau} \frac{\partial u_n}{\partial r}(r, i_n(X_r))\mathbf{1}_{\{X_r \in \Delta\}}\, dr \\ &\quad + \int_s^{t\wedge\tau} \frac{\partial \tilde{u}_n}{\partial x}(r, h_n(X_r))\sigma_0(h_n(X_r))\mathbf{1}_{\{X_r \in \Delta\}}\, dW_r \\ &\quad + \frac{1}{2}\int_{s\wedge\tau}^{t\wedge\tau} L_n u_n(r, i_n(X_r)) \frac{\sigma_0^2}{\sigma_{0,n}^2}\mathbf{1}_{\{X_r\in\Delta\}}\, d_r. \end{aligned}$$

Since the stochastic integrand with respect to the Brownian motion is bounded, its expectation is zero. Therefore, we get

$$\begin{aligned}&\mathbb{E}\Big(u_n(\tau\wedge T, h_n(X_{\tau\wedge T})) - u_n(0, i_n(X_s))\Big)\\ &= \int_{s\wedge\tau}^{T\wedge\tau} \mathbf{1}_{\{X_r\in D\}}\Big(\frac{\partial u_n}{\partial r}(r, i_n(X_r)) - L_n u_n(r, i_n(X_r))\frac{\sigma_0^2}{\sigma_{0,n}^2}(X_r)\Big)\, dr.\end{aligned}$$

We remark that the expectation exists since all integrands are bounded. Passing to the limit $n\to\infty$ and using $\sigma_0^2/\sigma_{0,n}^2 \to 1$ in C^0, we obtain

$$\mathbb{E}(u(\tau\wedge T, X_{\tau\wedge T})) - u(s,z)) = \mathbb{E}\left(\int_s^{T\wedge\tau} (\partial_t u - Lu)(r, X_r^{s,z})\, dr\right). \quad (3.28)$$

(i) For $N>0$, we set $\Delta := [-N,N], \tau = \tau^N$, the sequence (τ^N) defines a "suitable" sequence of stopping times in the sense defined before Remark 1.3. We let $N\to\infty$ in (3.28) and the result follows.

(ii) We set $\Delta := D$. According to the assumption we get

$$\begin{aligned}u(s,z) &= \mathbb{E}\,(u(\tau\wedge T, X_{\tau\wedge T}))\\ &= \int_{\{X_t^{s,z}\in D, \forall t\in[0,T]\}} u(T, X_T^{s,z})\, dP\\ &+ \int_{\{\tau^n\le T\}} \underbrace{u(\tau, X_\tau^{s,z})}_{0}\, dP\end{aligned}$$

This allows to conclude.

□

Corollary 3.22 *Let $u = (u(t,x), t\ge 0, x\in\mathbb{R})$ be of class C^1 and g continuous and bounded such that*

(i) $u(t,\cdot)\in\mathcal{D}_L$ for every $t\ge 0$,

(ii) $u(T,z) = g(z)$,

(iii) u solves the parabolic PDE

$$\frac{\partial u}{\partial t}(t,\cdot) = Lu(t,\cdot) \quad (3.29)$$

in the C^1-generalized sense.

Then we have

$$u(s,x) = \mathbb{E}(g(X_T^{s,z})),$$

where $X^{s,z}$ *solves the martingale problem related to* L *with initial condition* z *at* s.

Next, we are interested in the following situation. Let X be the solution to the martingale problem related to L with initial condition z. If the law of X has a density which kind of estimates can we obtain using analytical results?

To this end, we first suppose that L is of divergence type which means

$$b = \frac{\sigma^2}{2} \quad \text{such that} \quad Lg = (g' \frac{\sigma^2}{2})'. \tag{3.30}$$

We recall the fundamental lemma in this situation.

Lemma 3.23 *We suppose* $0 < c \le \sigma^2 \le C$. *Let* σ_n, $n \in \mathbb{N}$, *be smooth functions such that* $0 < c \le \sigma_n^2 \le C$ *and* $\sigma_n^2 \to \sigma^2$ *in* C^0 *as at the beginning of Section 2. We set* $L_n g = (\frac{\sigma_n^2}{2} g')'$. *There exists a family of probability measures* $(\nu_t(dx,y), t \ge 0, y \in \mathbb{R})$, *resp.* $(\nu_t^n(dx,y), t \ge 0, y \in \mathbb{R})$, *enjoying the following properties:*

(i) $\nu_t(dx,y) = p_t(x,y)\,dx$, $\nu_t^n(dx,y) = p_t^n(x,y)\,dy$.

(ii) (Aronson estimates) There exists $M > 0$ *with*

$$\frac{1}{M\sqrt{t}} \exp\left(-\frac{M|x-y|^2}{t}\right) \le p_t(x,y) \le \frac{M}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{Mt}\right).$$

(iii) $\frac{\partial}{\partial x} p_t(\cdot, y)$ *exists in the distributional sense and satisfies*

- $\sup_{y \in K} \int_0^T \left(\int_K \left| \frac{\partial}{\partial x} p_t(x,y) \right|^2 dx \right)^{\frac{1}{2}} dt < \infty.$
- $\int_{[0,T]\times K} \left| \frac{\partial}{\partial x} p_t(x,y) \right| \, dx\, dt < \infty$

for every compact interval K.

(iv) We have

$$\frac{\partial \nu_t}{\partial t}(\cdot, y) = L\nu_t(\cdot, y), \quad \nu_0(\cdot, y) = \delta_y \tag{3.31}$$

and

$$\frac{\partial \nu_t^n}{\partial t}(\cdot, y) = L_n \nu_t^n(\cdot, y), \quad \nu_0^n(\cdot, y) = \delta_y.$$

ν (resp. ν^n) is called the fundamental solution related to the previous parabolic linear equation.

(v) *The map $(t, x, y) \mapsto p_t(x, y)$ is continuous from $]0, \infty[\times\mathbb{R}^2$ to $\mathbb{R}$.*

(vi) *The p^n are smooth on $]0, \infty[\times\mathbb{R}^2$.*

(vii) *We have $\lim_{n\to\infty} p_t^n(x, y) = p_t(x, y)$ uniformly on each compact subset of $]0, \infty[\times\mathbb{R}^2$.*

(viii) *$p_t(x, y) = p_t(y, x)$ holds for every $t > 0$ and every $x, y \in \mathbb{R}$.*

(ix) *The semigroup property holds, i.e. for positive s, t we have*

$$\int p_t(x, y) p_s(y, z)\, dy = p_{t+s}(x, z).$$

Remark 3.24

(i) Aronson estimates, which were established in [1], imply in particular

$$\lim_{|y|\to\infty} p_t(x, y) = \lim_{|x|\to\infty} p_t(x, y) = 0.$$

(ii) The continuity of $y \mapsto p_t(x, y)$ $(t > 0,\ x \in \mathbb{R})$ entails that

$$t \mapsto \int \nu_t(dx, y)\, f(x)$$

is continuous for every bounded Borel function f.

(iii) (3.31) is to be understood in the following distributional way:

$$\int \nu_t(dx, y)\, f(x) = f(y) - \int_0^t ds \int dx\, \frac{\sigma^2}{2}(x) \frac{\partial}{\partial x}(p_s(x, y)) f'(x). \quad (3.32)$$

(iv) We can replace δ_y with any probability measure μ_0. The solution to (3.31) is then given by

$$\nu_t(dy) = \int d\mu_0(x)\, p_t(x, y).$$

(v) The maps $(t, y) \mapsto p_t(x, y)$ are in $L^2(]0, T] \times \mathbb{R}^2)$ again because of the Aronson estimates.

Proof. The proof of points (i), (ii), (iv), (v), (vi), (vii), (ix) is essentially contained in [49, ch. II.3] and the references therein. Statement (viii) is a consequence of the fact that L is self-adjoint.

The proof of (iii) is a little bit delicate and we have not found it explicitely in the literature, therefore we provide it.

Consider the following (classical) variational framework. All the definitions and properties recalled below without specific comments can be found in [25], [38], [50]. Let H and V be the Hilbert spaces

$$H = L^2(\mathbb{R}), \quad V = W^{1,2}(\mathbb{R}).$$

Let H' and V' be their dual spaces. Identifying H with H', we have the continuous dense injections

$$V \subset H \subset V'$$

and the dual pairing $\langle u, v \rangle_{V',V}$ between V' and V coincides with the scalar product in H, $\langle u, v \rangle_H$, when $u \in H$, $v \in V$.

Consider the bilinear form $a(.,.) : V \times V \to \mathbb{R}$ defined by

$$a(u, v) = \int_{\mathbb{R}} \frac{\sigma^2(x)}{2} u'(x) v'(x) \, dx, \quad u, v \in V.$$

It is symmetric, continuous and coercive. By Lax-Milgram theorem, there exists an isomorphism $L : V \to V'$ such that $a(u, v) = -\langle Lu, v \rangle_{V',V}$ for all $u, v \in V$. It is given by

$$Lu = \left(\frac{\sigma^2}{2} u' \right)', \quad u \in V,$$

where we have $\frac{\sigma^2}{2} u' \in H$ (by the assumptions on σ) and the subsequent derivative is taken in the distributional sense.

Let

$$D(L) = \{u \in V; \, Lu \in H\} = L^{-1}(H).$$

The linear (unbounded) operator $L : D(L) \subset H \to H$, restriction of the previously defined operator $L : V \to V'$, is selfadjoint (it is a consequence of the symmetry of $a(.,.)$) and generates an analytic semigroup e^{tL}, $t \geq 0$, in H (all selfadjoint negative definite operators in Hilbert spaces are infinitesimal generators of analytic semigroups). Among the regularity properties of e^{tL}, let us recall that $e^{tL}(H) \subset D(L)$ for all $t > 0$, hence in particular $e^{tL}(H) \subset V$ for all $t > 0$.

The operator L^{-1} is an isomorphism between H and the Hilbert space $D(L)$ endowed with the graph norm. The space $D(L)$ is dense in V and H.

It is possible to define the fractional powers $(-L)^{\alpha}$, $\alpha > 0$, as linear selfadjoint strictly positive operators in H with domains $D((-L)^{\alpha}) \subset H$. They are isomorphisms when the domains are endowed with the graph norm. We have $V = D\left((-L)^{\frac{1}{2}}\right)$. Hence $D\left((-L)^{\frac{1}{2}}\right) = W^{1,2}(\mathbb{R})$. By interpolation, $D((-L)^{\alpha}) = W^{2\alpha,2}(\mathbb{R})$ for all $\alpha \in \left(0, \frac{1}{2}\right)$. We remark that the fractional powers of $-L$ commute between themselves and with e^{tL} (with proper domains of the compositions).

By duality we have the continuous dense inclusions

$$V \subset D((-L)^{\alpha}) \subset H \subset D((-L)^{\alpha})' \subset V'$$

for all $\alpha \in \left(0, \frac{1}{2}\right)$. Since $(-L)^{\alpha}$ is selfadjoint, it can be extended to an isomorphism $(-L)^{\alpha} : H \to D((-L)^{\alpha})'$. We shall use its inverse $(-L)^{-\alpha} : D((-L)^{\alpha})' \to H$. Notice that $D((-L)^{\alpha})' = W^{-2\alpha,2}(\mathbb{R})$, for all $\alpha \in \left(0, \frac{1}{2}\right)$.

The semigroup e^{tL}, $t \geq 0$, can be restricted to an analytic semigroup in every Hilbert space $D((-L)^{\alpha})$, in particular in V. By duality, and taking into account that it is selfadjoint, it can be extended to an analytic semigroup in V'. We continue to denote these restrictions and extensions by e^{tL}, $t \geq 0$. Dualizing the regularity property $e^{tL}(H) \subset V$ for all $t > 0$, we get $e^{tL}(V') \subset H$ for all $t > 0$, and therefore, by composition (since $e^{tL} = e^{\frac{t}{2}L} e^{\frac{t}{2}L}$), we have $e^{tL}(V') \subset V$ for all $t > 0$.

Since $V \subset C^0(\mathbb{R})$ with continuous dense embedding, by Sobolev embedding theorem, for every $y \in \mathbb{R}$ the Dirac distribution δ_y at y belongs to V'. Therefore we can compute $e^{tL}\delta_y$, $t \geq 0$, that a priori is a continuous function of t with values in V'. By the previous regularity fact, we have $e^{tL}\delta_y \in V$

for all $t > 0$. By the properties of e^{tL}, the function $(t,x) \mapsto \left(e^{tL}\delta_y\right)(x)$ is the unique solution of the parabolic PDE $v' = Lv$ with initial condition $v(0) = \delta_y$, hence it coincides with $p_t(x,y)$.

More precisely, we have $W^{2\alpha,2}(\mathbb{R}) \subset C^0(\mathbb{R})$ with continuous dense embedding, for $\alpha > \frac{1}{4}$. Therefore $\delta_y \in W^{-2\alpha,2}(\mathbb{R})$. It follows that, for any given $\alpha > \frac{1}{4}$ and $y \in \mathbb{R}$, $(-L)^{-\alpha}\delta_y \in H$. Moreover, given a compact interval K, one can check by the definition of δ_y that $\sup_{y\in K}\|\delta_y\|_{W^{-2\alpha,2}(\mathbb{R})} \le C_K$, and therefore

$$\sup_{y\in K}\left\|(-L)^{-\alpha}\delta_y\right\|_H \le \tilde{C}_K.$$

Finally, recall the basic inequality for analytic semigroups

$$\left\|(-L)^{\beta} e^{tL} h\right\|_H \le \frac{C_{\beta,T}}{t^{\beta}}\|h\|_H$$

for every $t \in (0,T]$, $\beta > 0$, $h \in H$.

Take some $\alpha \in \left(\frac{1}{4},\frac{1}{2}\right)$. Given a compact interval K, we have

$$\begin{aligned}
\int_0^T \left\|e^{tL}\delta_y\right\|_V dt &= \int_0^T \left\|(-L)^{\alpha}(-L)^{-\alpha}e^{tL}\delta_y\right\|_V dt \\
&= \int_0^T \left\|(-L)^{\alpha}e^{tL}(-L)^{-\alpha}\delta_y\right\|_V dt \\
&\le const \int_0^T \left\|(-L)^{\frac{1}{2}}(-L)^{\alpha}e^{tL}(-L)^{-\alpha}\delta_y\right\|_H dt \\
&= const \int_0^T \left\|(-L)^{\frac{1}{2}+\alpha}e^{tL}(-L)^{-\alpha}\delta_y\right\|_H dt \\
&\le const \int_0^T \frac{1}{t^{\frac{1}{2}+\alpha}}\left\|(-L)^{-\alpha}\delta_y\right\|_H dt \\
&\le const \left\|(-L)^{-\alpha}\delta_y\right\|_H \le C'_K
\end{aligned}$$

for all $y \in K$. Using this bound, we finally have

$$\begin{aligned}
\sup_{y\in K}\int_{[0,T]\times K}\left|\frac{\partial}{\partial x}p_t(x,y)\right| dxdt &\le C_{T,K}\sup_{y\in K}\int_0^T\left(\int_K\left|\frac{\partial}{\partial x}p_t(x,y)\right|^2 dx\right)^{\frac{1}{2}} dt \\
&\le C'_{T,K}\sup_{y\in K}\int_0^T\left\|e^{tL}\delta_y\right\|_V dt \le C''_{T,K}.
\end{aligned}$$

Both claims (iii) are finally proved. □

Let X solve the martingale problem related to L with initial condition x. For $t \geq 0$, we denote the law of X_t by ν_t.

Our aim is now to show that its law has a density $(p_t(x,y), t > 0, x, y \in \mathbb{R})$ enjoying the property of Lemma 3.23.

Proposition 3.25 *Let L be of divergence type (see (3.30)), $g \in C_c^1 \cap \mathcal{D}_L$ such that $Lg \in C_c^0$. We use the same notation as in Lemma 3.23 and define*

$$v(t,z) = \int \nu_t(dx,z) g(x).$$

Then u: $(r,z) \mapsto v(T-t,z)$ belongs to $\mathcal{U}_L$. Moreover, $\partial_t u - Lu = 0$ holds in the C^0-generalized sense.

Proof. First of all, $u \in C^0([0,T] \times \mathbb{R})$ follows from Remark 3.24 ii) and v) because $v \in C^0([0,T] \times \mathbb{R})$. Moreover, v is bounded because of

$$\int p_t(x,y)\, dx = 1. \tag{3.33}$$

Let (g_n) such that $L_n g_n = Lg, g_n(0) = g(0), g_n'(0) = g'(0)$. Then by Lemma 2.7, g_n converges to g in C^0. We define

$$v_n(t,z) = \int \nu_t^n(dx,z)\, g_n(x) \tag{3.34}$$

are smooth because so are p^n. Moreover, by Lemma 3.23 (vii), we have

$$v_n(t,z) = \int p_t^n(x,z) g_n(x)\, dx \to \int p_t(x,z) g(x)\, dx \tag{3.35}$$

since g is bounded.

It remains to prove that $\partial_t v_n$ and $L_n v_n$ converge pointwise to v^a and v^b, respectively, for some continuous and bounded functions v^a and v^b on $[0,T] \times \mathbb{R}$. We calculate

$$\begin{aligned}
\partial_t \left(\int_{\mathbb{R}} p_t^n(x,z) g_n(x)\, dx \right) &= \int \partial_t p_t^n(z,x) g_n(x)\, dx \\
&= \int L_{n,x} p_t^n(z,x) g_n(x)\, dx \\
&= -\int_{\mathbb{R}} p_t^n(z,x) L_n g_n(x)\, dx.
\end{aligned}$$

This quantity converges pointwise to

$$v^a(t,z) = -\int_{[0,T]\times\mathbb{R}} p_t(z,x) Lg(x)\, dx. \tag{3.36}$$

Again v^a is bounded because of (3.33). Moreover, it is continuous.

The proof of the convergence of $L_n v_n$ to $v^b = v^a$ is included in the previous verification. Therefore, we have $v \in \mathcal{U}_L$ and $\partial_t u - Lu = 0$ in the C^0-generalized sense.

Corollary 3.26 *Let L be of divergence type as in (3.30) with $0 < c \le \sigma^2 \le C$. Let X be the solution to the martingale problem related to L with initial condition z. Then the law of X_t, $t \ge 0$, has a density which we denote by $p_t(x,z)$. Moreover, $p_t(x,z)$ coincides with the density introduced in Lemma 3.23.*

Proof. We start with $p_t(x,z)$ introduced in Lemma 3.23 and $g \in \mathcal{D}_L \cap C_c^1$ such that Lg has compact support. The function

$$\tilde{v}_T(t,z) = \int p_{T-t}(x,z) g(x)\, dx \tag{3.37}$$

coincides with

$$\mathbb{E}(g(X_t^z)) \tag{3.38}$$

by Theorem 3.21. Since $\{g \in \mathcal{D}_L \cap C_c^1 : Lg \in C_c^0\}$ is dense in $\mathcal{D}_L$ which is dense in C^1, the law of X_t^z is completely determined by equality (3.37) and (3.38). □

In the next chapter, we will show that the law of X_t, $t > 0$, has always a density if X solves the martingale problem related to any L satisfying the conditions of Section 2, with a supplementary assumption technical assumption.

4 Lyons-Zheng processes and Itô formula under weak conditions

In this section we will use the same notations as in Section 2. σ, b will be continuous functions such that $\sigma > 0$, σ_n^2, b_n will be regularizations of σ^2

and b with the same mollifier. L_n will stand for

$$L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'.$$

We suppose that

$$\Sigma(x) = \lim_{n \to \infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy \tag{4.1}$$

exists in C^0. We recall that, by Proposition 2.4, there is a unique $h \in C^1$ such that $Lh = 0$ and $h(0) = 0$, $h'(1) = 0$. It can be represented as

$$h'(x) = \exp(-\Sigma(x)). \tag{4.2}$$

A family $(p_t(x, \cdot), t > 0, x \in \mathbb{R})$ of probability densities is said to fulfill the *local Aronson estimates* if, for every continuous function χ with compact support, there is some $M > 0$ such that

$$\begin{aligned} & \frac{1}{M\sqrt{t}} \exp\left(-\frac{|x-y|^2 M}{t}\right) \chi(x-y) \\ \leq \; & p_t(x,y)\chi(x-y) \\ \leq \; & \frac{M}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{Mt}\right) \chi(x-y). \end{aligned} \tag{4.3}$$

Let X be the solution to the martingale problem related to L with initial condition x_0.

At this level, we need to formulate a technical assumption (TA). It will suppose there are positive constants c, C, such that

$$(TA) \qquad c \leq \frac{e^{\Sigma}}{\sigma} \leq C.$$

Remark 4.1 If the condition of non explosion (3.16) stated in Proposition 3.13 is fulfilled then an easy calculation will show that (TA) is verified. □.

We will show that, under (TA), for $t > 0$, the law of X_t admits a density fulfilling the local Aronson estimates. We already know that X is a Dirichlet process. The next result on time reversibility entails that X is a LZ process. We believe however that the assumption (TA) could be dropped.

Theorem 4.2 *Suppose that (TA) is verified.*

(i) For every $t > 0$, the law of X_t has a density $p = p_t(x_0, \cdot)$.

(ii) *For every $t > 0$, p satisfies the local Aronson estimates and $(t, x, y) \mapsto p_t(x, y)$ is continuous from $]0, \infty[\times \mathbb{R}^2$ to $\mathbb{R}$.*

(iii) *The process $Y = h(X)$ is a time reversible semimartingale.*

Remark 4.3

(i) X is a LZ process in view of Remark 1.5.

(ii) Fabes and Kenig ([17]) prove the existence of a diffusion (with inhomogenous diffusion term) whose law density is singular with respect to Lebesgue measure (even if it is non-atomic). Theorem 4.2 tells us that this is not possible in the case of homogeneous coefficients.

Proof of Theorem 4.2. Again, we use the notations of Section 2. In particular, we recall

$$\Sigma(x) = \lim_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma^2}(y)\,dy$$

and $h' = \exp(-\Sigma), h(0) = 0$. By Section 3, we know that, for $y_0 := h(x_0)$, we have $Y_t = y_0 + \int_0^t \sigma_0(Y_s)\,dW_s$, where W is a $\mathcal{F}_Y$-Brownian motion and $\sigma_0 = (\sigma h') \circ h^{-1}$. By the classical Itô formula, Y solves the martingale problem related to L^0, where

$$L^0 f = \frac{1}{2}\sigma_0^2 (h')^2 f''.$$

By Proposition 3.19, L^0 is also the infinitesimal generator of Y.

We denote by I the image set of h. We consider now a C^1 transformation k given by Lemma 2.20 but related to σ_0. We set $Z := j(Y), J = j(I)$, where j is determined by $j(0) = 0$ and $j'(y) = (1/\sigma_0^2)(y)$. We consider the formal PDE operator

$$L^1 f = (\frac{\sigma_1^2}{2} f')', \text{ where } \sigma_1 = (\sigma k') \circ k^{-1} = (\sigma_0 j) \circ j^{-1}.$$

The assumption (TA) on L implies that σ_1^2 is lower and upper bounded by a positive constant; so it fullfills the basic assumption of Lemma 3.23.

Lemma 4.4 *Z solves the martingale problem related to L^1 with initial condition $z_0 := k(y_0)$.*

Proof. Let $\tilde{f} \in \mathcal{D}_{L^1}(J)$. We know that $f = \tilde{f} \circ j \in C^2(I)$ by Proposition 2.12. Therefore we get

$$f(Y_t) = f(y_0) + \int_0^t f'(Y_s)\, dY_s + \int_0^t (L^0 f)(Y_s)\, ds.$$

Since $Y_t = j^{-1}(Z_t)$ and $\tilde{f} = f \circ j$, we conclude

$$\begin{aligned}\tilde{f}(Z_t) &= \tilde{f}(z_0) + \int_0^t (\tilde{f} \circ j)'(Y_s)\, dY_s \\ &\quad + \int_0^t L^1(\tilde{f} \circ j)(j^{-1}(Z_s))\, ds.\end{aligned}$$

which completes the proof. □.

The law of $Z_t^{z_0}$ $(t > 0)$ has a density $r_t(z_0, \cdot)$ by Corollary 3.26. Since $X = (h \circ j^{-1})(Z)$ and $Y = j^{-1}(Z)$, for $t > 0$ the law of X_t, resp. of Y_t, has a density $p_t(x_0, \cdot)$, resp. $q_t(y_0, \cdot)$, where $k(y_0) = z_0$, $(k \circ h)(x_0) = z_0$. Those densities can be calculated. In fact, if $f \in C^0$ is bounded and $Y^{y_0} = Y$, $Z^{z_0} = Z$, we get

$$\begin{aligned}\mathbb{E}(f(Y_t^{y_0})) &= \mathbb{E}(f \circ j^{-1}(Z_t^{z_0})) \\ &= \int (f \circ j^{-1})(z) r_t(z_0, z)\, dz \\ &= \int f(y) q_t(y_0, y)\, dy,\end{aligned}$$

where

$$q_t(y_0, y) = r_t(j(y_0), j(y)) j'(y) = r_t(j(y_0), j(y)) \frac{1}{\sigma_0^2(y)}. \tag{4.4}$$

In the same way, we verify

$$p_t(x_0, x) = r_t(j \circ h(x_0), j \circ h(x))(j \circ h)'(x). \tag{4.5}$$

This establishes (i) and (ii) of the theorem.

As for (iii), usual calculations about time reversal given for instance in [37, 33, 46] say that the time reversed process $(\hat{Y}_t, t \in [0, T])$ solves the stochastic differential equation

$$\hat{Y}_t = \hat{Y}_0 + \int_0^t \sigma_0(\hat{Y}_s)\, dB_s + \int_0^t \tilde{b}(T - s, \hat{Y}_s)\, ds,$$

where B is a classical $\mathcal{F}_{\hat{Y}}$- Brownian motion and

$$\tilde{b}(s,y) = -\Big(\frac{\partial}{\partial y}(\sigma_0^2(y)q_s(y_0,y))\Big)/q_s(y_0,y) \tag{4.6}$$

provided that (4.6) makes sense and

$$\int_0^T |\tilde{b}(s,Y_s)|\, ds < \infty \tag{4.7}$$

holds a.s. For this it is enough to show that

$$\mathbb{E}\left(\int_0^T ds\, |\tilde{b}(s,Y_s)| \mathbf{1}_{\{\sup_{t\in[0,1]}|Y_t|\le M\}}\right) < \infty$$

for some $M > 0$. Previous expression is bounded by

$$\int_0^T ds \int_{-M}^{M} dy \left|\frac{\partial}{\partial y}(\sigma_0^2(y)q_s(y_0,y))\right|. \tag{4.8}$$

(4.4) implies that

$$\begin{aligned} \sigma_0^2(y)q_s(y_0,y) &= \frac{\partial}{\partial y}(r_s(j(y_0),j(y))) \\ &= \left(\frac{\partial r_s}{\partial y}\right)(j(y_0),j(y))\, j'(y). \end{aligned} \tag{4.9}$$

(4.8) gives

$$\begin{aligned} &\int_0^T ds \int_{-M}^{M} dy \left|(\frac{\partial}{\partial y} r_t)(j(y_0),j(y))\right| j'(y) \\ &= \int_0^T ds \int_{[-j(M),j(M)]} dz \left|\left(\frac{\partial r_s}{\partial z}\right)(j(y_0),z)\right| \end{aligned}$$

which is finite because of Lemma 3.23 and Corollary 3.24.

In conclusion, Y is a time reversible semimartingale and so X a LZ process. $\square$.

At this level we would like to relax the technical assumption, but it is not completely possible. It will however be possible for the study of Itô formula.

For $M > 0$, and a real function f we set

$$f^M(x) = \begin{cases} f(x) & \text{if} \quad |x| \le M \\ f(M) & \text{if} \quad x \ge M \\ f(-M) & \text{if} \quad x \le -M \end{cases}$$

We can show that

$$\lim_{n\to\infty}\int_0^{\cdot} \frac{(b_n^M)'}{(\sigma_n^M)^2}(y)\,dy$$

is well-defined in C^0 (independently of the mollifier) and it equals Σ^M. It is obvious that for the PDE map $L(M)$, defined formally by

$$L(M)g = \frac{(\sigma^M)^2}{2} g'' + (b^M)' g',$$

the assumption (TA) is fulfilled.

We consider the event

$$\Omega_M = \{\omega : X_t(\omega) \in [-M, M], \forall t \in [0,T]\}$$

and the stopping time

$$\tau^M = \inf\{t \in [0,T] | X_t \notin [-M, M]\} \wedge (T+1)$$

(τ^M) is a "suitable" sequence of stopping times.

Remark 4.5 Let $M > 0$ such that $x_0 \in \;]-M, M[$. On Ω_M, the process X coincides with the stopped processes X^{τ^M}. On the same event, this one coincides with the stopped process $X(M)^{\tau^M}$ for the solution $X(M)$ to a martingale problem related to $L(M)$.

Indeed, for this, Proposition 3.3 allows us to consider the stochastic differential equation

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s)\, dW_s,$$

which is solved by $Y := h(X)$. The time changed process

$$B_t := Y_{T_t},$$

where $T_t = A_t^{-1}$ is the inverse of $A_t := \int_0^t \sigma_0^2(Y_s)\,ds$, is easily checked to be a Brownian motion. Furthermore, by [13, Proposition 5.2], we know

$$T_t = \int_0^t \frac{1}{\sigma_0^2}(B_s)\,ds.$$

Now we define

$$\sigma_0^{(M)}(y) = \begin{cases} \sigma_0(y) & \text{if} \quad |y| \le h(M) \\ \sigma_0(M) & \text{if} \quad y \ge h(M) \\ \sigma_0(-M) & \text{if} \quad y \le h(-M) \end{cases}$$

and consider

$$T_t^{(M)} := \int_0^t \frac{1}{(\sigma_0^{(M)})^2}(B_s)\,ds$$

and $A_t^{(M)} := T_t^{(M)^{-1}}$. By [13, Proposition 5.2], the process $Y(M)_t := B_{A_t^{(M)}}$ then solves the stochastic differential equation

$$Y(M)_t = Y_0 + \int_0^t \sigma_0^{(M)}(Y(M)_s)\,d\tilde{W}_s$$

for some Brownian motion $\tilde{W}$. From $B_t = Y_{T_t}$ we deduce $T_t^{(M)} = T_t$ on $\{t < A_{\tau_m}\}$, hence

$$A_t = A_t^{(M)} \quad \text{on} \quad \{t < \tau_m\}.$$

Thus, we conclude $Y_{t\wedge\tau_m} = Y(M)_{t\wedge\tau_m}$. For a more detailed discussion on construction of solutions to SDEs without drift we refer to [13].

The final result of this section is the Itô formula under weak conditions. We recall that similar formulas were first considered independently first by [21] and [44]; further extensions have been performed in [20] and [5], [34, 35].

Here, the innovation is that we deal with non-degenerate diffusion processes with non-smooth coefficients. For that purpose, the technical assumption (TA) is not required.

Theorem 4.6 *Let $\sigma \in C^0(\mathbb{R})$ with $\sigma > 0$, γ be a locally bounded function and X a diffusion process of the type*

$$X_t = x_0 + \int_0^t \sigma(X_s)\,dW_s + \int_0^t \gamma(X_s)\,ds. \tag{4.10}$$

Then the following Itô formula holds for every $f \in W^{1,2}_{loc}$:

$$f(X_t) = f(x_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}[f(X), X]_t. \tag{4.11}$$

Proof. According to the notations of Sections 2 and 3, X solves the martingale problem related to L, where

$$Lf = \frac{\sigma^2}{2}f'' + b'f'$$

and $b(x) = \int_0^x \gamma(y)\,dy$. In this case, we have

$$\Sigma(x) = 2\int_0^x \frac{\gamma}{\sigma^2}(y)\,dy.$$

In particular, Σ belongs to $W^{1,\infty}_{loc}$.

For a first step we assume again (TA). By Theorem 4.2, X is a LZ process. Since X is a semimartingale, $\hat{X}$ is also a semimartingale by Remark 1.5.

Remark 4.7 Using [44], under (TA), we have already shown (4.11) for every $f \in C^1$. □

In order to prove (4.11) for $f \in W^{1,2}_{loc}$, we need to work out explicitely the equation solved by $\hat{X}$. We set $Z := k(Y)$, where $k \in C^1(\mathbb{R})$ is defined in Lemma 2.20 which yields together with (4.5)

$$p_t(x_0, x) = r_t(k(x_0), k(x))\sigma^{-2}(x)\exp(\Sigma(x)). \tag{4.12}$$

As for (4.6), we have

$$\hat{X}_t = \hat{X}_0 + \int_0^t \sigma(\hat{X}_s)\,dB_s + \int_0^t \tilde{\gamma}(T-s, \hat{X}_s)\,ds, \tag{4.13}$$

where

$$\gamma(x) + \tilde{\gamma}(t,x) = \frac{-\frac{\partial}{\partial x}(\sigma^2(x)p_t(x_0,x))}{p_t(x_0,x)}$$

provided that

$$\int_0^T |\tilde{\gamma}(s, X_s)|\,ds < \infty \tag{4.14}$$

holds a.s. Since $\int_0^t \gamma(X_s)ds$ exists, (4.14) holds, if for every $M > 0$, we have

$$\mathbb{E}\left(\int_0^T |\gamma + \tilde{\gamma}|(X_s)1_{\{|X_s|\le M\}}ds\right) < \infty.$$

This happens if

$$\int_0^T ds \int_{-M}^M dx \left|\frac{\partial}{\partial x}(\sigma^2(x)p_s(x_0,x))\right| < \infty. \tag{4.15}$$

(4.15) will be proved later. For the moment, we observe

$$\begin{aligned}
\frac{\partial}{\partial x}(\sigma^2(x)p_t(x_0,x)) &= \frac{\partial}{\partial x}\Big(\exp(\Sigma(x))r_t(k(x_0),k(x))\Big) \\
&= \exp(\Sigma(x))\Big(\Sigma'(x)r_t(k(x_0),k(x)) \qquad (4.16) \\
&\quad + \frac{\partial}{\partial z}r_t(k(x_0),k(x))k'(x)\Big).
\end{aligned}$$

In order to conclude by the Banach-Steinhaus argument of [44] we have to check that

$$\int_0^t g(X)\, d^{\pm} X \tag{4.17}$$

exists for every $g \in L^2_{loc}$. The forward integral is known to coincide with the Itô integral

$$\int_0^t g(X_s)\, dX_s = \int_0^t g(X_s)\sigma(X_s)\, dW_s + \int_0^t g(X_s)\gamma(X_s)\, ds. \tag{4.18}$$

By Remark 1.2, the backward integral equals

$$-\int_{T-t}^T g(\hat{X}_s)\, d\hat{X}_s = -\int_{T-t}^T (g\sigma)(\hat{X}_s)\, ds - \int_{T-t}^T g(\hat{X}_s)\tilde{\gamma}(T-s, \hat{X}_s)\, ds \tag{4.19}$$

provided that the right members of (4.18) and (4.19) exist. For this, we have to verify

$$\int_0^T g^2(X_s)\sigma^2(X_s)\, ds \quad < \quad \infty, \tag{4.20}$$

$$\int_0^T |g(X_s)\gamma(X_s)|\, ds \quad < \quad \infty, \tag{4.21}$$

$$\int_0^T |g(X_s)\tilde{\gamma}(s, X_s)|\, ds \quad < \quad \infty \tag{4.22}$$

a.s. Since σ and γ are locally bounded, by Cauchy-Schwarz, (4.20) and (4.21) hold if

$$\int_0^T |g|^2(X_s)\, ds < \infty \quad \text{a.s.} \tag{4.23}$$

(4.22) will be verified if, for every $M > 0$,

$$\int_0^T ds \int_{-M}^M dx\, |g(x)| \left| \frac{\partial}{\partial x}(\sigma^2(x) p_s(x_0, x)) \right| < \infty \tag{4.24}$$

holds. Therefore proving (4.24) will justify (4.15) and (4.23), simultaneously.

In view of (4.16), expression (4.24) is bounded by

$$const \Big(\int_0^T ds \int_{-M}^M dx\, |g(x)| r_s(k(x_0), k(x))$$

$$+ \int_0^T ds \int_{[-k(M), k(M)]} dz \left| g(k^{-1}(z)) \left(\frac{\partial r_s}{\partial z} \right) (z_0, z) \right| \Big).$$

The first integral above is finite because $g \in L^2_{loc}$ and $(s,x) \mapsto r_s(k(x_0), k(x))$ is square integrable by Remark 3.24 (v).

The second integral is bounded again through Cauchy-Schwarz. It gives

$$\left(\int_{[-k(M),k(M)]} dz g^2(k^{-1}(z))\right)^{\frac{1}{2}} \int_0^T ds \left(\int_{[-k(M),k(M)]} dz \left(\frac{\partial r_s}{\partial z}\right)^2 (z_0, z)\right)^{\frac{1}{2}} \tag{4.25}$$

This quantity is bounded because of $g \in L^2_{loc}$ and Lemma 3.23 (iii).

This shows the result when (TA) is fulfilled.

Suppose, however, that (TA) is not necessarily fulfilled, Then, we know that on the event Ω_M defined just before Remark 4.5, $X = X(M)$.

Taking in account the definition of forward and backward integrals, (4.17) will exist if

$$\int_0^t g(X(M)) d^{\pm} X(M)$$

exist. This is obvious because $X(M)$ solves the martingale problem related to $L(M)$ and this fulfills assumption (TA). □

So far, we have not needed the finite dimensional distributions of a solution to the martingale problem related to L. Those distributions will also be useful for deeper calculations on brackets in Section 5. For simplicity, we will restrict ourselves to the case of two dimensional distributions.

Theorem 4.8 *Let $X = X^{x_0}$ be a solution to the martingale problem related to L with initial condition x_0 such that (TA) is realized. The joint law of $(X_s^{x_0}, X_T^{x_0})$, $0 < s < T$, has a density given by*

$$(x_1, x_2) \mapsto p_s(x_0, x_1) p_{T-s}(x_1, x_2).$$

Proof. Let $f \in C^0(\mathbb{R}^2)$ with compact support. We have to evaluate

$$\mathbb{E}(f(X_s, X_T)) = \mathbb{E}(\mathbb{E}(f(X_s, X_T)|X_s))$$

for $0 < s < T$. In order to calculate the previous conditional expectation we need some preliminary results. The first one is an adaptation of Theorem 3.21.

Lemma 4.9 *Let $u \in \mathcal{U}_L$ such that $(\partial_t - L)u = 0$ holds in the C^0-generalized sense. Then we have*

$$\mathbb{E}(u(T, X_T)|\mathcal{F}_s^X) = u(s, X_s)$$

a.s. In particular, $(u(t, X_t))$ is a $\mathcal{F}^X$-martingale.

Proof. The same as for Theorem 3.21, but we take conditional expectations instead of expectations on X^{x_0} starting from zero instead of s. □

We focus now on the case that L is of divergence type. If $p_t(x, y)$ is the fundamental solution associated with L, we set

$$u(x_1, t, z) := \begin{cases} \int dx\, p_{T-t}(x, z) f(x_1, x) & : \quad t < T; \\ f(x_1, z) & : \quad t = T. \end{cases}$$

We already know $u(x_1, \cdot) \in \mathcal{U}_L$ by Proposition 3.25 and $(\partial_t - L)u(x_1, t, \cdot) = 0$ in the C^0-generalized sense. Using the above lemma, we now have

$$\begin{aligned} \mathbb{E}(f(X_s, X_T)) &= \mathbb{E}(\mathbb{E}(f(X_s, X_T)|X_s)) \\ &= \mathbb{E}(u(X_s, s, X_s)) \\ &= \int dx_1\, p_s(z, x_1) u(x_1, s, x_1) \\ &= \int dx_1\, p_s(z, x_1) \int dx_2\, p_{T-s}(x_2, x_1) f(x_1, x_2). \end{aligned}$$

This proves the result if L is of divergence type.

The general case now follows using $(X_s, X_T) = (k(Z_s), k(Z_T))$. In fact, if $f \in C^0(\mathbb{R}^2)$ with compact support then we have

$$\begin{aligned} \mathbb{E}(f(X_s, X_T)) &= \mathbb{E}(f(k(Z_s), k(Z_T))) \\ &= \int dz_1\, dz_2\, r_s(z_0, z_1) r_{T-s}(z_1, z_2), \end{aligned} \tag{4.26}$$

where $(r_t(z, \cdot))$ is the law density of Z_t^z which solves the martingale problem related to L^1 which is of divergence type. We recall that L^1 has been defined in Lemma 2.20, (2.21).

(4.12) says that (4.26) equals

$$\int dz_1\, dz_2\, p_s(x_0, k^{-1}(z_1)) p_{T-s}(k^{-1}(z_1), k^{-1}(z_2)) k'(k^{-1}(z_1)) k'(k^{-1}(z_2)).$$

Using the change of variables $x_i = k^{-1}(z_i)$, $i = 1, 2$, we complete the proof.
□

5 The semimartingale characterization

Let L be again a PDE of the type considered in Section 2, $h \in C^1$, such that $h(0) = 0$ and $h'(x) = \exp(-\Sigma(x))$. In particular, we have $Lh = 0$.

Let X be a solution to the martingale problem related to L with initial condition x_0. By Remark 3.6, for $f \in C^1$,we have

$$f(X_t) = f(x_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \mathcal{A}(f). \tag{5.1}$$

For $f \in C^1$ it is well-known that $\mathcal{A}(f)$ is a zero quadratic variation process, see for instance [45] and Remark 1.3.

Remark 5.1 Under assumption (TA), previous results can be generalized to the case $W^{1,2}_{loc}$.

(i) Theorem 4.2 says that the law of X_s, $s > 0$, admits a density which fulfills the local Aronson estimates. Using arguments similar to those at the end of Section 4, we can show that

$$f \mapsto M^f_t := \int_0^\cdot (f'\sigma)(X_s)\, dW_s$$

is continuous from $W^{1,2}_{loc}$ to the space $\mathcal{C}$ of continuous processes. Therefore $\mathcal{A}$ and (5.1) can be extended continuously to $W^{1,2}_{loc}$.

(ii) The bracket $[f(X), g(X)]$ also exists and equals $\int_0^\cdot f'g'(X_s)d[X]_s$ for every $f, g \in W^{1,2}_{loc}$. In fact, by bilinearity, it is only necessary to consider the case $f = g$. For this we have to evaluate the ucp limit of

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^t (f(X_{s+\varepsilon}) - f(X_s))^2\, d[X]_s \\ &= \frac{1}{\varepsilon}\int_0^t ds \left(\int_0^1 d\alpha[f'((X_s + \alpha(X_{s+\varepsilon} - X_s))]\right)^2 (X_{s+\varepsilon} - X_s)^2 \\ &= I_1(\varepsilon, t) + I_2(\varepsilon, t), \end{aligned}$$

where

$$I_1(\varepsilon, t) = \frac{1}{\varepsilon} \int_0^t f'(X_s)^2 (X_{s+\varepsilon} - X_s)^2 \, ds$$

and I_2 is the remainder term. For estimating both terms, we need the law of $(X_s, X_{s+\varepsilon})$ which is derived in Theorem 4.8. I_2 will then converge to zero.

(iii) Actually, $\mathcal{A}(f)$ has zero quadratic variation even if $f \in W^{1,2}_{loc}$. For this, we proceed similarly as in [45]. We write

$$[\mathcal{A}(f)] = [f(X)] + [M^f] + 2[f(X), M^f], \tag{5.2}$$

where M^f has been defined at point (i). Using (ii) and similar considerations, it is possible to prove that all three brackets exist and equal

$$\int_0^t (f'\sigma)^2(X_s) \, ds.$$

(iv) We learn from the previous point that, if $f \in W^{1,2}_{loc}$, $f(X)$ is a $\mathbb{F}$-Dirichlet process with M^f as martingale part. On the other hand $f(X)$ admits obviously a LZ type decomposition given at Remark 1.5 e) with $M^1_f = M^f$. Remark 1.5 h) entails that it is a true LZ-process.

If assumption (TA) is not fulfilled, then $f(X)$ is still a Dirichlet process with local martingale part M^f. In fact consider the "suitable" sequence of stopping times (τ^M) and PDE operators $L(M)$, defined before the statement of Theorem 4.6.

For $M > 0$ the process $f(X)$ stopped at τ^M coincides with $f(X(M))$, where $X(M)$ solves a martingale probleme related to $L(M)$. $L(M)$ fulfills then the assumption (TA) and therefore it is Dirichlet. The considerations of Section 1 can be then applied.

The basic question of this section is the following. Which are the functions $f \in W^{2,1}_{loc}$ such that $f(X)$ is a semimartingale? In particular, this includes a necessary and sufficient condition for X to be a semimartingale. We recall that BV^1 is the set of functions $f \in W^{1,2}_{loc}$ such that $f' \exp \Sigma$ is locally of bounded variation.

Before stating the characterization formula, we provide a preliminary result which strenghtens the Bouleau-Yor property defined before Lemma 3.9.

Lemma 5.2 *Let X be a solution to a martingale problem with respect to L satisfying (TA). Then*

$$\int_0^\cdot g(X) d^- \mathcal{A}(f)$$

exists for every $g, f \in W^{1,2}_{loc}$. In particular, the mapping $f \to \int_0^\cdot g(X) d^- \mathcal{A}(f)$ is continuous from $W^{1,2}_{loc}$ to $\mathcal{C}$.

Proof. We know

$$\mathcal{A}(f) = f(X_t) - f(X_0) - \int_0^\cdot (f'\sigma)(X_s)\, dW_s.$$

Because $\int_0^\cdot (f'\sigma)(X_s)\, dW_s$ is a local martingale, the forward integral above exists if $\int_0^\cdot g(X)\, d^- f(X)$ exists for every $f, g \in W^{1,2}_{loc}$. Since Remark 5.1 (ii) ensures the existence of $[f(X), g(X)]$, the forward integrals exist if and only if

$$\int_0^\cdot g(X)\, d^0 f(X)$$

exists, see Remark 1.1. But since $g(X)$ is a finite quadratic variation process, Remark 1.6 tells us that the symmetric integral above equals the LZ type integral

$$\int_0^\cdot g(X) \circ df(X),$$

Now, the LZ integral is well-defined because $f(X)$ is a $(\mathbb{F}, \mathbb{H})$-LZ process and $g(X)$ is $(\mathbb{F}, \mathbb{H})$- adapted and square integrable. □.

Proposition 5.3 *$f(X)$ is a semimartingale if and only if $f \in BV^1$.*

Proof. We introduce again the notations included in Lemma 2.20 and just before.

k is a C^1 real function such that

$$k'(x) = \frac{e^{\Sigma(x)}}{\sigma^2(x)}, \quad L^1 g = \left(\frac{\sigma_1^2 g}{2}\right)', \quad \sigma_1 = (\sigma\, k') \circ k^{-1}.$$

We also set

$$q(x) = \int_0^T ds\, p_s(x_0, x) \tag{5.3}$$

where $(p_t(x_0, \cdot))$ is the density of the law of X_t, $t > 0$.

We recall that $Z_t = k(X_t)$ and the density of (Z_t) is a fundamental solution of $\partial_t u = L^1 u$. Let $r_t(x_1, \cdot)$, $x_1 = k(x_0)$, be such a density. We recall that by (4.12), we have

$$p_t(x_0, x) = r_t(x_1, k(x))\, k'(x). \tag{5.4}$$

We set

$$r(z) = \int_0^T ds\, r_s(x_1, z).$$

We have of course

$$q(x) = r(k(x))\, k'(x). \tag{5.5}$$

By Theorem 4.2(ii), q and r are continuous, hence locally bounded. Moreover they are strictly positive because of the Aronson estimates and Lemma 3.23 (iii) yields $r \in W^{1,1}_{\mathrm{loc}}$.

Let us consider the "suitable" sequence of stoppping times (τ^M) and processes $X(M)$ solving a martingale problem related to $L(M)$ as before Theorem 4.6.

Now, $f(X)$ is a semimartingale if and only if $f(X(M))$ is a semimartingale for every M. Therefore, we can suppose that X fulfills (TA).

We proceed now with the proof of necessity.

i) Let us suppose that $f(X)$ is a semimartingale for some $f \in W^{1,2}_{\mathrm{loc}}$. Then by (5.1), $\mathcal{A}(f)$ must be a semimartingale. Since it is a zero quadratic variation process, it is forced to be of bounded variation.

ii) We set $\bar{f} = f \circ k$. Clearly $\bar{f} \in W^{1,2}_{\mathrm{loc}}$; so $L^1 \bar{f}$ is a well-defined distribution. We first prove that $L^1 \bar{f}$ is a Radon measure.

By Corollary 2.10, there is a sequence (f_n) in $\mathcal{D}_L$ so that $f_n \to f$ in $W^{1,2}_{\mathrm{loc}}$. We define $\bar{f}_n = f_n \circ k$ which belong to $\mathcal{D}_{L^1}$.

Let $\phi \in C^0(\mathbb{R})$ with compact support such that

$$\tilde{\phi} = \frac{\phi \circ k}{r \circ k} \in C^1 \tag{5.6}$$

We observe that $\phi \in W^{1,2}_{loc}$. Since $\mathcal{L}^1$ is continuous from $W^{1,2}_{loc}$ to L^2_{loc}

$$\langle L^1 \bar{f}, \phi \rangle = -\int \frac{\sigma^2}{2} \bar{f}'(z) \phi'(z) dz$$

$$
\begin{aligned}
&= -\int dz\, \mathcal{L}^1 \bar{f}(z) \phi'(z) \\
&= \lim_{n\to\infty} -\int dz\, \mathcal{L}^1 \bar{f}_n(z) \phi'(z) \\
&= \lim_{n\to\infty} \int dz\ \phi(z)\, L^1 \bar{f}_n(z) \\
&= \lim_{n\to\infty} \int dy\, \phi \circ k(y)\, L\, f_n(y)\, k'(y) \\
&= \lim_{n\to\infty} \int dy\, \frac{\phi \circ k}{r \circ k}(y)\, L\, f_n(y)\, k'(y)\, r \circ k(y)
\end{aligned}
$$

Using (5.5), we obtain

$$
\begin{aligned}
& \lim_{n\to\infty} \int dy\, \tilde{\phi}(y)\, L\, f_n(y)\, q(y) \\
= & \lim_{n\to\infty} \int ds \int dy\, p_s(x_0, y)\, \tilde{\phi}(y) L f_n(y) \\
= & \lim_{n\to\infty} \mathbb{E}\left\{ \int_0^T ds\, L\, f_n(X_s)\, \tilde{\phi}(X_s) \right\}
\end{aligned}
$$

This equals

$$
\lim_{n\to\infty} \mathbb{E}\left\{ \int_0^T d^-\, \mathcal{A}(f_n)\, \tilde{\phi}(X) \right\} \tag{5.7}
$$

Lemma 5.2 and the fact that X is a LZ process imply that

$$
\int_0^{\cdot} \tilde{\phi}(X) d^- \mathcal{A}(f_n) \longrightarrow \int_0^{\cdot} \tilde{\phi}(X) d^- \mathcal{A}(f) = \int_0^{\cdot} \tilde{\phi}(X) d\mathcal{A}(f)
$$

holds in ucp. The equality above is explained by the fact that $\mathcal{A}(f)$ is a bounded variation process. Therefore (5.7) converges to

$$
\mathbb{E}\left(\int_0^T d\mathcal{A}(f) \tilde{\phi}(X) \right)
$$

provided that the sequence

$$
\int_0^T L f_n(X_s) \tilde{\phi}(X_s) = \int_0^T L^1 \bar{f}_n(Z_s) \phi_1(Z_s),
$$

where $\phi_1 = \frac{\phi}{r} = \phi \circ k^{-1}$ is uniformly integrable. This can be established by verifying that the sequence of the expectations of squares is bounded. In fact, using Theorem 4.8

$$
\mathbb{E}\left(\int_0^T ds L^1 \bar{f}_n(Z_s) \phi_1(Z_s) \right)^2
$$

$$
\begin{aligned}
&= 2\int_0^T ds_1 \int_{s_1}^T ds_2 \int dy_1 \int dy_2 L^1 \bar{f}_n(y_1)\phi_1(y_1) L^1 \bar{f}_n(y_2)\phi_1(y_2) \\
&\quad r_{s_1}(x_0, y_1) r_{s_2 - s_1}(y_1, y_2) \\
&= 2\int_0^T ds_1 \int_{s_1}^T ds_2 \int dy_1 \frac{\sigma_1^2}{2} \bar{f}_n'(y_1) \frac{\partial}{\partial y_1}\left(\phi_1(y_1) r_{s_1}(x_0, y_1)\right) \\
&\quad \int dy_2 \frac{\sigma_1^2}{2}(y_2) \bar{f}_n'(y_2) \frac{\partial}{\partial y_2}\left(\phi_1(y_2) r_{s_2 - s_1}(y_1, y_2)\right)
\end{aligned}
$$

Using Cauchy-Schwarz, this quantity is bounded by

$$
\begin{aligned}
&const(\tilde{\phi}) \int_0^T ds_1 \left\{ \int_{-M}^M dy_1 \bar{f}_n'^2(y_1) \int_{-M}^M \left[\left(\frac{\partial r_{s_1}}{\partial y_1}(x_0, y_1)\right)^2 + (r_{s_1}(x_0, y_1))^2 \right] dy_1 \right\}^{\frac{1}{2}} \\
&\quad \int_0^{T - s_1} ds_2 \left\{ \int_{-M}^M dy_2 \bar{f}_n'^2(y_2) \int_{-M}^M \left[\left(\frac{\partial r_{s_2}}{\partial y_2}(y_1, y_2)\right)^2 + (r_{s_2}(y_1, y_2))^2 \right] dy_2 \right\}^{\frac{1}{2}}
\end{aligned}
$$

for some $M > 0$ such that $[-M, M]$ contains the support of ϕ_1.

The latter quantity is bounded by

$$
\begin{aligned}
&const(\tilde{\phi}) \int_{-M}^M dy \bar{f}_n'^2(y) \int_0^T ds_1 \left\{ \int_{-M}^M \left[\left(\frac{\partial r_{s_1}}{\partial y_1}(x_0, y_1)\right)^2 + (r_{s_1}(x_0, y_1))^2 \right] dy_1 \right\}^{\frac{1}{2}} \\
&\qquad \sup_{y_1 \in \mathbb{R}} \int_0^T ds_2 \left\{ \int_{-M}^M \left[\left(\frac{\partial r_{s_2}}{\partial y_2}(y_1, y_2)\right)^2 + (r_{s_2}(y_1, y_2))^2 \right] dy_2 \right\}^{\frac{1}{2}}
\end{aligned}
$$

The fact that $\bar{f}_n' \to \bar{f}'$ in L^2_{loc} and lemma 3.23 (ii), (iii) imply that the quantity above is finite.

We have now established the identity

$$
\langle L^1 \bar{f}, \phi \rangle = \mathbb{E}\left(\int_0^T d\mathcal{A}(f) \tilde{\phi}(X) \right). \tag{5.8}
$$

Now, the right member can be extended by continuity to $\phi \in C^0$; in fact functions ϕ fulfilling (5.6) are dense in C^0. This shows that $L^1 \bar{f}$ is a Radon measure.

iii) $L^1 \bar{f}$ being a Radon measure, $\mathcal{L}^1 \bar{f}$ is of bounded variation; this means that $\frac{\sigma_1^2 \bar{f}'}{2}$ has bounded variation. This equals

$$
(\sigma k')^2 \circ k^{-1} (f \circ k^{-1})' = (\sigma^2 k' f') \circ k^{-1},
$$

which implies that

$$\sigma^2 k' f' = e^{\Sigma} f'$$

must be of bounded variation. This shows the necessity.

We now proceed to the converse implication. Let $f \in BV^1$. By Lemma 2.15, there is a sequence (f_n) in $\mathcal{D}_L$ such that $f_n \to f$ in BV^1 and so $Lf_n dx \to d\mu$ weakly- $*$ for some Radon measure μ.

Remark 2.14 d) guarantees the existence of a subsequence (n_k) such that

$$Lf_{n_k}^+ dx \to \nu^+ \text{ and } Lf_{n_k}^- \to \nu^-$$

weakly, where $\nu^\pm$ are two Radon measures on $\mathbb{R}$.

We want to prove that $\mathcal{A}(f)$ has bounded variation. By Remark 5.1, $f \mapsto \mathcal{A}(f)$ is continuous from C^1 to $\mathcal{C}$. Again by Lemma 2.15, the sequence (f_n) converges to f in $W^{1,2}_{loc}$ so that $\mathcal{A}(f_n) \to \mathcal{A}(f)$ holds in ucp. Now, we have

$$\mathcal{A}(f_n) = \int_0^{\cdot} (Lf_n)(X_s)\, ds = \int_0^{\cdot} (Lf_n)^+(X_s)\, ds - \int_0^{\cdot} (Lf_n)^-(X_s)\, ds.$$

We define $\tilde{f}_n^+$, $\tilde{f}_n^-$ in C^1 such that $\tilde{f}_n^\pm(0) = 0$ and

$$(\tilde{f}_n^\pm)' = h'(x)\left(2\int_0^x \frac{(Lf_n)^\pm}{(\sigma h')^2}(y)\, dy\right)$$

hold. Since $(Lf_{n_k})^\pm(y)\, dy$ converges weakly-$*$ to $\nu^\pm$, Remark 2.15 a) says that

$$\frac{(\tilde{f}_n^\pm)'(x)}{h'} \to 2\int_0^x \frac{d\nu^\pm}{(\sigma h')^2(y)}$$

holds in BV. We consider $\tilde{f}^\pm \in BV^1$ which are given by $\tilde{f}^\pm(0) = 0$ and

$$(\tilde{f}^\pm)'(x) = 2h'(x)\int_0^x \frac{d\nu^\pm}{(\sigma h')^2(y)}.$$

According to (5.1), we have

$$\int_0^{\cdot} Lf_n^\pm(X_s)\, ds = f_n^\pm(X_t) - f_n^\pm(X_0) - \int_0^t ((f_n^\pm)'\sigma)(X_s)\, dW_s. \tag{5.9}$$

Since $f_n^\pm \to f$ in $W^{1,2}_{loc}$ and $\mathcal{A}$ is continuous, the sequence of increasing processes $\int_0^{\cdot} Lf_{n_k}^\pm(X_s)\, ds$ in (5.9) converges to the increasing process

$$\tilde{f}^\pm(X_t) - \tilde{f}^\pm(X_0) - \int_0^t ((\tilde{f}^\pm)'\sigma)(X_s)\, dW_s = \mathcal{A}(\tilde{f}^\pm).$$

So $\mathcal{A}(f)$ is the difference of the increasing processes $\mathcal{A}(\tilde{f}^+)$ and $\mathcal{A}(\tilde{f}^-)$. □

Remark 5.4 If $f \in BV^1$ then, in particular, we have $\hat{\mathcal{L}}^{BV} f \in BV$.

Remark 5.5 For a Markov process X, [10] states necessary and sufficient conditions on f such that $f(X)$ is a semimartingale. Here, we specify in particular this result in a very simple way without using directly the Markov property. We observe that, for a standard Brownian motion X, we have $h \equiv id$ and $BV^1 = \{f \in W^{1,2}_{loc} : f' \in BV\}$. BV^1 is in this case constituted by functions which are a difference of convex functions.

Corollary 5.6 *Let us suppose that L is close to divergence type. Then $f(X)$ is a semimartingale if and only if $\hat{\mathcal{L}} f$ has bounded variation.*

Remark 5.7 Proposition 2.22 allows to extend continuously $\hat{\mathcal{L}}$ to $W^{1,2}_{loc}$ so that the statement makes sense.

Proof of the Corollary.

Let $f \in BV^1$. By Corollary 2.10 and Proposition 2.16, there is a sequence (f_n) in $\mathcal{D}_L$ converging to f in $W^{1,2}_{loc}$ such that $(\mathcal{L} f_n)$ converges in BV to some $g \in BV$. By continuity of $\hat{\mathcal{L}}$ in $W^{1,2}_{loc}$, we have

$$\hat{\mathcal{L}} f = \lim_{n \to \infty} \mathcal{L} f_n \text{ in } L^2_{loc}.$$

Since the convergence in L^2_{loc} and BV must agree, we have $\hat{\mathcal{L}} f = g \in BV$.

Conversely, if $\hat{\mathcal{L}} f = g$ is BV, by a usual regularization procedure, we find $g_n \in C^1$ satisfying $g_n \to g$ in BV. Let $f_n \in \mathcal{D}_L$ such that

$$f_n(0) = 0,\ f_n'(x) = h'(x) \int_0^x \frac{2}{\sigma^2(y)}\, dg_n(y).$$

Lemma 2.7 says that $\mathcal{L} f_n = g_n$. Taking the limit in the expression above and using the continuity of the extension of $\mathcal{L}$ (Proposition 2.21) we obtain that $\frac{f_n'}{h'}$ converge to $\frac{f'}{h'}$ in BV. Thus, we have $f \in BV^1$. □

Corollary 5.8 *Let X be a solution to the martingale problem related to L. Then X is a semimartingale if and only if Σ has bounded variation.*

Proof. By Proposition 5.3, X is a semimartingale if and only if $id \in BV^1$. This means $(h')^{-1} = \exp(\Sigma) \in BV$ or, equivalently $\Sigma \in BV$. □

Remark 5.9

(i) If $\sigma = 1$ then $\Sigma \equiv b$ and we discover the result of [47].

(ii) Let L be close to divergence type (see (2.22)) and

$$\Sigma(x) = \ln \frac{1}{\sigma^2} + 2\int_0^x \frac{d\beta}{\sigma^2}.$$

Then Σ is of bounded variation if and only if so is σ.

6 The backward and symmetric equations

In this section again we suppose that L fulfills the basic assumptions of Section 2. Let X be the solution to the martingale problem related to L with initial condition x_0.

We know by Corollary 3.12 that X solves the generalized stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + A\Big(\frac{\sigma^2}{2}\Big). \tag{6.1}$$

If $\sigma \in C^1$ then X is a semimartingale and

$$A\Big(\frac{\sigma^2}{2}\Big) = \int_0^t \Big(\frac{\sigma^2}{2}\Big)'(X_s)\, ds = \int_0^t (\sigma\sigma')(X_s)\, ds = [\sigma(X), X]_t.$$

Therefore, X solves the backward stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(X_s)\, d^+W_s \tag{6.2}$$

because of Remark 1.1. Immediately, the following question arises. If σ is not smooth is X still a solution to (6.2)? The answer will be yes if X is a semimartingale. In the general case, it does not seem to be true but we do not give a rigorous argument for this.

Let us suppose again our technical assumption (TA), which corresponds here to $0 < c \le \sigma^2 \le C < \infty$.

First of all, we would like to understand some features of the time reversal of the $\mathcal{F}^X$-Brownian motion W.

Let us recall that the time reversed process $\hat{Y}$ of Y with respect to some horizon $T > 0$ is a solution to the stochastic differential equation given before (4.6)

$$\hat{Y}_t = \hat{Y}_0 + \int_0^t \sigma_0(\hat{Y}_s)\, dB_s + \int_0^t \tilde{b}(T-s, \hat{Y}_s)\, ds, \tag{6.3}$$

where B is a $\mathcal{F}^Y$-Brownian motion and $\tilde{b}$ is given by (4.6), (4.4), and (4.10):

$$\tilde{b}(s,y) = -\Big(\frac{\partial}{\partial x}\log p_s\Big)(x_0, h^{-1}(y));$$

$(p_t(x,y))$ is the fundamental solution associated with L. We remark that in this case p coincides with r since L is already in divergence form. As in the proof of Proposition 3.3, if $f \in \mathcal{D}_L$ then we can apply Itô formula to $(f \circ h^{-1})(\hat{Y})$ which equals in fact $f(\hat{X})$. We get

$$\begin{aligned} f(\hat{X}_t) &= f(\hat{X}_0) + \int_0^t (f'\sigma)(\hat{X}_s)\, dB_s + \int_0^t (Lf)(\hat{X}_s)\, ds \\ &\quad - \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s) f'(\hat{X}_s)\, ds. \end{aligned} \tag{6.4}$$

Remark 6.1 In a sense to be precised $\hat{X}$ solves the martingale problem related to

$$\tilde{L}f = Lf - \frac{\partial}{\partial x}(\log p_{T-s})(x_0, x) f'.$$

Lemma 3.23 tells us that the additional term due to time reversal belongs to $L^1_{loc}([0,T] \times \mathbb{R})$ in (t,x).

Even in this time reversed concept it is possible to define $\tilde{\mathcal{A}}$ as the unique extension of the map $f \mapsto \int_0^\cdot Lf(\hat{X})\, ds$ to C^1. The map $\tilde{A} : C^0 \to \mathcal{C}$ will be the unique extension of $l \mapsto \int_0^\cdot l'(\hat{X}_s)\, ds$. It is clear that (6.4) can be extended to C^1 (and even to $W^{1,2}_{loc}$) by

$$\begin{aligned} f(\hat{X}_t) &= f(\hat{X}_0) + \int_0^t (f'\sigma)(\hat{X}_s)\, dB_s + \tilde{\mathcal{A}}(f)_t \\ &\quad - \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s) f'(\hat{X}_s)\, ds. \end{aligned} \tag{6.5}$$

Remark 6.2 For $f \in C^1$, $l \in C^0$, we have

(i) $\tilde{A}(f)_t = \mathcal{A}(f)_T - \mathcal{A}(f)_{T-t}$,

(ii) $\tilde{A}(l)_t = A(l)_T - A(l)_{T-t}$.

(iii) For $f = id$, (6.5) yields

$$\begin{aligned}\hat{X}_t &= \hat{X}_0 + \int_0^t \sigma(\hat{X}_s)\, dB_s - \hat{A}\Big(\frac{\sigma^2}{2}\Big)_t \\ &\quad + A\Big(\frac{\sigma^2}{2}\Big)_T - \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s)\, ds.\end{aligned}$$

By time reversal, we get

$$\begin{aligned}X_t &= X_T - \int_t^T \sigma(X_s)\, d^+\hat{B}_s - A\Big(\frac{\sigma^2}{2}\Big)_t \\ &\quad + A\Big(\frac{\sigma^2}{2}\Big)_T - \int_t^T \frac{\partial}{\partial x}(\log p_{T-s})(x_0, X_s)\, ds.\end{aligned}$$

Thus, we have

$$\begin{aligned}X_t &= x_0 - \int_0^t \sigma(X_s)\, d^+B_s + A\Big(\frac{\sigma^2}{2}\Big)_t \\ &\quad - \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, X_s)\, ds.\end{aligned} \tag{6.6}$$

We consider now $f_0 \in C^1$ such that $f_0(0) = f_0'(0) = 0$ and $f_0' = 1/\sigma$. By regularization, it is not difficult to see that $\mathcal{L}f_0 = \frac{\sigma}{2}$. We now apply (3.6) and obtain

$$W_t = f_0(X_t) - f_0(x_0) - \frac{1}{2}A(\sigma)_t. \tag{6.7}$$

Using (6.5) with $f = f_0$, we get

$$\begin{aligned}B_t &= f_0(\hat{X}_t) - f_0(\hat{X}_0) - \frac{1}{2}\tilde{A}(\sigma)_t \\ &\quad + \int_0^t \frac{\frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s)}{\sigma(\hat{X}_s)}\, ds.\end{aligned} \tag{6.8}$$

Proposition 6.3 *$\hat{W}_t - W_T$ is a $\mathbb{H}$ semimartingale if and only if σ is of bounded variation.*

Remark 6.4 Proposition 6.3 means that $\hat{W}_t - W_T$ is a $\mathbb{H}$ semimartingale if and only if X is a semimartingale. □

Proof of Proposition 6.3. From (6.7) and Remark 6.2 we get

$$\hat{W}_t = f_0(\hat{X}_t) - f_0(x_0) - \frac{1}{2}A(\sigma)_T + \frac{1}{2}\tilde{A}(\sigma)_t.$$

Subtracting (6.8) from (6.7), we obtain

$$\begin{aligned} B_t - \hat{W}_t &= f_0(x_0) - f_0(X_T) + \hat{A}(\sigma)_t - \frac{1}{2}A(\sigma)_T \\ &\quad + \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s)\frac{1}{\sigma(\hat{X}_s)}\, ds. \end{aligned} \tag{6.9}$$

Since $f_0(X_T) = f_0(x_0) + W_T + \frac{1}{2}A(\sigma)_T$, we get

$$B_t - \hat{W}_t + W_T = \hat{A}(\sigma)_t + \int_0^t \frac{\partial}{\partial x}\log p_{T-s}(x_0, \hat{X}_s)\frac{1}{\sigma(\hat{X}_s)}\, ds. \tag{6.10}$$

We recall that $(\hat{W}_t - W_T)$ and B are both $\mathbb{H}$-adapted. Since B is a $\mathbb{H}$-Brownian motion and $A(\sigma)$, hence $\hat{A}(\sigma)$, a zero quadratic variation process, (6.10) shows that $(\hat{W}_t - W_T)$ is a $\mathbb{H}$-Dirichlet process.

Now, $(\hat{W}_t - W_T)$ is a $\mathbb{H}$-semimartingale if and only if $A(\sigma)$ has bounded variation. By Section 5, this happens if and only if σ is of bounded variation. □

We go on with the study of the backward equation. Let X be a solution to the martingale problem related to L. Let us suppose that

$$\int_0^t g(X_s)\, d^+W_s \text{ exists for all } g \in C^0. \tag{6.11}$$

Then, using (6.10), we see

$$\begin{aligned} \int_0^t g(X_s)\, d^+W_s &= -\int_{T-t}^T g(\hat{X}_s)\, d^-\hat{W}_s \\ &= -\int_{T-t}^T g(\hat{X}_s)\, dB_s - \int_{T-t}^T g(\hat{X}_s)\, d^-\hat{A}(\sigma)_s \end{aligned} \tag{6.12}$$

$$\quad - \int_{T-t}^T \frac{\partial}{\partial x}\log p_{T-s}(x_0, \hat{X}_s)\frac{g(\hat{X}_s)}{\sigma(\hat{X}_s)}\, ds. \tag{6.13}$$

In particular, $\int_{T-t}^{T} g(\hat{X}_s)\, d^- \hat{A}(\sigma)_s$ exists. Therefore we realize the following

Remark 6.5 $\int_0^t g(X_s)\, d^+ W_s$ exists if and only if $\int_0^t g(X_s)\, d^+ A(\sigma)_s$ exists. We recall that, by Lemma 5.2, this is always true if $g \in W_{loc}^{1,2}$ because X is a LZ process. □

Proposition 6.6 *Under the assumption (6.11) X is a solution to*

$$X_t = x_0 + \int_0^t \sigma(X_s)\, d^+ W_s. \tag{6.14}$$

Corollary 6.7 *If X is a semimartingale then (6.11) is always verified.*

Proof of Corollary 6.7. If X is a semimartingale then $\hat{X}$ is a $\mathbb{H}$-semimartingale because X is also a LZ process. In this case $(\hat{W}_t - W_T)$ is a $\mathbb{H}$-semimartingale by Remark 6.4. At this point, $\int_0^t g(X_s)\, d^+ W_s = -\int_{T-t}^{T} g(\hat{X}_s)\, d\hat{W}_s$ is a classical Itô integral. □

Proof of Proposition 6.6. Relation (6.10) implies

$$\hat{B}_t - W_t + W_T = A(\sigma)_t + \int_t^T \frac{\partial}{\partial x} \log p_{T-s}(x_0, X_s)\, ds$$

so that

$$d^+ \hat{B}_t = d^+ W_t + d^+ A(\sigma)_t + \frac{\frac{\partial}{\partial x} \log p_{T-t}(x_0, X_t)}{\sigma(X_t)}\, dt \tag{6.15}$$

holds. By (6.15), we evaluate

$$\begin{aligned} \int_0^t \sigma(X_s)\, d^+ W_s \quad &= \quad \int_0^t \sigma(X_s)\, d^+ \hat{B}_s - \int_0^t \sigma(X_s)\, d^+ A(\sigma)_s \\ &\quad - \int_0^t \frac{\partial}{\partial x} \log(p_{T-s})(x_0, X_s)\, ds. \end{aligned} \tag{6.16}$$

Comparing (6.6) and (6.16) yields

$$X_t - \int_0^t \sigma(X_s)\, d^+ W_s = -A(\sigma^2/2)_t + \int_0^t \sigma(X_s)\, d^+ A(\sigma)_s. \tag{6.17}$$

□

Remark 6.8 X fulfills (6.17) without assumption (6.11) provided that $\int_0^t \sigma(X_s)\, d^+ W_s$ exists. □.

If (6.11) is realized then

$$A(\sigma^2/2)_t = \int_0^t \sigma(X_s)\, d^+ A(\sigma)_s \tag{6.18}$$

holds. Assumption (6.11), Remark 6.5 and Banach-Steinhaus theorem imply that

$$g \mapsto \int_0^{\cdot} g(X_s)\, d^+ A(\sigma)_s$$

is continuous from C^0 to $\mathcal{C}$. Therefore

$$\int_0^t \sigma(X_s)\, d^+ A(\sigma)_s = \lim_{n\to\infty} \int_0^t \sigma_n(X_s)\, d^+ A(\sigma)_s$$

holds for the usual regularizations of σ^2.

Since $\sigma_n(X)$ is a finite quadratic variation process and $A(\sigma)$ a zero quadratic variation process, we get

$$\int_0^t \sigma_n(X_s)\, d^+ A(\sigma)_s = \int_0^t \sigma_n(X_s)\, d^- A(\sigma)_s.$$

Using the fact that $\sigma_n \in C^2$, Lemma 3.10 says that previous integral equals $A(\Phi(\sigma_n, \sigma))_t$, where $\Phi(g,l)(x) = (gl)(x) - gl(0) - \int_0^x g' l(y)\, dy$. So, we have

$$\int_0^t \sigma_n(X_s)\, d^+ A(\sigma)_s = A(\Phi(\sigma_n, \sigma))_t.$$

The problem here is that $(\Phi(\sigma_n, \sigma))$ does not necessarily tend to σ^2. Using the additivity of A, we get

$$\begin{aligned} A(\Phi(\sigma_n,\sigma)) &= A(\sigma_n\sigma) - A\left(\int_0^{\cdot} (\frac{\sigma_n^2}{2})(y)\, dy\right) \\ &= A(\frac{\sigma_n^2}{2}) - \int_0^{\cdot} \sigma_n'(\sigma - \sigma_n)(X_s)\, ds. \end{aligned}$$

Clearly $A(\sigma_n^2) \to A(\sigma^2)$ ucp so that $A(\Phi(\sigma_n,\sigma))$ converge to $A(\frac{\sigma^2}{2})$ if and only if

$$\int_0^t \sigma_n'(\sigma - \sigma_n)(X_s) ds \to 0 \tag{6.19}$$

holds in probability for any t.

Proposition 6.9 *If X is a semimartingale then (6.19) holds.*

Remark 6.10 In the general case there is no reason for (6.19) to be fulfilled.

Proof of the Proposition. Using localization techniques, we may assume σ, σ_n to have compact support. By Theorem 4.8, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t \sigma_n'(\sigma-\sigma_n)(X_s)\,ds\right)^2 \\
&= \int_0^t ds_1 \int_0^t ds_2\, \mathbb{E}\left(\sigma_n'(\sigma-\sigma_n)(X_{s_1})\sigma_n'(\sigma-\sigma_n)(X_{s_2})\right) \\
&= 2\int_0^t ds_1 \int_0^{s_1} ds_2 \int dy_1\, dy_2\, \sigma_n'(y_1)\sigma_n'(y_2)(\sigma-\sigma_n)(y_1)(\sigma-\sigma_n)(y_2) \\
&\quad p_{s_2}(x_0,y_2)p_{s_1-s_2}(y_2,y_1).
\end{aligned}$$

This equals

$$\begin{aligned}
&2\int_0^t ds_1 \int_0^{s_1} ds_2 \int d\sigma_n(y_1)d\sigma_n(y_2)(\sigma-\sigma_n)(y_1)(\sigma-\sigma_n)(y_2) \\
&\quad p_{s_2}(x_0,y_2)p_{s_1-s_2}(y_2,y_1) \\
&= 2\int d\sigma_n(y_1)\int d\sigma_n(y_2)(\sigma-\sigma_n)(y_1)(\sigma-\sigma_n)(y_2) \\
&\quad \int_0^t ds_1 \int_0^{s_1} ds_2 p_{s_2}(x_0,y_2)p_{s_1-s_2}(y_2,y_1).
\end{aligned}$$

By Corollary 5.8 and Remark 5.9 ii), σ is of bounded variation, so $\sigma_n \to \sigma$ in BV since $\sigma_n^2 \to \sigma^2$ in BV. The fact that $d\sigma_n \to$ weak-$*$ and that $\sigma_n \to \sigma$ in C^0, implies that the expression above converges to zero. □

We finish the paper with some remarks on the symmetric case; this corresponds to the case $\alpha = \frac{1}{2}$ in Remark 2.6a).

We consider

$$Lf = \frac{\sigma^2}{2}f'' + \frac{\sigma^{2\prime}}{4}f'.$$

Let X be a solution to the martingale problem related to L.

Proposition 6.11 *X is the unique solution to the stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s)\circ dW_s. \tag{6.20}$$

Proof. We first prove that (6.20) has a unique solution given by $F(W_t, X_0)$, where F is the deterministic flow given by

$$\frac{\partial F}{\partial r}(r, x_0) = \sigma(F(r, x_0)), \quad F(0, x_0) = x_0. \tag{6.21}$$

We recall that $F \in C^1(\mathbb{R}_+ \times \mathbb{R})$.

Let $H : \mathbb{R}_+ \times \mathbb{R}$ be the inverse flow. Let X be a $(\mathbb{F}, \mathbb{H})$-adapted process solving (6.20). Then the Itô formula of Remark 1.8 yields

$$H(W_t, X_t) = X_0 + \int_0^t \frac{\partial H}{\partial r}(W_s, X_s) \circ dW_s + \int_0^t \frac{\partial H}{\partial x}(W_s, X_s) \circ dX_s.$$

Remark 1.7, (6.20) and the fact that $\frac{\partial u}{\partial r}(r, x) = -\sigma(x)\frac{\partial u}{\partial x}(r, x)$ show that $H(W_t, X_t) \equiv X_0$. Therefore, the solution X must be equal to $F(W_t, X_0)$ and hence unique.

The fact that $F(W_t, X_0)$ solves (6.20) is a direct consequence of the one-dimensional LZ Itô formula of Remark 1.7.

Remark 6.11 The proof of Proposition 6.11 yields something more. If W is a $(\mathbb{F}, \mathbb{H})$ reversible semimartingale then there exists a unique $(\mathbb{F}, \mathbb{H})$ adapted solution to (6.20). $\square$.

In order to conclude the proof of Proposition (6.11) we have to show that $F(W_t, X_0)$ solves the martingale problem.

First of all, we observe that in this case, setting $h(0) = x_0$, $h'(r) = \sigma^{-1}(r)$, we get $h^{-1}(r) = F(r, x_0)$. This means that the process Y of Section 3 is in fact the Brownian motion W. We recall that $L^0 f = f''/2$. Proposition 2.12 says that

$$\mathcal{D}_L = \{f \in C^1 : f \circ h^{-1} \in C^2\}.$$

Moreover, $Lf = \frac{1}{2}(f \circ h^{-1})''$ holds. For $f \in \mathcal{D}_L$, the Itô formula yields

$$\begin{aligned}
f(X_t) &= (f \circ h^{-1})(W_t) \\
&= f(x_0) + \int_0^t (f \circ h^{-1})'(W_s)\, dW_s + \frac{1}{2}\int_0^t (f \circ h^{-1})''(W_s)\, ds \\
&= f(x_0) + M_t + \frac{1}{2}\int_0^t L^0 f \circ h^{-1}(W_s)\, ds \\
&= f(x_0) + M_t + \int_0^t (Lf) \circ h^{-1}(W_s)\, ds.
\end{aligned}$$

Therefore, X solves the martingale problem related to L. □

Remark 6.12 X solves also the symmetric equation

$$X_t = x_0 + \int_0^t \sigma(X_s)\, d^0 W_s \tag{6.22}$$

because of Remark 1.6.

Conversely, let X be a $(\mathbb{F}, \mathbb{H})$-adapted process and W a Brownian motion (or a more general $(\mathbb{F}, \mathbb{H})$ semimartingale). If X solves (6.22) then it also solves (6.20). However, other solutions to (6.22) may exist.

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References

[1] Aronson, D. G., *Bounds on the fundamental solution of a parabolic equation.* Bull. Amer. Math. Soc. **73**, 890-896 (1967).

[2] Albeverio, S., Hoegh-Krohn R., *Dirichlet forms and diffusion processes on rigged Hilbert spaces.* Z. Wahrsch. verw. Geb. **40**, 1-57 (1977).

[3] Assing, S., Schmidt, W.M., *Continuous strong Markov processes in dimension one.* Lecture Notes in Math. 1688 (1998).

[4] Bertoin, J., *Les processus de Dirichlet en tant qu'espace de Banach.* Stochastics **18**, 155-168 (1986).

[5] Bardina I Simorra, X., Jolis M., *An extension of Itô formula for elliptic diffusions.* Stochastic processes and its appl. **60**, 83-109 (1997).

[6] Bouleau, N., Yor, M., *Sur la variation quadratique des temps locaux de certaines semimartingales.* C. R. Acad. Sci. Paris, Série I **292**, 491-494 (1981).

[7] Cattiaux, P., Léonard, C., *Minimisation of the Kullback information of diffusion processes.* Ann. IHP **30(1)**, 83-132.

[8] Cattiaux, P.,Léonard, C., *Minimisation of the Kullback information for general Markov processes.* Sém. Prob. XXX, Lect. Notes Math. , 283-311, (1995)

[9] Chitasvili, R., Mania, M., *On functions transforming a Wiener process into a semimartingale.* Probab. Theory Rel. Fields **109**, 57-76 (1997).

[10] Cinlar, E., Jacod, J., Protter, P., Sharpe, M.J., *Semimartingales and Markov processes.*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **54**, 161-219 (1980).

[11] Doss, H., *Liens entre équations différentielles stochastiques et ordinaires.* Ann. IHP **13**, 99-125 (1977).

[12] Dunford, N., Schwartz, J.T., *Linear Operators, Part I, General Theory.* New York: Wiley (1967).

[13] Engelbert, H.J., Schmidt, W., *On solutions of one-dimensional stochastic differential equations without drift.* Z. Wahrscheinlichkeitstheorie verw. Gebiete **68**, 287-314 (1985).

[14] Engelbert, H.J., Schmidt, W., *Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations III.* Math. Nachr. **151**, 149-197 (1991).

[15] Engelbert, H.J., Wolf, J., *Strong Markov local Dirichlet processes and stochastic differential equations.* Teorija Veroyatnost. i ee Primenen. **43**, 331-348 (1998).

[16] Errami, M., Russo, F., Vallois, P., *Itô formula for $C^{1,\lambda}$-functions of a càdlàg process and related calculus.* Preprint 99-06, Paris 13 (1999). To appear: Probab. Theory Rel. Fields.

[17] Fabes, E., Kenig, *Examples of singular parabolic measures and singular transition probability densities.* Duke Mathematical Journal, Vol **48**, No 4, (1981).

[18] Flandoli, F., *An elementary proof of pathwise uniqueness, and existence of stochastic flows, for a one dimensional stochastic differential equation.* Preprint Pisa (1998).

[19] Föllmer, H., *Calcul d'Itô sans probabilités.* Sém. Prob. XV, Lect. Notes Math. **850**, 143-150 (1981).

[20] Föllmer, H., Protter, P., *On Itô's formula for multidimensional Brownian motion.* Probab. Theory Rel. Fields **116**, 1-20 (2000).

[21] Föllmer, H., Protter, P., Shiryaev, A.N., *Quadratic covariation and an extension of Itô's formula.* Bernoulli **1**, 149-169 (1995).

[22] Fukushima, M., *Dirichlet Forms and Markov processes.* North Holland (1990).

[23] Fukushima, M., Oshima, Y., Takeda, T., *Dirichlet forms and symetric Markov processes.* Walter de Gruyter (1994).

[24] Karatzas, I., Shreve, S.E., *Brownian Motion and Stochastic Calculus.* Springer Verlag, Second edition (1991).

[25] J. L. Lions, E. Magenes, *Problemes aux Limites non Homogenes et Applications*, Vol. I, II, Dunod, Paris 1968.

[26] Lyons, T.J., Zhang, T.S., *Decomposition of Dirichlet processes and its applications.* Ann. Probab. **22**, 494-524 (1994).

[27] Lyons, T.J., Zheng, W., *A crossing estimate for the canonical process on a Dirichlet space and tightness result.* Colloque Paul Lévy sur les processus stochastiques. Astérisque **157-158**, 249-271 (1988).

[28] Ma, Z.M., Röckner, M., *An introduction to (non-symmetric) Dirichlet forms.* Springer-Verlag, Berlin (1991).

[29] Mania, M., *Semimartingale characterizations of generalized derivatives.* Stochastics and Stoch. Reports **61**, 35-66 (1997).

[30] Martiano, J.J., *Sur quelques processus à dérive singulière et leurs applications en mécanique stochastique.* Thèse, Orsay 1999.

[31] Mathieu, P., *Zero white noise limit through Dirichlet forms, with applications to diffusions in a random medium.* Probab. Theory Rel. Fields **99**, 549-580, (1994).

[32] Mathieu, P., *Limit theorem for diffusions with a random potential.* Stoch. Processes and their Applications **60**, 105-111 (1995).

[33] Millet, A., Nualart, D., Sanz, M., *Integration by parts and time reversal of diffusion processes.* Ann. Probab. **17**, 208-238 (1989).

[34] Moret, S., Nualart, D., *Quadratic covariation and Itô's formula for smooth nondegenerate martingales.* Preprint Barcelona (1998), J. Th. Probability, to appear.

[35] Moret, S., Nualart, D., *Generalization of Itô's formula for smooth nondegenerate martingales.* Preprint 1999, Barcelona.

[36] Oshima, Y., *On a construction of Markov processes associated with time dependent Dirichlet spaces.* Forum Mathematicum **4**, 395-415 (1992).

[37] Pardoux, E., *Grossissement d'une filtration et retournement d'une diffusion.* Sém. Prob. XX, Lect. Notes Math. **1204**, (1986).

[38] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[39] Portenko, N.I., *Generalized diffusion processes*, American mathematical society. **Vol. 83**, (1990).

[40] Rozkosz, A., *Stochastic representation of diffusions corresponding to divergence form operators*, Stoc. Processes and their Applications **63**, 11-33 (1996).

[41] Revuz, D., Yor, M., *Continuous Martingales and Brownian Motion.* Berlin, New York: Springer-Verlag (1994).

[42] Russo, F., Vallois, P., *Forward, backward and symmetric stochastic integration.* Probab. Theory Relat. Fields **97**, 403-421 (1993).

[43] Russo, F., Vallois, P., *The generalized covariation process and Itô formula.* Stochastic Processes and their Applications **59**, 81-104 (1995).

[44] Russo, F., Vallois, P., *Itô formula for C^1-functions of semimartingales.* Probab. Theory Relat. Fields **104**, 27-41 (1996).

[45] Russo, F., Vallois, P., *Stochastic calculus with respect to a finite variation process.* Preprint BiBoS (1998). To appear: Stochastics and Stochastic Reports.

[46] Russo, F., Vallois, P., Wolf, J., *A generalized class of Lyons-Zheng processes.* Preprint Paris 13 (1999). To appear: Journal of the Bernoulli Society.

[47] Seignourel, P., *Processus dans un milieu irrégulier. Une approche par les formes de Dirichlet.* Preprint RI, 388, CMAP-Ecole Polytechnique (1998).

[48] Stannat, W., *The theory of generalized Dirichlet forms and its applications in analysis and stochastics.* To appear: Memoirs of the Americal Mathematical Society (1999).

[49] Stroock, D., *Diffusion processes corresponding to uniformly elliptic divergence form operators.* Lect. Notes Math. **1321**, 316-347 (1980)

[50] H. Tanabe, *Equations of Evolution*, Pitman, London 1979.

[51] Wolf, J., *Transformations of semimartingales and local Dirichlet processes.* Stochastics and Stochastics Reports **62**, 95-101 (1997).

[52] Wolf, J., *An Itô formula for continuous local Dirichlet processes.* Stochastics and Stochastics Reports, **62**, 103-115 (1997).

[53] Wolf, J., *A representation theorem for continuous additive functionals of zero quadratic variation.* Prob. Math. Stat. **18**, 367-379 (1998)

[54] Zvonkin, *A transformation of the phase space of a diffusion process that removes the drift.* Math. Sbornik, **T 93 (135)** No 1, (1974).